Self-dual noncommutative ϕ^4-theory in four dimensions is a non-perturbatively solvable and non-trivial quantum field theory

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Introduction

There are few examples of rigorously constructed quantum field theories in four dimensions. Candidates are either

- trivial (i.e. free fields), or
- perturbative (formal power series with zero radius of convergence), or
- too difficult (cf. Yang-Mills theory).

We show: The simple 4D-ϕ^4-model can be rigorously constructed if put on a noncommutative manifold.

- It has different symmetries, and the rôle of time is obscure.
- Otherwise it is an honest quantum field theory, with infinitely many divergent but renormalisable Feynman graphs.
- The guiding mathematical structure waits for discovery.

Field-theoretical matrix models

matrix = compact operator on Hilbert space $H = L^2(I, \mu)$

- realise as integral kernel operators:
  $M = (M_{ab}) \in L^2(I \times I, \mu \times \mu)$ (for Hilbert-Schmidt)
  - product: $(MN)_{ab} = \int_I d\mu(c)\, M_{ac} M_{cb}$
  - trace: $\mathrm{tr}(M) = \int_I d\mu(a)\, M_{aa}$
  - adjoint: $(M^*)_{ab} = \overline{M_{ba}}$

- action = non-linear functional S for $\phi = \phi^*$:

$$S[\phi] = \mathrm{tr}(E\phi^2) + V[\phi]\,, \quad V[\phi] = \mathrm{tr}(P[\phi])$$

where E is external matrix
(positive selfadjoint operator with compact resolvent),
$P[\phi]$ – polynomial in ϕ with scalar coefficients

Euclidean quantum field theory

- action with source term $\longrightarrow$ partition function

$$\mathcal{Z}[J] = \int \mathcal{D}\phi \; \exp(-S[\phi] + \mathrm{tr}(\phi J))$$

where $\mathcal{D}\phi = \prod_{a,b \in I} d\phi_{ab}$ (Lebesgue on finite-rank op's)

- connected correlation functions obtained from $\mathcal{W}[J] = \ln \mathcal{Z}[J]$ as

$$\langle \varphi_{a_1 b_1} \dots \varphi_{a_n b_n} \rangle = \frac{\partial^n \mathcal{W}[J]}{\partial J_{b_1 a_1} \dots \partial J_{b_n a_n}} \Big|_{J=0}$$

- unless I is finite, the resulting index sums may diverge and require a renormalisation

Ward identity

- unitary transformation $\phi \mapsto \tilde{\phi} = U\phi U^*$
  $U \in (\mathcal{K}(H))^1$ with $UU^* = U^*U = \mathrm{id}$, leaves $\mathcal{K}_{s.a.}$ invariant:

$$\int \mathcal{D}\phi\ \exp(-S[\phi] + \mathrm{tr}(\phi J)) = \int \mathcal{D}\tilde{\phi}\ \exp(-S[\tilde{\phi}] + \mathrm{tr}(\tilde{\phi} J))$$

- measure unitarily invariant: $\mathcal{D}\tilde{\phi} = \mathcal{D}\phi$:

$$0 = \int \mathcal{D}\phi \Big[\exp(-S[\phi] + \mathrm{tr}(\phi J)) - \exp(-S[\tilde{\phi}] + \mathrm{tr}(\tilde{\phi} J))\Big]$$

  note: $[\ ] \neq 0$ because $\mathrm{tr}(E\phi^2)$, $\mathrm{tr}(\phi J)$ not unitarily invariant!

- linearisation $\longrightarrow$ Ward identity (matricial equation!)

$$0 = \int \mathcal{D}\phi \Big[E\phi\phi - \phi\phi E - J\phi + \phi J\Big] \exp(-S[\phi] + \mathrm{tr}(\phi J))$$

- choose a reference frame where E is diagonal (but J is not)
- use functional derivative $\phi_{ab} = \frac{\partial}{\partial J_{ba}}$

Proposition

The partition function $\mathcal{Z}[J]$ of the matrix model defined by the external matrix E satisfies the $|I| \times |I|$ Ward identities

$$0 = \sum_{n \in I} \Big((E_a - E_p) \frac{\partial^2 \mathcal{Z}}{\partial J_{an} \partial J_{np}} + J_{pn} \frac{\partial \mathcal{Z}}{\partial J_{an}} - J_{na} \frac{\partial \mathcal{Z}}{\partial J_{np}} \Big)$$

Main assumption

- $m \mapsto E_m > 0$ injective

Note: always the case if we pass to equivalence classes $[m]$ which have the same E_m; then $\sum_{m \in I} f(m) = \sum_{[m] \in [I]} \mu([m]) f([m])$

We will see: These Ward identities and the choice of $V[\phi]$ determine the QFT of the matrix model non-perturbatively!

