Construction of a 4D quantum field theory

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(based on joint work with Harald Grosse,
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Introduction

- Quantum field theory (QFT) is the theory that describes Nature at very high energy density.
- One famous such experiment measures the magnetic moment g of the electron: $\frac{g_{experiment}}{2} = 1.001\,159\,652\,180\,7$
- QFT predicts that number in terms of the electron charge e measured to $e^{-2} = 137.035\,999\,084$:

$$\begin{aligned}\frac{g_{QFT}}{2} &= 1 + \frac{1}{2}\frac{e^2}{\pi} + \Big\{\frac{197}{144} + \Big(\frac{1}{2} - 3\log 2\Big)\zeta(2) + \frac{3}{4}\zeta(3)\Big\}\Big(\frac{e^2}{\pi}\Big)^2 \\ &+ \Big\{\frac{28259}{5184} + \Big(\frac{17101}{235} - \frac{596}{2}\cdot\log 2\Big)\zeta(2) + \frac{139}{18}\zeta(3) + \frac{100}{3}\mathrm{Li}_4\Big(\frac{1}{2}\Big) \\ &+ \frac{25}{18}\Big(\log^4 2 - \pi^2\log^2 2\Big) - \frac{239\zeta(4) - 166\zeta(2)\cdot\zeta(3) + 215\zeta(5)}{24}\Big\}\Big(\frac{e^2}{\pi}\Big)^3 \\ &+ \Big\{\dots\Big\}\Big(\frac{e^2}{\pi}\Big)^4 \\ &= 1.001\,159\,652\,153\,5\end{aligned}$$

Constructive quantum field theory

- Constructive quantum field theory tries to give a meaning to such expansions.

  This involves convergence and analyticity in reasonable domains.

- Unfortunately, this also fails for QED due to the Landau ghost problem.

  It is expected to work for non-Abelian gauge theories because of asymptotic freedom.

- But these theories are too complicated (millennium prize problem), and simpler 4D-models (such as ϕ_4^4) cannot be constructed.

A main difficulty is the non-linearity of field equations.

A toy model

Non-linear problems sometimes solvable by fixed point methods

- We propose a toy model of a 4D Euclidean QFT inspired by noncommutative geometry.
- This is essentially a matrix model. The action of the $U(\infty)$ group induces an infinite number of Ward identities.
- The Ward identities turn the Schwinger-Dyson equations into a fixed point problem which is solvable in the infinite volume limit.
- We find numerical evidence for a phase structure, phase transitions and critical phenomena.
- The infinite volume limit restores Euclidean covariance and symmetry. In one of the phases we find numerical evidence for reflection positivity of the 2-point function.

Field-theoretical matrix models

- A matrix is for us a compact (Hilbert-Schmidt) operator on Hilbert space $H = L^2(I, \mu)$.
- realise as integral kernel operators: $\phi = (\phi_{ab}) \in L^2(I\times I, \mu\times\mu)$
- action = non-linear functional S for $\phi = \phi^*$ in volume V:

$$S[\phi] = V\,\mathrm{tr}(E\phi^2 + P[\phi])$$

E – unbounded positive selfadjoint op. with compact resolvent,
$P[\phi]$ – polynomial in ϕ with scalar coefficients

Euclidean quantum field theory

- partition function $\mathcal{Z}[J] = \int \mathcal{D}[\phi]\ \exp(-S[\phi] + V\,\mathrm{tr}(\phi J))$
- For $P[\phi] \equiv 0$, $\mathcal{D}[\phi]e^{-V\mathrm{tr}(E\phi^2)}/\mathcal{Z}[0]$ is Gaußian measure (of covariance determined by E) on random s.a. matrices.
- Our aim is to construct $\mathcal{D}[\phi]e^{-V\mathrm{tr}(E\phi^2+\frac{\lambda}{4}\phi^4)}/\mathcal{Z}[0]$.

Ward identity

- Unitary transformation $\phi \mapsto U\phi U^*$ leads to Ward identity

$$0 = \int \mathcal{D}\phi \, \Big[E\phi\phi - \phi\phi E - J\phi + \phi J\Big] \exp(-S[\phi] + V\,\mathrm{tr}(\phi J))$$

that describes how E, J break the invariance of the action.

