Solution of the quartic matrix model

Raimar Wulkenhaar

Mathematisches Institut, Westfälische Wilhelms-Universität Münster

(based on joint work with Harald Grosse,
arXiv: 1205.0465v4)

Field-theoretical matrix models

- classical scalar field $\phi \in \mathcal{C}_0(\mathbb{R}^d) \subset \mathcal{B}(H)$, with $\frac{m^2}{2}\int_{\mathbb{R}^d} dx\ \phi^2(x) < \infty$
- translates to $\mathrm{tr}(\phi^2) < \infty$, i.e. nc scalar field is Hilbert-Schmidt compact operator on Hilbert space $H = L^2(I, \mu)$
- realise as integral kernel operators: $M = (M_{ab}) \in L^2(I \times I, \mu \times \mu)$
  - product: $(MN)_{ab} = \int_I d\mu(c)\ M_{ac} M_{cb}$
  - trace: $\mathrm{tr}(M) = \int_I d\mu(a)\ M_{aa}$
  - adjoint: $(M^*)_{ab} = \overline{M_{ba}}$
- action = non-linear functional S for $\phi = \phi^*$ in volume V:

$$S[\phi] = V\ \mathrm{tr}(E\phi^2 + P[\phi])$$

E – unbounded positive selfadjoint op. with compact resolvent,
$P[\phi]$ – polynomial in ϕ with scalar coefficients

Euclidean quantum field theory

- action with source term $\longrightarrow$ partition function

$$\mathcal{Z}[J] = \int \mathcal{D}\phi\ \exp(-S[\phi] + V\,\mathrm{tr}(\phi J))$$

where $\mathcal{D}\phi = \prod_{a,b\in I} d\phi_{ab}$ (Lebesgue on finite-rank op's)

- We cannot expect that $\mathcal{Z}$ has a limit for $V \to \infty$.
The aim is to prove existence of $\lim_{V\to\infty} \frac{1}{volume(V)} \log \mathcal{Z}[J]$.

- $\mathcal{W}[J] = \log \mathcal{Z}[J]$ gives rise to connected correlation functions

$$\langle \varphi_{a_1 b_1} \dots \varphi_{a_N b_N} \rangle_c = \frac{\partial^N \mathcal{W}[J]}{\partial J_{b_1 a_1} \dots \partial J_{b_N a_N}}\Big|_{J=0}$$

- unless I is finite, the resulting index sums may diverge and require a renormalisation

Topological expansion

- Connected Feynman graphs in matrix models are ribbon graphs.
- Viewed as simplicial complexes, they encode the topology (B, g) of a genus-g Riemann surface with B boundary components (or punctures, marked points, holes, faces).
- Matrix index conserved along each strand of the ribbon graph: Right index of J_{ab} is left index of another J_{bc}, or of the same J_{bb}.
- The k^{th} boundary component carries a cycle $J^{N_k}_{p_1 \dots p_{N_k}} := \prod_{j=1}^{N_k} J_{p_j p_{j+1}}$ of N_k external sources, $N_k + 1 \equiv 1$.
- Expand $\log \mathcal{Z}[J] = \sum \frac{1}{S} V^{2-B} G_{|p_1 \dots p_{N_1}| \dots |q_1 \dots q_{N_B}|} J^{N_1}_{p_1 \dots p_{N_1}} \cdots J^{N_B}_{q_1 \dots q_{N_B}}$ according to the cycle structure.

The first terms for $P[\phi] = P[-\phi]$ are:

$$\begin{aligned}
\frac{\mathcal{W}[J]}{V^2} &= \frac{\mathcal{W}[0]}{V^2} + \frac{1}{V}\sum_{p,q\in I} G_{|pq|}\Big(\frac{J_{pq}J_{qp}}{2}\Big) + \frac{1}{2!V^2}\sum_{p,q\in I} G_{|p|q|}\Big(\frac{J_{pp}}{1}\Big)\Big(\frac{J_{qq}}{1}\Big)\\
&+ \frac{1}{V}\sum_{p,q,r,s\in I} G_{|pqrs|}\Big(\frac{J_{pq}J_{qr}J_{rs}J_{sp}}{4}\Big) + \frac{1}{V^2}\sum_{p,q,r,s\in I} G_{|p|qrs|}\Big(\frac{J_{pp}}{1}\Big)\Big(\frac{J_{qr}J_{rs}J_{sq}}{3}\Big)\\
&+ \frac{1}{2!V^2}\sum_{p,q,r,s\in I} G_{|pq|rs|}\Big(\frac{J_{pq}J_{qp}}{2}\Big)\Big(\frac{J_{rs}J_{sr}}{2}\Big)\\
&+ \frac{1}{2!V^3}\sum_{p,q,r,s\in I} G_{|p|q|rs|}\Big(\frac{J_{pp}}{1}\Big)\Big(\frac{J_{qq}}{1}\Big)\Big(\frac{J_{rs}J_{sr}}{2}\Big)\\
&+ \frac{1}{4!V^4}\sum_{p,q,r,s\in I} G_{|p|q|r|s|}\Big(\frac{J_{pp}}{1}\Big)\Big(\frac{J_{qq}}{1}\Big)\Big(\frac{J_{rr}}{1}\Big)\Big(\frac{J_{ss}}{1}\Big) + \mathcal{O}(J^6)
\end{aligned}$$

Attention: $G_{|pp|}J_{pp}J_{pp}$ is topologically different from $G_{|p|p|}J_{pp}J_{pp}$!

Ward identity

- unitary transformation $\phi \mapsto \tilde{\phi} = U\phi U^*$
  $U \in (\mathcal{K}(H))^1$ with $UU^* = U^*U = \mathrm{id}$, leaves $\mathcal{K}_{s.a.}$ invariant:

$$\int \mathcal{D}\phi \; \exp(-S[\phi]+V\,\mathrm{tr}(\phi J)) = \int \mathcal{D}\tilde{\phi} \; \exp(-S[\tilde{\phi}]+V\,\mathrm{tr}(\tilde{\phi} J))$$

- measure unitarily invariant: $\mathcal{D}\tilde{\phi} = \mathcal{D}\phi$:

$$0 = \int \mathcal{D}\phi \Big[\exp(-S[\phi] + V\,\mathrm{tr}(\phi J)) - \exp(-S[\tilde{\phi}] + V\,\mathrm{tr}(\tilde{\phi} J))\Big]$$

note: $[\;\;] \neq 0$ because $\mathrm{tr}(E\phi^2)$, $\mathrm{tr}(\phi J)$ not unitarily invariant!