Decomposition into cycles

From perturbative expansion into ribbon graphs: right index of J_{ab} is left index of another J_{bc}, or of the same J_{bb}.

Decomposition of $\mathcal{W}[J]$ for even $V[\phi]$ into J-cycles:

$$\begin{aligned}\mathcal{W}[J] = \mathcal{W}[0] &+ \frac{1}{2}\sum_{p,q\in I} G_{pq}(J_{pq}J_{qp}) + \frac{1}{2}\sum_{p,q\in I} G_{p|q}(J_{pp})(J_{qq}) \\ &+ \frac{1}{4}\sum_{p,q,r,s\in I} G_{pqrs}(J_{pq}J_{qr}J_{rs}J_{sp}) + \frac{1}{3}\sum_{p,q,r,s\in I} G_{pqr|s}(J_{pq}J_{qr}J_{rp})(J_{ss}) \\ &+ \frac{1}{8}\sum_{p,q,r,s\in I} G_{pq|rs}(J_{pq}J_{qp})(J_{rs}J_{sr}) + \frac{1}{4}\sum_{p,q,r,s\in I} G_{pq|r|s}(J_{pq}J_{qp})(J_{rr})(J_{ss}) \\ &+ \frac{1}{24}\sum_{p,q,r,s\in I} G_{p|q|r|s}(J_{pp})(J_{qq})(J_{rr})(J_{ss}) + \mathcal{O}(J^6)\end{aligned}$$

Attention: $G_{pp}J_{pp}J_{pp}$ is topologically different from $G_{p|p}J_{pp}J_{pp}$!

A new Ward identity for injective E

We turn the previous Ward identity

$$0 = \sum_{n \in I} \Big((E_a - E_p) \frac{\partial^2 \mathcal{Z}}{\partial_{an} \partial J_{np}} + J_{pn} \frac{\partial \mathcal{Z}}{\partial J_{an}} - J_{na} \frac{\partial \mathcal{Z}}{\partial J_{np}} \Big)$$

for injective E into a formula for

$$\sum_{n \in I} \frac{\partial^2 \mathcal{Z}[J]}{\partial J_{an} \partial J_{np}} = \sum_{n \in I} \Big(\frac{\partial^2 \mathcal{W}[J]}{\partial J_{an} \partial J_{np}} + \frac{\partial \mathcal{W}[J]}{\partial J_{an}} \frac{\partial \mathcal{W}[J]}{\partial J_{np}} \Big) \mathcal{Z}[J]$$

The J-cycle structure in $\mathcal{W}[J]$ creates

- singular contributions $\sim \delta_{ap}$
- regular contributions present for all a, p

Theorem (Ward identity for injective E)

$$\sum_{n\in I} \frac{\partial^2 \mathcal{Z}[J]}{\partial J_{an}\partial J_{np}}$$

$$= \delta_{ap}\Big(\sum_{n\in I} G_{an} + G_{a|a}$$

$$+ \sum_{r,s\in I} \Big\{ G_{a|r}G_{a|s} + \frac{1}{2}G_{a|a|r|s} + G_{rar|s} + \frac{1}{2}\sum_{n\in I} G_{r|s|an}\Big\} J_{rr}J_{ss}$$

$$+ \sum_{r,s\in I} \Big\{ \frac{1}{2}G_{a|a|rs} + G_{rars} + \frac{1}{2}\sum_{n\in I} G_{rs|an}\Big\} J_{rs}J_{sr} + \mathcal{O}(J^4)\Big)\mathcal{Z}[J]$$

$$-\frac{1}{E_a - E_p}\sum_{n\in I}\Big(J_{pn}\frac{\partial \mathcal{Z}[J]}{\partial J_{an}} - J_{na}\frac{\partial \mathcal{Z}[J]}{\partial J_{np}}\Big)$$

The $\mathcal{O}(J^4)$ terms are explicitly known. Injectivity of E is crucial!