…choose E (but not J) diagonal, use $\phi_{ab} = \frac{\partial}{V\partial J_{ba}}$:

Proposition [Disertori-Gurau-Magnen-Rivasseau, 2007]

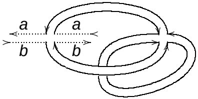
The partition function $\mathcal{Z}[J]$ of the matrix model defined by the external matrix E satisfies the $|I| \times |I|$ Ward identities

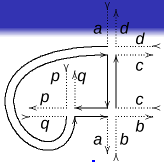
$$0 = \sum_{n\in I} \Big(\frac{(E_a - E_p)}{V} \frac{\partial^2 \mathcal{Z}}{\partial J_{an} \partial J_{np}} + J_{pn} \frac{\partial \mathcal{Z}}{\partial J_{an}} - J_{na} \frac{\partial \mathcal{Z}}{\partial J_{np}} \Big)$$

For E of compact resolvent we can always assume that $m \mapsto E_m > 0$ is injective!

Topological expansion

- Feynman graphs in matrix models are ribbon graphs.





- Viewed as simplicial complexes, they encode the topology (B, g) of a genus-g Riemann surface with B boundary components (or punctures, marked points, holes, faces).

- The k^{th} boundary component carries a cycle $J^{N_k}_{p_1 \dots p_{N_k}} := \prod_{j=1}^{N_k} J_{p_j p_{j+1}}$ of N_k external sources, $N_k + 1 \equiv 1$.

- Expand $\log \mathcal{Z}[J] = \sum \frac{1}{S} V^{2-B} G_{|p_1 \dots p_{N_1}|\dots|q_1 \dots q_{N_B}|} J^{N_1}_{p_1 \dots p_{N_1}} \cdots J^{N_B}_{q_1 \dots q_{N_B}}$ according to the cycle structure.

- QFT of matrix models determines the weights of Riemann surfaces with $\mathbb{N}$-decorated punctures compatible with gluing and symmetry.

Schwinger-Dyson equations

Write $S = \frac{V}{2}\sum_{a,b}(E_a + E_b)\phi_{ab}\phi_{ba} + VS_{int}[\phi]$.
The E_a are the eigenvalues of E.

- Functional integration yields, up to irrelevant constant,

$$\mathcal{Z}[J] = e^{-VS_{int}[\frac{\partial}{V\partial J}]}e^{\frac{V}{2}\langle J,J\rangle_E}\ , \qquad \langle J,J\rangle_E := \sum_{m,n\in I}\frac{J_{mn}J_{nm}}{E_m+E_n}$$

- Much better than the perturbative expansion of $e^{-VS_{int}[\frac{\partial}{V\partial J}]}$ is to apply J-derivatives to $\mathcal{Z}[J]$.
  $\to$ Choose them to give $G_{\dots}$ on the lhs.
- These external derivatives combine on rhs with internal derivatives from $S_{int}[\frac{\partial}{V\partial J}]$ to certain identities for $G_{\dots}$.

These Schwinger-Dyson equations are often of little use because they express an N-point function in terms of $(N+2)$-point functions.

Schwinger-Dyson equations (for $S_{int}[\phi] = \frac{\lambda}{4}\mathrm{tr}(\phi^4)$)

Ward identity and reality $\mathcal{Z} = \overline{\mathcal{Z}}$ let the usually infinite tower of Schwinger-Dyson equations collapse:

after genus expansion $G_{\dots} = \sum_{g=0}^{\infty} V^{-2g} G^{(g)}_{\dots}$:

1. A closed non-linear equation for $G^{(0)}_{ab}$ (planar+regular)

$$G^{(0)}_{|ab|} = \frac{1}{E_a + E_b} - \frac{\lambda}{(E_a + E_b)} \frac{1}{V} \sum_{p \in I} \Big(G^{(0)}_{|ab|} G^{(0)}_{|ap|} - \frac{G^{(0)}_{|pb|} - G^{(0)}_{|ab|}}{E_p - E_a} \Big)$$

2. For $N \geq 4$ a universal algebraic recursion formula

$$G^{(0)}_{|b_0 b_1 \dots b_{N-1}|} = (-\lambda) \sum_{l=1}^{\frac{N-2}{2}} \frac{G^{(0)}_{|b_0 b_1 \dots b_{2l-1}|} G^{(0)}_{|b_{2l} b_{2l+1} \dots b_{N-1}|} - G^{(0)}_{|b_{2l} b_1 \dots b_{2l-1}|} G^{(0)}_{|b_0 b_{2l+1} \dots b_{N-1}|}}{(E_{b_0} - E_{b_{2l}})(E_{b_1} - E_{b_{N-1}})}$$

Restriction to genus $g = 0$ is exact for $V \to \infty$ and $\frac{1}{V} \sum_{p \in I}$ finite

Renormalisation theorem

Theorem [Grosse-W., 2013]

Given a real quartic matrix model $S = V\,\mathrm{tr}(E\phi^2 + \frac{\lambda}{4}\phi^4)$ with E of compact resolvent.