- linearisation $\longrightarrow$ Ward identity (matricial equation!)

$$0 = \int \mathcal{D}\phi \Big[E\phi\phi - \phi\phi E - J\phi + \phi J\Big] \exp(-S[\phi] + V\,\mathrm{tr}(\phi J))$$

- choose a reference frame where E is diagonal (but J is not)
- use functional derivative $\phi_{ab} = \frac{\partial}{V\partial J_{ba}}$

Proposition [Disertori-Gurau-Magnen-Rivasseau, 2006]

The partition function $\mathcal{Z}[J]$ of the matrix model defined by the external matrix E satisfies the $|I| \times |I|$ Ward identities

$$0 = \sum_{n\in I}\Big(\frac{(E_a - E_p)}{V}\frac{\partial^2 \mathcal{Z}}{\partial J_{an}\partial J_{np}} + J_{pn}\frac{\partial \mathcal{Z}}{\partial J_{an}} - J_{na}\frac{\partial \mathcal{Z}}{\partial J_{np}}\Big)$$

Main assumption

- $m \mapsto E_m > 0$ injective

Note: always the case if we pass to equivalence classes $[m]$ which have the same E_m; then $\sum_{m\in I} f(m) = \sum_{[m]\in[I]} \mu([m])f([m])$

We will see: These Ward identities (and the choice of $P[\phi]$) determine the QFT of the matrix model non-perturbatively!

We turn the Ward identity for E injective into formula for $\sum_{n\in I} \frac{\partial^2 \mathcal{Z}[J]}{\partial J_{an}\partial J_{np}}$. The J-cycle structure in $\log \mathcal{Z}$ creates

- singular contributions $\sim \delta_{ap}$
- regular contributions present for all a, p

Theorem (Ward identity for injective E)

$$\sum_{n\in I} \frac{\partial^2 \mathcal{Z}[J]}{\partial J_{an}\partial J_{np}} = \delta_{ap}\Big\{V^2 \sum_{(K)} \frac{J_{P_1}\cdots J_{P_K}}{S_K}\Big(\sum_{n\in I} \frac{G_{|an|P_1|\dots|P_K|}}{V^{|K|+1}} + \frac{G_{|a|a|P_1|\dots|P_K|}}{V^{|K|+2}}$$

$$+\sum_{r\geq 1}\sum_{q_1,\dots,q_r\in I} \frac{G_{|q_1 a q_1\dots q_r|P_1|\dots|P_K|}J^r_{q_1\dots q_r}}{V^{|K|+1}}\Big)$$

$$+V^4 \sum_{(K),(K')} \frac{J_{P_1}\cdots J_{P_K}J_{Q_1}\cdots J_{Q_{K'}}}{S_K S_{K'}} \frac{G_{|a|P_1|\dots|P_K|}}{V^{|K|+1}} \frac{G_{|a|Q_1|\dots|Q_{K'}|}}{V^{|K'|+1}}\Big\}\mathcal{Z}[J]$$

$$+\frac{V}{E_p - E_a}\sum_{n\in I}\Big(J_{pn}\frac{\partial \mathcal{Z}[J]}{\partial J_{an}} - J_{na}\frac{\partial \mathcal{Z}[J]}{\partial J_{np}}\Big)$$

There are four sources of a singular contribution $\sim \delta_{ap}$:

1. $$\sum_n \frac{\partial^2}{\partial J_{an}\partial J_{np}} \sum_{q_1,q_2,\dots,} G_{\dots|q_1q_2|\dots}\Big(\frac{\overset{\downarrow}{J}_{q_1q_2}\overset{\downarrow}{J}_{q_2q_1}}{2}\Big)\prod J$$

2. $$\sum_n \frac{\partial^2}{\partial J_{an}\partial J_{np}} \sum_{q_1,q_2,\dots,} G_{\dots|q_1|\dots|q_2|\dots}\Big(\frac{\overset{\downarrow}{J}_{q_1q_1}}{1}\Big)\Big(\frac{\overset{\downarrow}{J}_{q_2q_2}}{1}\Big)\prod J$$

3. $$\sum_n \frac{\partial}{\partial J_{an}}\frac{\partial}{\partial J_{np}} \sum_{q_0,\dots,q_{r+1},\dots,} G_{\dots|q_0q_1\dots q_rq_{r+1}|\dots}\Big(\frac{J_{q_0q_1}J_{q_1q_2}\cdots J_{q_rq_{r+1}}\overset{\downarrow}{J}_{q_{r+1}q_0}}{r+2}\Big)\prod J$$
$$= \sum_n \frac{\partial}{\partial J_{an}} \sum_{q_1,\dots,q_r,\dots,} G_{\dots|pq_1\dots q_rn|\dots}\Big(\overset{\downarrow}{J}_{pq_1}J_{q_1q_2}\cdots J_{q_rn}\Big)\prod J$$

4. $$\sum_n \frac{\partial^2}{\partial J_{an}\partial J_{np}}\Big[\sum_{q_1\dots,} G_{\dots|q_1|\dots}\Big(\frac{\overset{\downarrow}{J}_{q_1q_1}}{1}\Big)\prod J\Big]\Big[\sum_{q_2\dots,} G_{\dots|q_2|\dots}\Big(\frac{\overset{\downarrow}{J}_{q_2q_2}}{1}\Big)\prod J\Big]$$

- All other derivatives persist for $a \neq p$ and therefore must be captured by the Ward identity for $a \neq p$.
- $\frac{V}{E_p-E_a}\sum_{n\in I}\Big(J_{pn}\frac{\partial\mathcal{Z}[J]}{\partial J_{an}} - J_{na}\frac{\partial\mathcal{Z}[J]}{\partial J_{np}}\Big)$ has a meaningful limit $a \to p$!

Schwinger-Dyson equations

Write $S = \frac{V}{2}\sum_{a,b}(E_a + E_b)\phi_{ab}\phi_{ba} + VS_{int}[\phi]$.
The E_a are the eigenvalues of E.