How to use the Ward identity

Functional integration yields, up to irrelevant constant,

$$\mathcal{Z}[J] = e^{-V[\frac{\partial}{\partial J}]} e^{\frac{1}{2}\langle J,J\rangle_E}\;, \qquad \langle J,J\rangle_E := \sum_{m,n\in I} \frac{J_{mn}J_{nm}}{E_m+E_n}$$

Example: G_{ab} (for $a \neq b$)

$$G_{ab} = \frac{1}{\mathcal{Z}[0]} \frac{\partial^2 \mathcal{Z}[J]}{\partial J_{ba}\partial J_{ab}}\Big|_{J=0} = \frac{1}{\mathcal{Z}[0]} \Big\{ \frac{\partial}{\partial J_{ba}} e^{-V\left[\frac{\partial}{\partial J}\right]} \frac{\partial}{\partial J_{ab}} e^{\frac{1}{2}\langle J,J\rangle_E} \Big\}_{J=0}$$

$$= \frac{1}{E_a+E_b} + \frac{1}{(E_a+E_b)\mathcal{Z}[0]} \Big\{ \Big(\phi_{ab} \frac{\partial(-V)}{\partial \phi_{ab}} \Big) \Big[\frac{\partial}{\partial J} \Big] \Big\} \mathcal{Z}[J]\Big|_{J=0}$$

$\frac{\partial(-V)}{\partial \phi_{ab}}$ contains, for any V, the twofold derivative $\sum_n \frac{\partial^2}{\partial J_{an}\partial J_{np}}$

Schwinger-Dyson equations (for $V[\phi] = \frac{\lambda_4}{4}\mathrm{tr}(\phi^4)$)

further expansion of connected functions $G_{\dots} = \sum_{g=0}^{\infty} G^g_{\dots}$ into components of equal genus g leads to a short system of Schwinger-Dyson equations:

1. A closed non-linear equation for G^0_{ab} (planar+regular):

$$G^0_{ab} = \frac{1}{E_a + E_b} - \frac{\lambda_4}{E_a + E_b} \sum_{p \in I} \Big(G^0_{ab} G^0_{ap} - \frac{G^0_{pb} - G^0_{ab}}{E_p - E_a} \Big)$$

2. For every other $G^g_{a_1 \dots a_N}$ an equation which only depends on

- $G^g_{a_1 \dots a_k}$ for $k \leq N$,
- $G^h_{a_1 \dots a_k}$ with $h < g$ and $k \leq N + 2$;

this dependence is linear in the top degree (N, g)

Some $G_{\dots}$ need renormalisation of E, ϕ, and λ_n!

Noncommutative $\mathbb{R}^d$ (a.k.a. Moyal space)

$\mathcal{A}_\Theta$ = Rieffel deformation of $\mathcal{S}(\mathbb{R}^d)$ for action of translation group:

$$(f \star g)(x) = \int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{dx\, dk}{(2\pi)^d} f(x + \tfrac{1}{2}\Theta k)\, g(x+y)\, e^{\mathrm{i}\langle k, y\rangle}$$

(with $\Theta = -\Theta^t \in GL(d, \mathbb{R})$)

- non-unital pre-C^* algebra
- left action as subspace of $\mathcal{K}(H_1) \otimes \mathrm{id}_{H_2}$ on GNS-Hilbert space $H_{GNS} = H_1 \otimes H_2$
- thus gives rise to $(E, P[\phi])$-matrix model

We take $d = 4$, $V[\phi] = \sqrt{\det(2\pi\Theta)}\mathrm{tr}(\frac{\lambda_4}{4}\phi^4)$ and

$$\sqrt{\det(2\pi\Theta)}\mathrm{tr}(E\phi^2) = \int_{\mathbb{R}^d} dx\, \frac{1}{2}\phi(x)\Big(-\Delta + \langle 2\Theta^{-1}x, 2\Theta^{-1}x\rangle + \mu^2\Big)\phi(x)$$

Renormalised noncommutative ϕ^4_4-theory

ϕ^4-theory on 4*D*-Moyal space w/ harmonic oscillator potential

$$S[\phi] = \int d^4x \Big(\frac{Z}{2}\phi \star \big(-\Delta + \Omega^2(2\Theta^{-1}x)^2 + \mu_0^2\big)\phi + \frac{\lambda_0 Z^2}{4}\phi \star \phi \star \phi \star \phi \Big)(x)$$

- renormalisable as formal power series in λ_0 [Grosse-W., 2004] (renormalisation of $\mu_0^2, \lambda_0, Z \in \mathbb{R}_+$ and $\Omega \in [0,1]$) means: well-defined perturbative quantum field theory
- Langmann-Szabo duality: theories at Ω and $\Omega^* = \frac{1}{\Omega}$ are the same; self-dual case $\Omega = 1$ is matrix model
- β-function vanishes to all orders in λ_0 for $\Omega = 1$ [Disertori-Gurau-Magnen-Rivasseau, 2006] means: almost scale-invariant

Is the self-dual (critical) model integrable?