Assume that the selfconsistency equation for $G_{|ab|}$ has a finite solution after affine renormalisation $E \mapsto Z(E + C\mathbb{1})$ and $\lambda \mapsto Z^2\lambda$. Then

- All higher functions $G_{|b_0 \dots b_{N-1}|}$ with $N \geq 4$ are automatically finite without further need of a renormalisation of λ.
- All quartic matrix models (with renormalisable $G_{|ab|}$) have vanishing β-function (i.e. are almost scale-invariant).
- The perturbative observation $\beta = 0$ for Moyal [Disertori-Gurau-Magnen-Rivasseau, 2007] is generic!

(Similar statements hold for $B \geq 2$)

ϕ^4_4 on Moyal space with harmonic propagation

$$S[\phi]=64\pi^2\int_{\mathbb{R}^4} dx\Big(\frac{Z}{2}\phi\star(-\Delta+\Omega^2(2\Theta^{-1}x)^2+\mu^2_{bare})\phi+\frac{\lambda Z^2}{4}\phi\star\phi\star\phi\star\phi\Big)(x)$$

with Moyal product $(f\star g)(x)=\int_{\mathbb{R}^4\times\mathbb{R}^4}\frac{dy\,dk}{(2\pi)^4}f(x+\tfrac{1}{2}\Theta k)\,g(x+y)\,e^{\mathrm{i}\langle k,y\rangle}$

takes at $\Omega=1$ in matrix basis $f_{\underline{mn}}(x)=f_{m_1n_1}(x^0,x^1)f_{m_2n_2}(x^3,x^4)$

$$f_{mn}(y^0,y^1)=2(-1)^m\sqrt{\frac{m!}{n!}}\Big(\sqrt{\tfrac{2}{\theta}}y\Big)^{n-m}L_m^{n-m}\Big(\frac{2|y|^2}{\theta}\Big)e^{-\frac{|y|^2}{\theta}}$$

due to $f_{\underline{mn}}\star f_{\underline{kl}}=\delta_{\underline{nk}}f_{\underline{ml}}$ and $\int dx\,f_{\underline{mn}}(x)=64\pi^2V\delta_{\underline{mn}}$ the form

$$S[\phi]=V\Big(\sum_{\underline{m},\underline{n}\in\mathbb{N}^2_{\mathcal{N}}}E_{\underline{m}}\,\phi_{\underline{mn}}\phi_{\underline{nm}}+\frac{Z^2\lambda}{4}\sum_{\underline{m},\underline{n},\underline{k},\underline{l}\in\mathbb{N}^2_{\mathcal{N}}}\phi_{\underline{mn}}\phi_{\underline{nk}}\phi_{\underline{kl}}\phi_{\underline{lm}}\Big)$$

$$E_{\underline{m}}=Z\Big(\frac{|\underline{m}|}{\sqrt{V}}+\frac{\mu^2_{bare}}{2}\Big)\,,\qquad |\underline{m}|:=m_1+m_2\leq\mathcal{N}$$

- $V=\left(\frac{\theta}{4}\right)^2$ is for $\Omega=1$ the volume of the noncommutative manifold which is sent to ∞ in the thermodynamic limit.

Scaling limit $\mathcal{N}, \mu^4 V \to \infty$ with $\frac{\mathcal{N}}{\sqrt{\mu^4 V}} = \Lambda^2(1+\mathcal{Y})$ fixed

- $E_{\underline{p}} \mapsto E(p)$ continuous distribution of eigenvalues over $[0, \Lambda^2]$
  $G^{(0)}_{|\underline{ab}|} \mapsto \mu^{-2} G_{ab}$ function of "continuous indices" $a, b \in [0, \Lambda^2]$
  achieves: NON-linear integral equation for G_{ab}
- $\Lambda \to \infty$: Renormalisation $\mu_{bare} \mapsto \mu$ and $Z \mapsto (1+\mathcal{Y})$ by normalisation $G_{00} = 1$ and $\frac{dG_{ab}}{db}\Big|_{a=b=0} = -(1+\mathcal{Y})$