- Functional integration yields, up to irrelevant constant,

$$\mathcal{Z}[J] = e^{-VS_{int}[\frac{\partial}{V\partial J}]} e^{\frac{V}{2}\langle J,J\rangle_E} \,, \qquad \langle J,J\rangle_E := \sum_{m,n\in I} \frac{J_{mn}J_{nm}}{E_m + E_n}$$

- Much better than the perturbative expansion of $e^{-VS_{int}[\frac{\partial}{V\partial J}]}$ is to apply J-derivatives to $\mathcal{Z}[J]$.
  $\to$ Choose them to give $G_{\dots}$ on the lhs.
- These external derivatives combine on rhs with internal derivatives from $S_{int}[\frac{\partial}{V\partial J}]$ to certain identities for $G_{\dots}$.

These Schwinger-Dyson equations are often of little use because they express an N-point function in terms of $(N+2)$-point functions.

How to use the Ward identity: $G_{|ab|}$ for $a \neq b$

$$G_{|ab|} = \frac{1}{V\mathcal{Z}[0]} \frac{\partial^2 \mathcal{Z}[J]}{\partial J_{ba} \partial J_{ab}}\Big|_{J=0}$$

disconnected part of $\mathcal{Z}$ does not contribute for $a \neq b$

$$= \frac{1}{V\mathcal{Z}[0]} \Big\{ \frac{\partial}{\partial J_{ba}} e^{-VS_{int}\left[\frac{\partial}{V\partial J}\right]} \frac{\partial}{\partial J_{ab}} e^{\frac{V}{2}\langle J,J\rangle_E} \Big\}_{J=0}$$

$$= \frac{1}{(E_a+E_b)\mathcal{Z}[0]} \Big\{ \frac{\partial}{\partial J_{ba}} e^{-VS_{int}\left[\frac{\partial}{V\partial J}\right]} J_{ba} e^{\frac{V}{2}\langle J,J\rangle_E} \Big\}_{J=0}$$

$$= \frac{1}{E_a+E_b} + \frac{1}{(E_a+E_b)\mathcal{Z}[0]} \Big\{ \Big(\phi_{ab} \frac{\partial(-VS_{int})}{\partial \phi_{ab}} \Big) \Big[\frac{\partial}{V\partial J}\Big] \Big\} \mathcal{Z}[J]\Big|_{J=0}$$

- $\frac{\partial(-VS_{int})}{\partial \phi_{ab}}$ contains, for any $P[\phi]$, the derivative $\sum_n \frac{\partial^2}{\partial J_{an} \partial J_{np}}$
- For $P[\phi] = \frac{\lambda}{4}\phi^4$: $\frac{\partial(-VS_{int})}{\partial \phi_{ab}} = -\lambda V \sum_{n,p \in I} \phi_{bp}\phi_{pn}\phi_{na}$

$$\Rightarrow \quad \Big(\phi_{ab} \frac{\partial(-VS_{int})}{\partial \phi_{ab}} \Big) \Big[\frac{\partial}{V\partial J}\Big] = -\frac{\lambda}{V^3} \sum_{p,n \in I} \frac{\partial^4}{\partial J_{pb} \partial J_{ba} \partial J_{an} \partial J_{np}}$$

$$G_{|ab|} = \frac{1}{E_a+E_b} - \frac{\lambda}{V^3(E_a+E_b)\mathcal{Z}[0]} \sum_{p\in I} \frac{\partial^2}{\partial J_{pb}\partial J_{ba}} \sum_{n\in I} \frac{\partial^2 \mathcal{Z}}{\partial J_{an}\partial J_{np}}\Big|_{J=0}$$

$$= \frac{1}{E_a+E_b} - \frac{\lambda}{V(E_a+E_b)\mathcal{Z}[0]} \frac{\partial^2}{\partial J_{ab}\partial J_{ba}} \Big\{ \Big(\sum_{n\in I} \frac{G_{|an|}}{V} + \sum_{n,q,r\in I} \frac{G_{|an|qr|}}{V^2} \frac{J_{qr}J_{rq}}{2} + \sum_{n,q,r\in I} \frac{G_{|an|q|r|}}{V^3} \frac{J_{qq}}{1} \frac{J_{rr}}{1}$$

$$+ \frac{G_{|a|a|}}{V^2} + \sum_{q,r\in I} \frac{G_{|a|a|qr|}}{V^3} \frac{J_{qr}J_{rq}}{2} + \sum_{q,r\in I} \frac{G_{|a|a|q|r|}}{V^4} \frac{J_{qq}}{1} \frac{J_{rr}}{1}$$

$$+ \sum_{q,r\in I} \frac{G_{|qaqr|}}{V} J_{qr}J_{rq} + V^2 \frac{G_{|a|q|}}{V^2} \frac{J_{qq}}{1} \frac{G_{|a|r|}}{V^2} \frac{J_{rr}}{1} \Big) \mathcal{Z}[J] \Big\}\Big|_{J=0}$$

$$- \frac{\lambda}{V^2(E_a+E_b)\mathcal{Z}[0]} \sum_{p\in I} \frac{\left(\frac{\partial^2 \mathcal{Z}[J]}{\partial J_{ab}\partial J_{ba}} + \delta_{pb} \frac{\partial^2 \mathcal{Z}[J]}{\partial J_{aa}\partial J_{bb}} - \frac{\partial^2 \mathcal{Z}[J]}{\partial J_{pb}\partial J_{bp}} \right)}{E_p - E_a}\Big|_{J=0}$$

$$G_{|ab|} = \frac{1}{E_a+E_b} - \frac{\lambda}{V^3(E_a+E_b)\mathcal{Z}[0]} \sum_{p\in I} \frac{\partial^2}{\partial J_{pb}\partial J_{ba}} \sum_{n\in I} \frac{\partial^2 \mathcal{Z}}{\partial J_{an}\partial J_{np}}\Big|_{J=0}$$

$$= \frac{1}{E_a+E_b} - \frac{\lambda}{V(E_a+E_b)\mathcal{Z}[0]} \frac{\partial^2}{\partial J_{ab}\partial J_{ba}} \Big\{ \Big(\sum_{n\in I} \frac{G_{|an|}}{V} + \sum_{n,q,r\in I} \frac{G_{|an|qr|}}{V^2}\frac{J_{qr}J_{rq}}{2} + \sum_{n,q,r\in I}\frac{G_{|an|q|r|}}{V^3}\frac{J_{qq}}{1}\frac{J_{rr}}{1}$$