The closed equation for (unrenormalised) G^0_{ab}

$$G^0_{ab} = \frac{1}{Z\left(\frac{4}{\theta}(a+b)+\mu_0^2\right)} - \frac{\lambda}{\left(\frac{4}{\theta}(a+b)+\mu_0^2\right)} \sum_{p=0}^{\Lambda} \left(\frac{4}{\theta}\right)^2 (p+1)\left(Z G^0_{ab} G^0_{ap} - \frac{G^0_{pb}-G^0_{ab}}{\frac{4}{\theta}(p-a)}\right)$$

with: $a,b,p \in \{0,1,2,\dots,\Lambda\}$, measure: $d\mu_p = \binom{p+\frac{d}{2}-1}{\frac{d}{2}-1}$

Renormalisation:

- pass to 1PI functions $G^0_{ab} =: (Z(a+b+\mu_0^2) - \Gamma_{ab})^{-1}$ and Taylor-expand $\Gamma_{ab} = Z\mu_0^2 - \mu^2 + (Z-1)(a+b) + \Gamma^{ren}_{ab}$
- normalisation conditions: $\Gamma^{ren}_{00} = 0$ and $(\partial\Gamma^{ren})_{00} = 0$ determine μ_0^2 and Z
- no renormalisation of $\lambda = \frac{\lambda_0}{64\pi^2}$ because of $\beta_\lambda = 0$

The limit $\theta \to \infty$

$$G^0_{ab} = \frac{1}{Z\big(\frac{4}{\theta}(a+b)+\mu_0^2\big)} - \frac{\lambda}{\big(\frac{4}{\theta}(a+b)+\mu_0^2\big)} \sum_{p=0}^{\Lambda} \Big(\frac{4}{\theta}\Big)^2 (p+1)\Big(Z G^0_{ab} G^0_{ap} - \frac{G^0_{pb}-G^0_{ab}}{\frac{4}{\theta}(p-a)}\Big)$$

- sum converges for $\theta \to \infty$ to Riemann integral:
  $x_a = \frac{4}{\theta}a, \quad x_b = \frac{4}{\theta}b, \quad x_p = \frac{4}{\theta}p, \quad dx_p = \frac{4}{\theta}, \quad \Lambda_\infty = \frac{4}{\theta}\Lambda,$
  $$\sum_{p=0}^{\Lambda} \Big(\frac{4}{\theta}\Big)^2 (p+1) f\big(\tfrac{4}{\theta}p\big) = \int_0^{\Lambda_\infty} dx_p \, x_p f(x_p)$$
- express G by Γ^{ren}, impose normalisation conditions
- revert to $\frac{(1-\alpha)(1-\beta)}{1-\alpha\beta} G_{\alpha\beta} = \frac{\mu^2}{x_a+x_b+\mu^2-\Gamma^{ren}_{x_a x_b}}$
  $\alpha = \frac{x_a}{x_a+\mu^2}, \quad \xi = \frac{\Lambda_\infty}{\Lambda_\infty+\mu^2}$
- express $a, b, p \in [0, \Lambda_\infty]$ by $\alpha, \beta, \rho \in [0, \xi] \subset [0, 1[$

Cubic equation for (renormalised) $G_{\alpha\beta}$

$$
\begin{aligned}
&(G_{\alpha\beta}-1)-(Z^{-1}-1)+(Z^{-1}-1)G_{\alpha\beta}\frac{(1-\alpha)(1-\beta)}{(1-\alpha\beta)} \\
&=-\lambda G_{\alpha\beta}\frac{(1-\alpha)(1-\beta)}{(1-\alpha\beta)}\frac{1}{Z^{-1}}\int_0^\xi \frac{\rho\, d\rho}{(1-\rho)^2}\Big(\frac{(1-\alpha)G_{\alpha\rho}}{1-\alpha\rho}-G_{0\rho}\Big) \\
&+\lambda\frac{(1-\alpha)(1-\beta)}{(1-\alpha\beta)}\int_0^\xi \frac{d\rho}{(1-\rho)}\Big(\frac{G_{\rho\beta}}{1-\rho\beta}-G_{\alpha\beta}G_{\rho 0}+\alpha\frac{G_{\rho\beta}-G_{\alpha\beta}}{\rho-\alpha}\Big)
\end{aligned}
$$