LINEAR integral equation for $D_{ab} := \frac{a}{b}(G_{ab} - G_{a0})$

Introduce Hilbert transform $\mathcal{H}_a[f(\bullet)] := \frac{1}{\pi} \lim_{\epsilon \to 0} \Big(\int_0^{a-\epsilon} + \int_{a+\epsilon}^{\infty} \Big) \frac{f(q)\,dq}{q-a}$

and fix $\mathcal{Y} = -\lambda\pi\mathcal{H}_0[D_{\bullet 0}]$. Then

$$\Big(\frac{b}{a} + \frac{1 + \lambda\pi a \mathcal{H}_a\big[G_{\bullet 0}\big]}{a G_{a0}} \Big) D_{ab} - \lambda\pi \mathcal{H}_a\big[D_{\bullet b}\big] = -G_{a0}$$

The Carleman solution

- The equation for D_{ab} is a singular integral equation of Carleman type.
- Its solution theory [Carleman, 1922; Tricomi, 1957] expresses D_{ab}, hence G_{ab}, in terms of its boundary:

Theorem

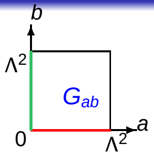
$$G_{ab}=\frac{\sin(\tau_b(a))}{|\lambda|\pi a}e^{\mathrm{sign}(\lambda)(\mathcal{H}_0[\tau_0(\bullet)]-\mathcal{H}_a[\tau_b(\bullet)])}\begin{cases}1 & \text{for } \lambda<0\\ \left(1+\frac{Ca+bF(b)}{\Lambda^2-a}\right) & \text{for } \lambda>0\end{cases}$$

where
$$\tau_b(a) := \underset{[0,\pi]}{\arctan}\left(\frac{|\lambda|\pi a}{b+\frac{1+\lambda\pi a\mathcal{H}_a[G_{\bullet 0}]}{G_{a0}}}\right)$$

- Consequence: $G_{ab}\geq 0$ (at least for $\lambda<0$)!
- The freedom for $\lambda>0$ corresponds to a winding number.

The self-consistency equation

Given boundary value G_{a0},
Carleman computes G_{ab}, in particular G_{0b}.
Symmetry forces $G_{b0} = G_{0b}$:



Master equation

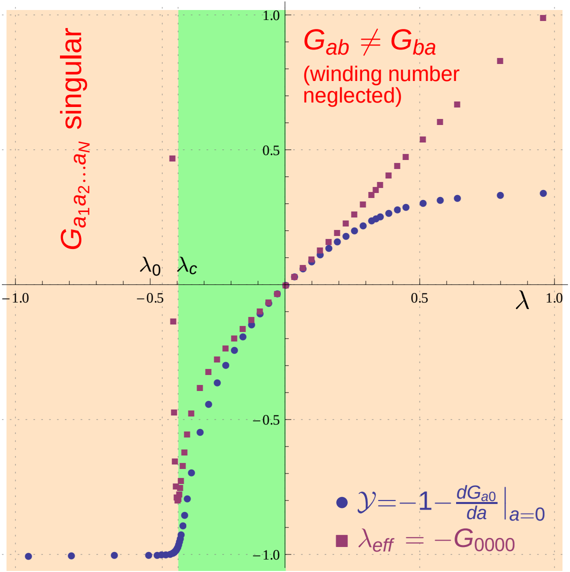
The theory is completely determined by the solution of the fixed point equation $G = TG$

$$G_{b0} = \frac{\begin{Bmatrix} 1 & \text{for } \lambda<0 \\ 1+bF(b) & \text{for } \lambda>0\end{Bmatrix}}{1+b} \exp\left(-\lambda\int_0^b dt \int_0^\infty \frac{dp}{(\lambda\pi p)^2 + \left(t+\frac{1+\lambda\pi p \mathcal{H}_p[G_{\bullet 0}]}{G_{p0}}\right)^2}\right)$$

Theorem

For $\lambda > 0$ but $F(b) = 0$, the master equation has a solution $G_{\bullet 0} \in \mathcal{C}_0^1(\mathbb{R}_+)$ by the Schauder fixed point theorem. The solution is automatically smooth and monotonously decreasing.

Computer simulation: evidence for phase transitions



- G_{ab} for $\Lambda^2 = 10^7$ with 2000 sample points
- $\mathcal{Y}'$ discontinuous at $\lambda_c = -0.396$
- λ_{eff} singular at $\lambda_0 = -0.455$ where $\mathcal{Y} = -1$
- Nothing particular at pole $\lambda_b = -\frac{1}{72} = 0.014$ of Borel resummation
- A key property for Schwinger functions is precisely realised in $[\lambda_c, 0]$, not outside!