$$+ \frac{G_{|a|a|}}{V^2} + \sum_{q,r\in I}\frac{G_{|a|a|qr|}}{V^3}\frac{J_{qr}J_{rq}}{2} + \sum_{q,r\in I}\frac{G_{|a|a|q|r|}}{V^4}\frac{J_{qq}}{1}\frac{J_{rr}}{1}$$

$$+ \sum_{q,r\in I}\frac{G_{|qaqr|}}{V}J_{qr}J_{rq} + V^2\frac{G_{|a|q|}}{V^2}\frac{J_{qq}}{1}\frac{G_{|a|r|}}{V^2}\frac{J_{rr}}{1}\Big) \overset{\downarrow}{\mathcal{Z}}[J] \Big\}\Big|_{J=0}$$

$$- \frac{\lambda}{V^2(E_a+E_b)\mathcal{Z}[0]} \sum_{p\in I} \frac{\Big(\frac{\partial^2\mathcal{Z}[J]}{\partial J_{ab}\partial J_{ba}} + \delta_{pb}\frac{\partial^2\mathcal{Z}[J]}{\partial J_{aa}\partial J_{bb}} - \frac{\partial^2\mathcal{Z}[J]}{\partial J_{pb}\partial J_{bp}}\Big)}{E_p - E_a}\Big|_{J=0}$$

$$G_{|ab|} = \frac{1}{E_a+E_b} - \frac{\lambda}{V^3(E_a+E_b)\mathcal{Z}[0]} \sum_{p\in I} \frac{\partial^2}{\partial J_{pb}\partial J_{ba}} \sum_{n\in I} \frac{\partial^2 \mathcal{Z}}{\partial J_{an}\partial J_{np}}\Big|_{J=0}$$

$$= \frac{1}{E_a+E_b} - \frac{\lambda}{V(E_a+E_b)\mathcal{Z}[0]} \frac{\partial^2}{\partial J_{ab}\partial J_{ba}} \Big\{$$

$$\Big(\sum_{n\in I} \frac{G_{|an|}}{V} + \sum_{n,q,r\in I} \frac{G_{|an|qr|}}{V^2}\frac{J_{qr}J_{rq}}{2} + \sum_{n,q,r\in I} \frac{G_{|an|q|r|}}{V^3}\frac{J_{qq}}{1}\frac{J_{rr}}{1}$$

$$+ \frac{G_{|a|a|}}{V^2} + \sum_{q,r\in I} \frac{G_{|a|a|qr|}}{V^3}\frac{J_{qr}J_{rq}}{2} + \sum_{q,r\in I} \frac{G_{|a|a|q|r|}}{V^4}\frac{J_{qq}}{1}\frac{J_{rr}}{1}$$

$$+ \sum_{q,r\in I} \frac{G_{|qaqr|}}{V} J_{qr}J_{rq} + V^2 \frac{G_{|a|q|}}{V^2}\frac{J_{qq}}{1}\frac{G_{|a|r|}}{V^2}\frac{J_{rr}}{1}\Big) \mathcal{Z}[J] \Big\}\Big|_{J=0}$$

$$- \frac{\lambda}{V^2(E_a+E_b)\mathcal{Z}[0]} \sum_{p\in I} \frac{\left(\frac{\partial^2 \mathcal{Z}[J]}{\partial J_{ab}\partial J_{ba}} + \delta_{pb} \frac{\partial^2 \mathcal{Z}[J]}{\partial J_{aa}\partial J_{bb}} - \frac{\partial^2 \mathcal{Z}[J]}{\partial J_{pb}\partial J_{bp}} \right)}{E_p - E_a}\Bigg|_{J=0}$$

With $\frac{\partial^2 \mathcal{Z}[J]}{\partial J_{pb} \partial J_{bp}} = VG_{|pb|} + \delta_{pb} G_{|p|b|}$:

Schwinger-Dyson equation for 2-point function

$$G_{|ab|} = \frac{1}{E_a + E_b} - \frac{\lambda}{E_a + E_b} \frac{1}{V} \sum_{p \in I} \Big(G_{|ab|} G_{|ap|} - \frac{G_{|pb|} - G_{|ab|}}{E_p - E_a} \Big)$$
$$- \frac{\lambda}{V^2 (E_a + E_b)} \Big(G_{|a|a|} G_{|ab|} + \frac{1}{V} \sum_{n \in I} G_{|an|ab|} + G_{|aaab|} + G_{|baba|} - \frac{G_{|b|b|} - G_{|a|b|}}{E_b - E_a} \Big)$$
$$- \frac{\lambda}{V^4 (E_a + E_b)} G_{|a|a|ab|}$$

- Genus expansion $G_{\ldots} = \sum_{g=0}^{\infty} V^{-2g} G^{(g)}_{\ldots}$ (only Borel summable?)

With $\frac{\partial^2 \mathcal{Z}[J]}{\partial J_{pb}\partial J_{bp}} = VG_{|pb|} + \delta_{pb}G_{|p|b|}$:

Schwinger-Dyson equation for 2-point function

$$G_{|ab|} = \frac{1}{E_a+E_b} - \frac{\lambda}{E_a+E_b}\frac{1}{V}\sum_{p\in I}\Big(G_{|ab|}G_{|ap|} - \frac{G_{|pb|}-G_{|ab|}}{E_p-E_a}\Big)$$

$$g\mapsto g+1 \left\{ -\frac{\lambda}{V^2(E_a+E_b)}\Big(G_{|a|a|}G_{|ab|} + \frac{1}{V}\sum_{n\in I}G_{|an|ab|} + G_{|aaab|} + G_{|baba|} - \frac{G_{|b|b|}-G_{|a|b|}}{E_b-E_a}\Big)\right.$$

$$g\mapsto g+2 \left\{ -\frac{\lambda}{V^4(E_a+E_b)}G_{|a|a|ab|}\right.$$

- Genus expansion $G_{\dots} = \sum_{g=0}^{\infty} V^{-2g}G^{(g)}_{\dots}$ (only Borel summable?)