$$
Z^{-1}=1-\lambda\int_0^\xi d\rho\frac{G_{\rho 0}}{1-\rho}+\lambda\int_0^\xi d\rho\Big(G_{\rho 0}-\frac{\frac{d}{d\sigma}G_{\rho\sigma}\big|_{\sigma=0}}{1-\rho}\Big)
$$

This equation can be solved!

- blue term expressed by eq. at $\beta=0$ $\quad$ ($\Rightarrow$ limit $\xi\to 1$ exists)
- red term by Hilbert transform if $G_{\alpha\beta}$ is Hölder-continuous

Result:

$$\Big(\frac{\beta(1-\alpha)}{\alpha(1-\beta)} + \frac{1+\lambda\mathcal{Y}+\lambda\pi\alpha\mathcal{H}_\alpha\big[G_{\bullet 0}\big]}{\alpha G_{\alpha 0}}\Big)D_{\alpha\beta} - \lambda\pi\mathcal{H}_\alpha\big[D_{\bullet\beta}\big] = -G_{\alpha 0},$$

$$-\lambda\pi\mathcal{H}_0[D_{\bullet 0}] = \lambda\mathcal{Y}$$

where $D_{\alpha\beta} := \dfrac{\alpha(1-\beta)}{\beta(1-\alpha)}\Big(\dfrac{(1-\beta)}{1-\alpha\beta}G_{\alpha\beta} - G_{\alpha 0}\Big)$

Finite Hilbert transform $\mathcal{H}_\alpha[f(\bullet)] := \dfrac{1}{\pi}\lim_{\epsilon\to 0}\Big(\int_0^{\alpha-\epsilon} + \int_{\alpha+\epsilon}^1\Big)\dfrac{f(\rho)}{\rho-\alpha}$

- preserves $L^p[0,1]$ for $p>1$, not for $p=1$ [M. Riesz]
- does not preserve $\mathcal{C}[0,1]$
- preserves locally-Hölder* spaces $(L^p \cap H_\eta)(]0,1[)$ [Okada-Elliott]

$f \in H_\eta[a,b] \;\Leftrightarrow\; \|f\|_\eta = \sup_{a\le\alpha\le b}|f(\alpha)| + \sup_{a\le\alpha<\beta\le b}\dfrac{|f(\beta)-f(\alpha)|}{(\beta-\alpha)^\eta} < \infty$

The Carleman equation

Theorem [Carleman 1922, Tricomi 1957]

The singular linear integral equation

$$a(x)y(x) - \lambda\pi\mathcal{H}_x[y] = f(x)\,, \qquad x \in [-1,1]$$

is for $a(x)$ continuous + Hölder near ± 1 and $f \in L^p$ solved by

$$y(x) = \frac{\sin(\theta(x))}{\lambda\pi}\Big(f(x)\cos(\theta(x)) + e^{\mathcal{H}_x[\theta]}\mathcal{H}_x\Big[e^{-\mathcal{H}_\bullet[\theta]}f(\bullet)\sin(\theta(\bullet))\Big] + \frac{Ce^{\mathcal{H}_x[\theta]}}{1-x}\Big)$$

$$\theta(x) = \arctan_{[0,\pi]}\Big(\frac{\lambda\pi}{a(x)}\Big)\,, \quad \sin(\theta(x)) = \frac{|\lambda\pi|}{\sqrt{(a(x))^2 + (\lambda\pi)^2}}$$

where C is an arbitrary constant.