Translation to 4D Euclidean QFT model

- infinite volume limit $\mu^4 V \to \infty$ requires densities
- $\lim_{V\mu^4\to\infty} \frac{1}{V\mu^4} \log \mathcal{Z}[J]$ produces matrix model at $(g{=}0, B{=}1)$

DIFFERENT limit: Schwinger functions

$$\mu^N S_c(\mu x_1, \dots, \mu x_N)$$

$$:= \lim_{V\mu^4\to\infty} \sum_{\underline{m}_1,\underline{n}_1,\dots,\underline{m}_N,\underline{n}_N\in\mathbb{N}^2} f_{\underline{m}_1\underline{n}_1}(x_1)\cdots f_{\underline{m}_N\underline{n}_N}(x_N) \frac{\mu^{4N}\partial^N \mathcal{F}[J]}{\partial J_{\underline{m}_1\underline{n}_1}\dots\partial J_{\underline{m}_N\underline{n}_N}}\Big|_{J=0}$$

$$\mathcal{F}[J] := \frac{1}{64\pi^2 V^2\mu^8} \log\left(\frac{\int \mathcal{D}[\phi]\, e^{-S[\phi]+V\sum_{\underline{a},\underline{b}\in\mathbb{N}^2}\phi_{\underline{ab}}J_{\underline{ba}}}}{\int \mathcal{D}[\phi]\, e^{-S[\phi]}}\right)_{\substack{Z\mu^2_{bare}\mapsto\mu^2\\ Z\mapsto(1+\mathcal{Y})}}$$

- J-cycle structure in $\mathcal{F}$ produces $f_{\underline{mn}}$-cycles for every face: $\sum_{\underline{m}_1,\dots,\underline{m}_j} f_{\underline{m}_1\underline{m}_2}\cdots f_{\underline{m}_{j-1}\underline{m}_j} f_{\underline{m}_j\underline{m}_1} G_{|\dots|\underline{m}_1\dots\underline{m}_j|\dots|}$
- Trace of even number of $f_{\underline{mn}}$ proportional to V

Proposition [Grosse-W., 2013]

$$S_c(\mu x_1,\dots,\mu x_N) = \frac{1}{64\pi^2} \sum_{\substack{N_1+\dots+N_B=N \\ N_\beta \text{ even}}} \sum_{\sigma\in\mathcal{S}_N} \Bigg(\prod_{\beta=1}^{B} \frac{4^{N_\beta}}{N_\beta} \int_{\mathbb{R}^4} \frac{d^4p_\beta}{4\pi^2\mu^4}\, e^{\mathrm{i}\left\langle \frac{p_\beta}{\mu}, \sum_{i=1}^{N_\beta}(-1)^{i-1}\mu x_{\sigma(N_1+\dots+N_{\beta-1}+i)}\right\rangle}\Bigg)$$
$$\times G_{\underbrace{\frac{\|p_1\|^2}{2\mu^2(1+\mathcal{Y})},\dots,\frac{\|p_1\|^2}{2\mu^2(1+\mathcal{Y})}}_{N_1}\Big|\dots\Big|\underbrace{\frac{\|p_B\|^2}{2\mu^2(1+\mathcal{Y})},\dots,\frac{\|p_B\|^2}{2\mu^2(1+\mathcal{Y})}}_{N_B}}$$

- Schwinger functions are symmetric and invariant under the full Euclidean group (this is limit $\theta\to\infty$!)
- Only a restricted sector of the matrix model contributes to position space: All faces have common matrix indices.
- Most interesting sector: every face has $N_\beta=2$ indices. It is described by the functions $G_{\frac{\|p_1\|^2}{2\mu^2(1+\mathcal{Y})}\frac{\|p_1\|^2}{2\mu^2(1+\mathcal{Y})}\Big|\dots\Big|\frac{\|p_B\|^2}{2\mu^2(1+\mathcal{Y})}\frac{\|p_B\|^2}{2\mu^2(1+\mathcal{Y})}}$
- This sector describes propagation and interaction of B particles without any momentum exchange.

Attempts of interpretation

- This is acceptable for a 2D-model. In 4D, absence of momentum transfer is a sign of triviality!
- Triviality proofs rely on clustering, analyticity in Mandelstam representation, absence of bound states. Needs verification.
- Clustering is violated: The $(N_1 + \ldots + N_B)$-point functions are insensitive to the distance of different boundaries.
  $\rightarrow$ non-trivial topological sectors

Intuitive picture: Noncommutative geometrical substructure

- Have exactly solvable matrix model describing extreme NCG at $\theta \rightarrow \infty$. Non-trivial but violates Euclidean symmetry.
- Projection to diagonal matrices restores Euclidean symmetry.
- Can the projection be arranged to respect QFT axioms?