  Precisely the first line preserves g, the others increase it!
- Limit $V\to\infty$ with $\frac{1}{V}\sum_{p\in I}$ finite of exact Schwinger-Dyson equation coincides with its restriction to planar sector $g=0$
- This is a closed non-linear equation for $G_{|ab|}$ alone!

$$G^{(0)}_{|ab|} = \frac{1}{E_a + E_b} - \frac{\lambda}{E_a + E_b} \frac{1}{V} \sum_{p \in I} \Big(G^{(0)}_{|ab|} G^{(0)}_{|ap|} - \frac{G^{(0)}_{|pb|} - G^{(0)}_{|ab|}}{E_p - E_a} \Big)$$

- We have derived in 2007/08 this selfconsistency equation for the Moyal model by the graphical method invented by [Disertori-Gurau-Magnen-Rivasseau, 2007].
- In this form, equation is meaningless: $\sum_{p \in I}$ diverges.
- In 2009 we solved the renormalisation problem. This is a renormalisation of infinitely many Feynman graphs at once!
- Renormalisation increases the non-linearity. In [arXiv:0909.1389] perturbative solution to $\mathcal{O}(\lambda^3)$.
- No progress with non-perturbative solution 2010+2011.
- Breakthrough 2012: Equation is linear in difference $G^{(0)}_{|ab|} - G^{(0)}_{|a0|}$ to the boundary and non-linear only in $G^{(0)}_{|a0|}$!

Higher N-point functions

$$G_{|ab_1\dots b_{N-1}|} = -\frac{\lambda}{E_a+E_{b_1}}\Big\{\frac{1}{V}\sum_{p\in I}\Big(G_{|ap|}G_{|ab_1\dots b_{N-1}|} - \frac{G_{|pb_1\dots b_{N-1}|}-G_{|ab_1\dots b_{N-1}|}}{E_p-E_a}\Big) - \sum_{l=1}^{\frac{N-2}{2}} G_{|b_1\dots b_{2l}|}\frac{G_{|b_{2l+1}\dots b_{N-1}a|}-G_{|b_{2l+1}\dots b_{N-1}b_l|}}{E_{b_{2l}}-E_a}\Big\}$$

$$g\mapsto g+1\Big\{-\frac{\lambda}{V^2(E_a+E_{b_1})}\Big(G_{|a|a|}G_{|ab_1\dots b_{N-1}|} + \sum_{k=1}^{N-1} G_{|b_1\dots b_k ab_k\dots b_{N-1}a|} + G_{|aaab_1\dots b_{N-1}|} + \frac{1}{V}\sum_{n\in I} G_{|an|ab_1\dots b_{N-1}|} - \sum_{k=1}^{N-1}\frac{G_{|b_1\dots b_k|b_{k+1}\dots b_{N-1}b_k|}-G_{|b_1\dots b_k|b_{k+1}\dots b_{N-1}a|}}{E_{b_k}-E_a}\Big)$$

$$g\mapsto g+2\Big\{-\frac{\lambda}{V^4(E_a+E_{b_1})}G_{|a|a|ab_1\dots b_{N-1}|}$$

- The planar sector $G^{(0)}_{|ab_1\dots b_{N-1}|}$, exact for $V\to\infty$ with $\frac{1}{V}\sum_{p\in I}$ finite, is a linear inhomogeneous equation with inductively known parameters!

Reality $\phi = \phi^*$

$\mathcal{Z} = \overline{\mathcal{Z}}$ implies invariance under orientation reversal

$$G_{|p_0^1 p_1^1 \dots p_{N_1-1}^1|\dots|p_0^B p_1^B \dots p_{N_B-1}^B|} = G_{|p_0^1 p_{N_1-1}^1 \dots p_1^1|\dots|p_0^B p_{N_B-1}^B \dots p_1^B|}$$

- empty for $G_{|ab|}$
- cancellations in $(E_a + E_{b_1}) G_{ab_1 b_2 \dots b_{N-1}} - (E_a + E_{b_{N-1}}) G_{ab_{N-1}\dots b_2 b_1}$

Theorem (universal algebraic recursion formula)

$$\begin{aligned} &G_{|b_0 b_1 \dots b_{N-1}|} \\ &= (-\lambda) \sum_{l=1}^{\frac{N-2}{2}} \frac{G_{|b_0 b_1 \dots b_{2l-1}|} G_{|b_{2l} b_{2l+1} \dots b_{N-1}|} - G_{|b_{2l} b_1 \dots b_{2l-1}|} G_{|b_0 b_{2l+1} \dots b_{N-1}|}}{(E_{b_0} - E_{b_{2l}})(E_{b_1} - E_{b_{N-1}})} \\ &+ \frac{(-\lambda)}{V^2} \sum_{k=1}^{N-1} \frac{G_{|b_0 b_1 \dots b_{k-1}|b_k b_{k+1} \dots b_{N-1}|} - G_{|b_k b_1 \dots b_{k-1}|b_0 b_{k+1} \dots b_{N-1}|}}{(E_{b_0} - E_{b_k})(E_{b_1} - E_{b_{N-1}})} \end{aligned}$$

(Last line increases the genus and is absent in $G^{(0)}_{|b_0 b_1 \dots b_{N-1}|}$)

Example: $G_{|abcd|} = G_{|adcb|}$

$$
\begin{aligned}
&(E_a+E_b)G_{|abcd|}\\
&\quad= -\lambda\Big\{\frac{1}{V}\sum_{p\in I}\Big(G_{|ap|}G_{|abcd|}-\frac{G_{|pbcd|}-G_{|abcd|}}{E_p-E_a}\Big)-G_{|bc|}\frac{G_{|da|}-G_{|dc|}}{E_c-E_a}\Big\}\\
&\quad-\frac{\lambda}{V^2}\Big\{G_{|a|a|}G_{|abcd|}+G_{|babcda|}+G_{|bcacda|}+G_{|bcdada|}+G_{|aaabcd|}+\frac{1}{V}\sum_{n\in I}G_{|an|abcd|}\\
&\qquad\qquad-\frac{G_{|b|cdb|}-G_{|b|cda|}}{E_b-E_a}-\frac{G_{|bc|dc|}-G_{|bc|da|}}{E_c-E_a}-\frac{G_{|bcd|d|}-G_{|bcd|a|}}{E_d-E_a}\Big\}\\
&\quad-\frac{\lambda}{V^4}\Big\{G_{|a|a|abcd|}\Big\}
\end{aligned}
$$