Assumption: $C = 0$

The breakthrough

Theorem

$$\frac{(1-\beta)}{1-\alpha\beta}\frac{G_{\alpha\beta}}{1+\lambda\mathcal{Y}} = \frac{\sin(\theta_\beta(\alpha))}{|\lambda|\pi\alpha} e^{\mathcal{H}_\alpha[\theta_\beta(\bullet)]-\mathcal{H}_0[\theta_0(\bullet)]+\mathcal{H}_1[\theta_0(\bullet)-\theta_\beta(\bullet)]}$$

$$\frac{\lambda\mathcal{Y}}{1+\lambda\mathcal{Y}} = \int_0^1 d\rho\, \frac{\sin^2(\theta_0(\rho))}{\lambda\pi^2\rho^2}$$

$$\theta_\beta(\alpha) = \underset{[0,\pi]}{\arctan}\left(\frac{\lambda\pi\alpha}{\frac{\beta(1-\alpha)}{1-\beta}+\frac{1+\lambda\mathcal{Y}+\lambda\pi\alpha\mathcal{H}_\alpha[G_{\bullet 0}]}{G_{\alpha 0}}}\right) \qquad (*)$$

Consequence: $G_{\alpha\beta} \geq 0$!

Main steps of the proof:

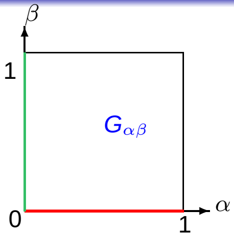
1. (*) is Carleman eq. $\lambda\pi\cot\theta_0(\alpha)G_{\alpha 0} - \lambda\pi\mathcal{H}_\alpha[G_{\bullet 0}] = \frac{1+\lambda\mathcal{Y}}{\alpha}$
2. Tricomi's identity

$$e^{-\mathcal{H}_\alpha[\theta_\beta]}\cos(\theta_\beta(\alpha)) + \mathcal{H}_\alpha\Big[e^{-\mathcal{H}_\bullet[\theta_\beta]}\sin(\theta_\beta(\bullet))\Big] = 1$$

The self-consistency equation

Given boundary value $G_{\alpha 0}$,
Carleman computes $G_{\alpha\beta}$,
in particular $G_{0\beta}$

symmetry forces $G_{\beta 0} = G_{0\beta}$



Master equation

The theory is completely determined by the solution of

$$G_{\beta 0} = \frac{1+\lambda\mathcal{Y}}{1+(1-\beta)\lambda\mathcal{Y}} \times \exp\left(-\lambda \int_0^{\frac{\beta}{1-\beta}} dt \int_0^1 \frac{d\rho}{(\lambda\pi\rho)^2 + \left(t(1-\rho) + \frac{1+\lambda\mathcal{Y}+\lambda\pi\rho\mathcal{H}_\rho[G_{\bullet 0}]}{G_{\rho 0}}\right)^2}\right)$$

(provided it exists, together with eq. for $\lambda\mathcal{Y}$)

Corollary $\lambda > 0$

$\frac{(1+(1-\beta)\lambda\mathcal{Y})}{1+\lambda\mathcal{Y}} G_{\beta 0} \in \mathcal{C}^1([0,1[)$, monotonously decreasing, positive;
G_{10} exists, $G_{\beta 0} \in \mathcal{C}[0,1]$

Corollary $\lambda < 0$

$\frac{(1+(1-\beta)\lambda\mathcal{Y})}{1+\lambda\mathcal{Y}} G_{\beta 0} \in \mathcal{C}^1([0,1[)$, monotonously increasing, positive;
$G_{\beta 0}$ unbounded at $\beta = 1$.

Some non-perturbative results

Lemma

Let $\lambda > 0$, $G = TG$ the master equation and $F \in H_\lambda[0,1]$.

Recall $Z^{-1}(G) = 1 + \lambda \mathcal{Y}_G - \lambda \int_0^1 d\rho \, \frac{G_{\rho 0}}{1-\rho}$

1. If $F(1) \neq 0$, then $(TF)(1) = 0$.
2. If $Z^{-1}(F) \geq \delta > 0$, then $(TF)(1) \geq \epsilon > 0$.
3. If $Z^{-1}(F) < 0$, then $(TF)(1) = 0$.

Consequently, $G_{10} = 0$ and $Z^{-1}(G) \leq 0$.

But ▸ original equation gives

$$G_{\alpha 0} = 0 \quad \Rightarrow \quad 1 + \lambda \mathcal{Y} + \lambda \pi \alpha \mathcal{H}_\alpha \big[G_{\bullet 0}\big] = 0$$

For $\alpha = 1$ this means $Z^{-1}(G) = 0$.