Osterwalder-Schrader reflection positivity

- For the 2-point function, analyticity + reflection positivity boil down to require that $a \mapsto G_{aa}$ is a Stieltjes function

  (consequence of Källén-Lehmann spectral representation)

- $f : \mathbb{R}_+ \to \mathbb{R}$ is Stieltjes iff $f(x) = c + \int_0^\infty \frac{d(\rho(t))}{x+t}$ with $c \geq 0$ and ρ positive and non-decreasing.

Theorem [Widder, 1938]

$f : \mathbb{R}_+ \to \mathbb{R}$ is Stieltjes iff $\mathcal{C}^\infty$, positive and $L_{k,t}[f(\bullet)] \geq 0$, where

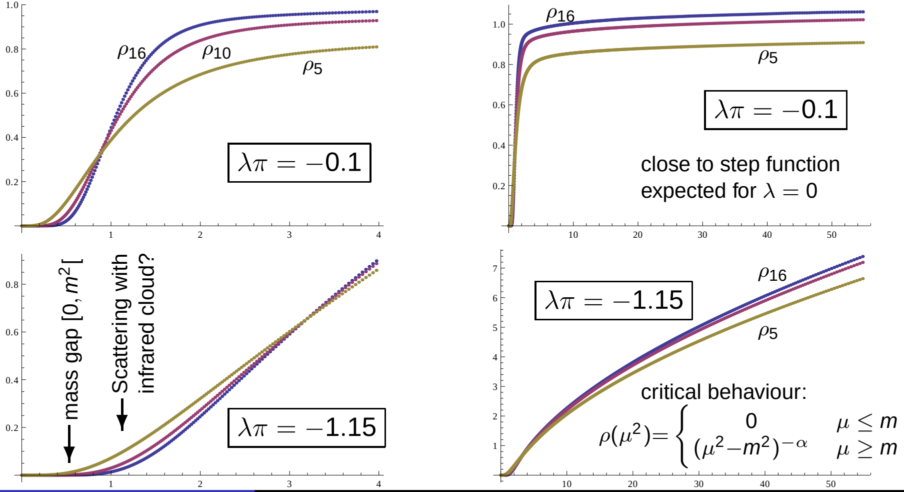
$$L_{k,t}[f(\bullet)] := \frac{(-t)^{k-1}}{c_k} \frac{d^{2k-1}}{dt^{2k-1}} \big(t^k f(t)\big) , \qquad c_1 = 1,\ c_{k>1} = k!(k-2)!$$

- In this case $\rho'(t) = \lim_{k\to\infty} L_{k,t}[f(\bullet)]$ (weakly and a.e.).

Perturbative argument: $a \mapsto G_{aa}$ cannot be Stieltjes for $\lambda > 0$!

Integrated "mass densities" $\rho_k(m^2) = \int_0^{m^2} dt\, L_{k,t}[G_{\bullet 0}]$

$$\frac{(\log G_{a0})^{(\ell)}}{(\ell-1)!} = \frac{(-1)^\ell}{(1+a)^\ell} + (-1)^\ell \,\mathrm{sign}(\lambda)\, \mathcal{H}_0^\Lambda\Big[\sin\left(\ell\tau_a(\bullet)\right)\Big(\frac{\sin\tau_a(\bullet)}{|\lambda|\pi\bullet}\Big)^\ell\Big]$$



Summary

1. The quartic matrix model $\mathcal{Z} = \int dM \exp(\mathrm{tr}(JM - EM^2 - \frac{\lambda}{4}M^4))$ is exactly solvable in terms of solution of a non-linear equation.

2. (Self-dual) ϕ^4_4-theory on Moyal space is of that type.
   For extreme noncommutativity $\theta \to \infty$, the non-linear equation is reduced to a fixed-point problem.

   Unique non-perturbative and non-trivial solution for $\lambda < 0$.

3. Projection to Schwinger functions for scalar field on $\mathbb{R}^4$.
   - Full Euclidean symmetry, but unusual properties.
   - Possibly trivial, but 2-point function shows scattering remnants due to NCG substructure.

4. Some axioms do not fail immediately. Why?
   Needs verification.