$$
\begin{aligned}
&(E_a+E_d)G_{|adcb|}\\
&\quad= -\lambda\Big\{\frac{1}{V}\sum_{p\in I}\Big(G_{|ap|}G_{|adcb|}-\frac{G_{|pdcb|}-G_{|adcb|}}{E_p-E_a}\Big)-G_{|dc|}\frac{G_{|ba|}-G_{|bc|}}{E_c-E_a}\Big\}\\
&\quad-\frac{\lambda}{V^2}\Big\{G_{|a|a|}G_{|adcb|}+G_{|dadcba|}+G_{|dcacba|}+G_{|dcbaba|}+G_{|aaadcb|}+\frac{1}{V}\sum_{n\in I}G_{|an|adcb|}\\
&\qquad\qquad-\frac{G_{|d|cbd|}-G_{|d|cba|}}{E_d-E_a}-\frac{G_{|dc|bc|}-G_{|dc|ba|}}{E_c-E_a}-\frac{G_{|dcb|b|}-G_{|dcb|a|}}{E_b-E_a}\Big\}\\
&\quad-\frac{\lambda}{V^4}\Big\{G_{|a|a|adcb|}\Big\}
\end{aligned}
$$

Example: $G_{|abcd|} = G_{|adcb|}$

$$(E_a+E_b)G_{|abcd|} = -\lambda\Big\{\frac{1}{V}\sum_{p\in I}\Big(G_{|ap|}G_{|abcd|}-\frac{G_{|pbcd|}-G_{|abcd|}}{E_p-E_a}\Big)-G_{|bc|}\frac{G_{|da|}-G_{|dc|}}{E_c-E_a}\Big\}$$
$$-\frac{\lambda}{V^2}\Big\{G_{|a|a|}G_{|abcd|}+G_{|babcda|}+G_{|bcacda|}+G_{|bcdada|}+G_{|aaabcd|}+\frac{1}{V}\sum_{n\in I}G_{|an|abcd|}$$
$$-\frac{G_{|b|cdb|}-G_{|b|cda|}}{E_b-E_a}-\frac{G_{|bc|dc|}-G_{|bc|da|}}{E_c-E_a}-\frac{G_{|bcd|d|}-G_{|bcd|a|}}{E_d-E_a}\Big\}$$
$$-\frac{\lambda}{V^4}\Big\{G_{|a|a|abcd|}\Big\}$$

$$(E_a+E_d)G_{|adcb|} = -\lambda\Big\{\frac{1}{V}\sum_{p\in I}\Big(G_{|ap|}G_{|adcb|}-\frac{G_{|pdcb|}-G_{|adcb|}}{E_p-E_a}\Big)-G_{|dc|}\frac{G_{|ba|}-G_{|bc|}}{E_c-E_a}\Big\}$$
$$-\frac{\lambda}{V^2}\Big\{G_{|a|a|}G_{|adcb|}+G_{|dadcba|}+G_{|dcacba|}+G_{|dcbaba|}+G_{|aaadcb|}+\frac{1}{V}\sum_{n\in I}G_{|an|adcb|}$$
$$-\frac{G_{|d|cbd|}-G_{|d|cba|}}{E_d-E_a}-\frac{G_{|dc|bc|}-G_{|dc|ba|}}{E_c-E_a}-\frac{G_{|dcb|b|}-G_{|dcb|a|}}{E_b-E_a}\Big\}$$
$$-\frac{\lambda}{V^4}\Big\{G_{|a|a|adcb|}\Big\}$$

Example: $G_{|abcd|} = G_{|adcb|}$

$$(E_b - E_d)G_{|abcd|}$$

$$= -\lambda\Big\{ \qquad - G_{|bc|}\frac{G_{|da|} - G_{|dc|}}{E_c - E_a}\Big\}$$

$$-\frac{\lambda}{V^2}\Big\{ \qquad -\frac{G_{|b|cdb|} - G_{|b|cda|}}{E_b - E_a} - \frac{G_{|bc|dc|} - G_{|bc|da|}}{E_c - E_a} - \frac{G_{|bcd|d|} - G_{|bcd|a|}}{E_d - E_a}\Big\}$$

$$+\lambda\Big\{ \qquad - G_{|dc|}\frac{G_{|ba|} - G_{|bc|}}{E_c - E_a}\Big\}$$

$$+\frac{\lambda}{V^2}\Big\{ \qquad -\frac{G_{|d|cbd|} - G_{|d|cba|}}{E_d - E_a} - \frac{G_{|dc|bc|} - G_{|dc|ba|}}{E_c - E_a} - \frac{G_{|dcb|b|} - G_{|dcb|a|}}{E_b - E_a}\Big\}$$

Example: $G_{|abcd|} = G_{|adcb|}$

$$(E_b - E_d)G_{|abcd|}$$

$$= -\lambda\Big\{ - G_{|bc|}\frac{G_{|da|} - G_{|dc|}}{E_c - E_a}\Big\}$$

$$-\frac{\lambda}{V^2}\Big\{ -\frac{G_{|b|cdb|} - G_{|b|cda|}}{E_b - E_a} - \frac{G_{|bc|dc|} - G_{|bc|da|}}{E_c - E_a} - \frac{G_{|bcd|d|} - G_{|bcd|a|}}{E_d - E_a}\Big\}$$

$$+\lambda\Big\{ - G_{|dc|}\frac{G_{|ba|} - G_{|bc|}}{E_c - E_a}\Big\}$$

$$+\frac{\lambda}{V^2}\Big\{ -\frac{G_{|d|cbd|} - G_{|d|cba|}}{E_d - E_a} - \frac{G_{|dc|bc|} - G_{|dc|ba|}}{E_c - E_a} - \frac{G_{|dcb|b|} - G_{|dcb|a|}}{E_b - E_a}\Big\}$$

$$= \lambda\Big\{ \frac{G_{|ab|}G_{|cd|} - G_{|ad|}G_{|cb|}}{E_a - E_c} - \frac{G_{|b|cda|} - G_{|a|cdb|}}{E_b - E_a} - \frac{G_{|bc|da|} - G_{|ba|dc|}}{E_c - E_a} - \frac{G_{|bcd|a|} - G_{|bca|d|}}{E_d - E_a}\Big\}$$

Renormalisation theorem

The renormalisation leaves algebraic equations invariant:

Theorem

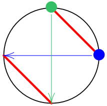
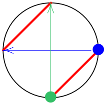
Given a real quartic matrix model with $S = V\,\mathrm{tr}(E\phi^2 + \frac{\lambda}{4}\phi^4)$ and $m \mapsto E_m$ injective, which determines the set $G_{|p_1^1 \dots p_{N_1}^1| \dots |p_1^B \dots p_{N_B}^B|}$ of $(N_1 + \dots + N_B)$-point functions.