The planar regular four-point function

Schwinger-Dyson equation for $G_{abcd} = G_{ab}G_{bc}G_{cd}G_{da}\Gamma_{abcd}$ becomes again a Carleman equation:

$$\lambda\pi \cot(\theta_\beta(\alpha))\mathcal{G}_{\alpha\beta\gamma\delta} - \lambda\pi\mathcal{H}_\alpha[\mathcal{G}_{\bullet\beta\gamma\delta}] = \lambda f_{\beta\gamma\delta}(\alpha)$$

where

$$\mathcal{G}_{\alpha\beta\gamma\delta} = \alpha G_{\gamma\delta}\frac{(1-\beta)G_{\alpha\beta}}{1-\alpha\beta}\frac{(1-\delta)G_{\delta\alpha}}{1-\delta\alpha}\Gamma_{\alpha\beta\gamma\delta}$$

$$f_{\beta\gamma\delta}(\alpha) = \frac{(1-\gamma)(1-\alpha\delta)G_{\gamma\delta} - (1-\alpha)(1-\gamma\delta)G_{\alpha\delta}}{(1-\delta\alpha)(\alpha-\gamma)}$$

Effective (“physical”) coupling constant

$$\Gamma_{0000} = \frac{|\lambda|}{1+\lambda\mathcal{Y}}\exp\Big(\int_0^1 \frac{d\rho}{\pi\rho}\,\underset{[0,\pi]}{\arctan}\,\frac{\lambda\pi\rho G_{\rho 0}}{1+\lambda\mathcal{Y}+\lambda\pi\rho\mathcal{H}_\rho[G_{\bullet 0}]}\Big)$$

Only finite renormalisation of λ in response to an infinite change of scales! Theory is almost (but not exactly!) scale-invariant.

Numerical iteration of T

We approximate G by piecewise-linear function sampled at $0 = x_0 < x_1 < \cdots < x_N = 1$ and study $f \mapsto Tf$ numerically.
We find for $\lambda > 0$ independently of starting point f:

1. $\frac{\lambda\mathcal{Y}}{1+\lambda\mathcal{Y}} = \ldots$ has unique solution
2. Tf is monotonously decreasing
3. $Tf \in H_\eta[0,1]$ for $\eta = \min(\lambda, \frac{1}{\pi})$
4. $T^n f$ converges in $H_\eta[0,1]$ to fixed-point solution $G = TG$
5. G is L^2-close to $(1-\beta^{\frac{1}{c}})^c$ for $c \approx \eta$
6. $Z^{-1} > 0$ for $0 < \lambda < \frac{1}{\pi}$, but $Z^{-1} \to 0$ for more sample points
   $Z^{-1} < 0$ for $\lambda > \frac{1}{\pi}$, stable in number of sample points

Something happens at $\lambda^* = \frac{1}{\pi}$!

The remaining part of the proof

Fact

Let $0 < \lambda \leq \frac{1}{\pi}$ and $K \subset \mathcal{C}[0,1]$ be a convex subset, with

– $\forall F \in K$: $F \in H_\lambda[0,1]$ and $\sqrt{1-\beta^2} \leq F(\beta) \leq 1$

– maybe F non-increasing, convex, ...

Assume for all $F \in K$:

1. $$\frac{Y_F}{1+Y_F} = \int_0^1 d\rho\, \frac{\lambda}{(\lambda\pi\rho)^2 + \left(\frac{1+Y_F+\lambda\pi\rho\mathcal{H}_\rho[F(\bullet)]}{F(\rho)}\right)^2}$$
   has unique solution which depends continuously on F
2. The following function TF is monotonously decreasing:

$$(TF)(\beta)=\frac{1+Y_F}{1+(1-\beta)Y_F}\exp\left(-\int_0^{\frac{\beta}{1-\beta}} dt \int_0^1 \frac{\lambda\, d\rho}{(\lambda\pi\rho)^2 + \left(t(1-\rho) + \frac{1+Y_F+\lambda\pi\rho\mathcal{H}_\rho[F(\bullet)]}{F(\rho)}\right)^2}\right)$$

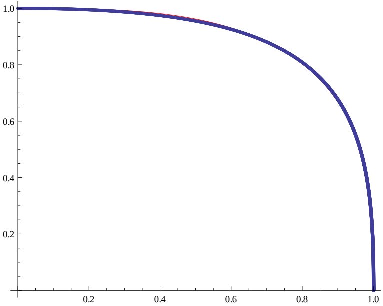
3. $\|TF\|_\lambda \leq M$ uniformly on K.