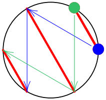
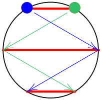
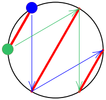
Assume the basic functions with all $N_i \leq 2$ are turned finite by $E_a \mapsto Z(E_a + \frac{\mu^2}{2} - \frac{\mu_{bare}^2}{2})$ and $\lambda \mapsto Z^2\lambda$.

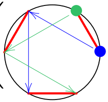
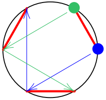
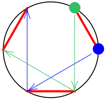
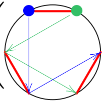
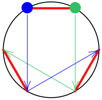
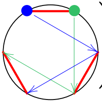
Then all functions with one $N_i \geq 3$

1. are finite without further need of a renormalisation of λ, i.e. all renormalisable quartic matrix models have vanishing β-function. The observation $\beta = 0$ for Moyal is generic!
2. are given by algebraic recursion formulae in terms of renormalised basic functions with $N_i \leq 2$.

Graphical realisation $(B = 1, g = 0)$

$$G_{b_0 b_1 b_2 b_3} = (-\lambda)\frac{G_{b_0 b_1} G_{b_2 b_3} - G_{b_0 b_3} G_{b_2 b_1}}{(E_{b_0} - E_{b_2})(E_{b_1} - E_{b_3})} = -\lambda\Big\{$$

$+$

$\Big\}$

$$G_{b_0 \dots b_5} = \lambda^2 \Big\{$$

$+$

$+$


$+\Big($

$+$

$+$

$\Big) + \Big($

$+$

$+$

$\Big)\Big\}$

b_i ——— b_j $= G_{b_i b_j}$ leads to non-crossing chord diagrams; these are counted by the Catalan number $C_{\frac{N}{2}} = \frac{N!}{(\frac{N}{2}+1)!\,\frac{N}{2}!}$

b_i ⟶ b_j $= \frac{1}{E_{b_i} - E_{b_j}}$ leads to rooted trees connecting the even or odd vertices, intersecting the chords only at vertices

Open Problem (Combinatorics): Which trees arise for a given chord diagram?

$B = 2$ boundary components: basic functions

$$G_{|a|c|} = -\frac{\lambda}{E_a + E_a}\Big\{ \frac{1}{V}\sum_{p\in I}\Big(G_{|ap|}G_{|a|c|} - \frac{G_{|p|c|} - G_{|a|c|}}{E_p - E_a}\Big) - \frac{G_{|cc|} - G_{|ac|}}{E_c - E_a} \Big\}$$

$$g\mapsto g+1\Big\{ -\frac{\lambda}{V^2(E_a + E_a)}\Big(3G_{|a|a|}G_{|a|c|} + G_{|a|cac|} + G_{|c|aaa|} + \frac{1}{V}\sum_{n\in I} G_{|a|c|an|}$$

$$g\mapsto g+2\Big\{ -\frac{\lambda}{V^4(E_a + E_a)} G_{|a|a|a|c|}$$

$$G_{|ab|cd|} = -\frac{\lambda}{E_a + E_b}\Big\{ \frac{1}{V}\sum_{p\in I}\Big((G_{|ap|}G_{|ab|cd|} + G_{|ab|}G_{|ap|cd|}) - \frac{G_{|pb|cd|} - G_{|ab|cd|}}{E_p - E_a}\Big)$$

$$+ G_{|ab|}(G_{|cacd|} + G_{|dadc|}) - \frac{G_{|cbcd|} - G_{|cbad|}}{E_c - E_a} - \frac{G_{|dbdc|} - G_{|dbac|}}{E_d - E_a}\Big\}$$

$$g\mapsto g+1\Big\{ -\frac{\lambda}{V^2(E_a + E_b)}\Big(G_{|a|a|}G_{|ab|cd|} + G_{ab}G_{|a|a|cd} + \frac{1}{V}\sum_{n\in I} G_{|an|ab|cd|}$$

$$+ G_{|cd|aaab|} + G_{|cd|baba|} + G_{|ab|cacd|} + G_{|ab|cddad|} - \frac{G_{|b|a|cd|} - G_{|b|b|cd|}}{E_b - E_a}\Big)$$

$$g\mapsto g+2\Big\{ -\frac{\lambda}{V^4(E_a + E_b)} G_{|a|a|ab|cd|}$$

$B=2$ boundary components: recursion formulae (odd case)

$$
\begin{aligned}
&G_{|b_0\dots b_{2l}|c_1\dots c_{N-2l-1}|}\\
&\quad= -\lambda\sum_{k=1}^{N-2l-1}\frac{G_{|c_1\dots c_{k-1}b_0b_1\dots b_{2l}c_kc_{k+1}\dots c_{N-2l-1}|}-G_{|c_1\dots c_{k-1}c_kb_1\dots b_{2l}b_0c_{k+1}\dots c_{N-2l-1}|}}{(E_{b_1}-E_{b_{2l}})(E_{b_0}-E_{c_k})}\\
&-\lambda\sum_{j=1}^{l}\frac{G_{|b_0b_1\dots b_{2j-2}|c_1\dots c_{N-2l-1}|}G_{|b_{2j-1}b_{2j}\dots b_{2l}|}-G_{|b_{2j-1}b_1\dots b_{2j-2}|c_1\dots c_{N-2l-1}|}G_{|b_0b_{2j}\dots b_{2l}|}}{(E_{b_1}-E_{b_{2l}})(E_{b_0}-E_{b_{2j-1}})}\\
&-\lambda\sum_{j=1}^{l}\frac{G_{|b_0b_1\dots b_{2j-1}|}G_{|b_{2j}b_{2j+1}\dots b_{2l}|c_1\dots c_{N-2l-1}|}-G_{|b_{2j}b_1\dots b_{2j-1}|}G_{|b_0b_{2j+1}\dots b_{2l}|c_1\dots c_{N-2l-1}|}}{(E_{b_1}-E_{b_{2l}})(E_{b_0}-E_{b_{2j}})}\\
&-\frac{\lambda}{V^2}\sum_{k=1}^{2l}\frac{G_{|b_0b_1\dots b_{k-1}|b_kb_{k+1}\dots b_{2l}|c_1\dots c_{N-2l-1}|}-G_{|b_kb_1\dots b_{k-1}|b_0b_{k+1}\dots b_{2l}|c_1\dots c_{N-2l-1}|}}{(E_{b_1}-E_{b_{2l}})(E_{b_0}-E_{b_k})}
\end{aligned}
$$

(Last line increases the genus and is absent in $G^{(0)}_{|b_0b_1\dots b_{2l}|c_1\dots C_{N-2l-1}|}$)

$B=2$ boundary components: recursion formulae (even case)

$$
\begin{aligned}
&G_{|ab_1\dots b_{2l-1}|c_1\dots c_{N-2l}|}\\
&= -\lambda\sum_{j=1}^{l-1}\frac{G_{|b_1\dots b_{2j-1}a|c_1\dots c_{N-2l}|}G_{|b_{2j}b_{2j+1}\dots b_{2l-1}|}-G_{|b_1\dots b_{2j-1}b_{2j}|c_1\dots c_{N-2l}|}G_{|ab_{2j+1}\dots b_{2l-1}|}}{(E_{b_1}-E_{b_{2l-1}})(E_a-E_{b_{2j}})}\\
&-\lambda\sum_{j=1}^{l-1}\frac{G_{|b_1\dots b_{2j-1}a|}G_{|b_{2j}b_{2j+1}\dots b_{2l-1}|c_1\dots c_{N-2l}|}-G_{|b_1\dots b_{2j-1}b_{2j}|}G_{|ab_{2j+1}\dots b_{2l-1}|c_1\dots c_{N-2l}|}}{(E_{b_1}-E_{b_{2l-1}})(E_a-E_{b_{2j}})}\\
&-\lambda\sum_{k=1}^{N-2l}\frac{G_{|c_1\dots c_{k-1}ab_1\dots b_{2l-1}c_kc_{k+1}\dots c_{N-2l}|}-G_{|c_1\dots c_{k-1}c_kb_1\dots b_{2l-1}ac_{k+1}\dots c_{N-2l}|}}{(E_{b_1}-E_{b_{2l-1}})(E_a-E_{c_k})}\\
&-\frac{\lambda}{V^2}\sum_{k=1}^{2l-1}\frac{G_{|b_1\dots b_{k-1}a|b_kb_{k+1}\dots b_{2l-1}|c_1\dots c_{N-2l}|}-G_{|b_1\dots b_{k-1}b_k|ab_{k+1}\dots b_{2l-1}|c_1\dots c_{N-2l}|}}{(E_{b_1}-E_{b_{2l-1}})(E_a-E_{b_k})}
\end{aligned}
$$

(Last line increases the genus and is absent in $G^{(0)}_{|b_0b_1\dots b_{2l-1}|c_1\dots C_{N-2l}|}$)

Digression: 2D quantum gravity

1. 2D quantum gravity is the enumeration of random triangulations of surfaces.
   - Its asymptotic behaviour is captured by the matrix model partition function

$$\mathcal{Z} = \int dM \; \exp\Big(-\mathcal{N}\sum_n t_n \,\mathrm{tr}(M^n)\Big)\;, \qquad M = M^* \in M_{\mathcal{N}}(\mathbb{C})$$

   - For $\mathcal{N} \to \infty$, this series in (t_n) is evaluated in terms of the τ-function for the Korteweg-de Vries (KdV) hierarchy.
2. 2D topological quantum gravity has correlation functions which are intersection numbers of complex curves.
   - They can be arranged into a generating functional with series parameters (t_n).

[Witten, 1990] conjectured that both (t_n)-series are the same.

The Kontsevich model

- [Kontsevich, 1992] computed the intersection numbers in terms of weighted sums over ribbon graphs.
- He proved these graphs to be generated from the Airy function matrix model (Kontsevich model)

$$\mathcal{Z}[E] = \frac{\int dM \; \exp\big(-\frac{1}{2}\mathrm{tr}(EM^2) + \frac{\mathrm{i}}{6}\mathrm{tr}(M^3)\big)}{\int dM \; \exp\big(-\frac{1}{2}\mathrm{tr}(EM^2)\big)}, \qquad M{=}M^*{\in}M_{\mathcal{N}}(\mathbb{C})$$

for $E = E^* > 0$ and $t_n = (2n{-}1)!!\mathrm{tr}(E^{-(2n-1)})$.

- Limit $\mathcal{N} \to \infty$ of $\mathcal{Z}[E]$ gives the KdV evolution equation, thus proving Witten's conjecture.

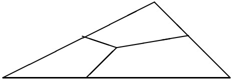
The quartic matrix model

- We have seen that also the quartic matrix model

$$\mathcal{Z}[E,J,\lambda] = \frac{\int dM\ \exp\big(-\mathrm{tr}(EM^2) + \mathrm{tr}(JM) - \frac{\lambda}{4}\mathrm{tr}(M^4)\big)}{\int dM\ \exp\big(-\mathrm{tr}(EM^2) - \frac{\lambda}{4}\mathrm{tr}(M^4)\big)}$$

is exactly solvable in terms of the solution of a non-linear eq.

- 2D quantum gravity should have equivalent descriptions as cubic and quartic matrix model.



- Quartic models show positivity and boundedness from below. They admit techniques from constructive QFT (loop vertex expansion) not possible in cubic model.
- Coloured tensor models extend these methods to quantum gravity in $D \geq 3$. They have Schwinger-Dyson equations and action of $U(\infty)$ group.