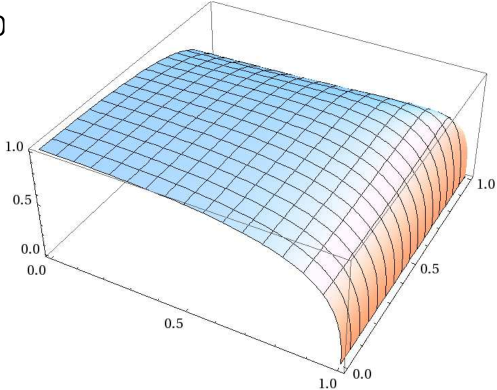
Then there is a fixed-point solution $G = TG \in K$.

Proof. Arzelà-Ascoli + Schauder fixed-point theorem

Some plots: $\lambda = \frac{1}{\pi} = 0.318310\ldots$

$x_k = \sin(\frac{\pi}{2}\sin(\frac{\pi}{2}\frac{k}{N}))$, $N = 2000$





eff. coupling: $\Gamma_{0000} = 0.333359$

$Z^{-1} = 1 * 10^{-5}$

$\|G - F\|_2 = 6 * 10^{-3}$,

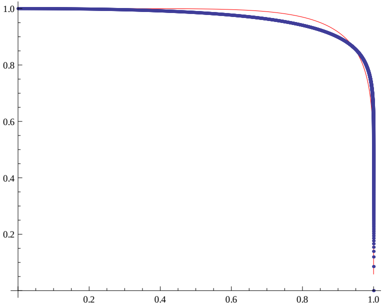
$F = (1 - \beta^p)^{\frac{1}{p}}$, $p = 3.1395$

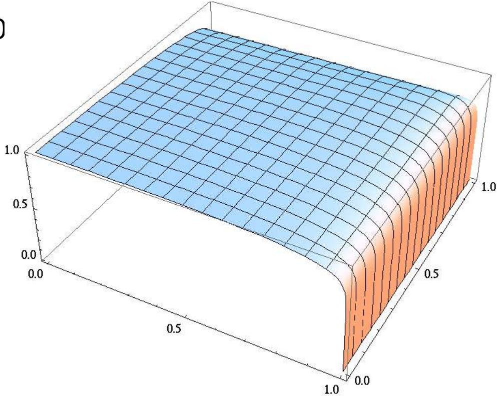
asymmetry

$\max_{\alpha,\beta} |G_{\alpha\beta} - G_{\beta\alpha}| = 4 * 10^{-6}$

$\lambda = \frac{1}{4\pi} = 0.079577\ldots$

$x_k = \sin(\frac{\pi}{2}\sin(\frac{\pi}{2}\frac{k}{N}))$, $N = 2000$





eff. coupling: $\Gamma_{0000} = 0.0798164$

$Z^{-1} = 9.7 * 10^{-2}$

$\|G - F\|_2 = 2 * 10^{-2}$,

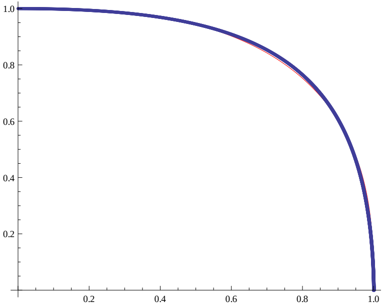
$F = (1 - \beta^p)^{\frac{1}{p}}$, $p = 7.2396$

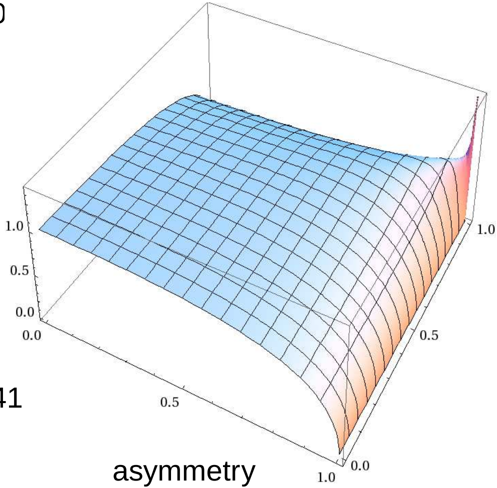
asymmetry

$\max_{\alpha,\beta} |G_{\alpha\beta} - G_{\beta\alpha}| = 2 * 10^{-4}$

$\lambda = \frac{3}{2\pi} = 0.477465\ldots$

$x_k = \sin(\frac{\pi}{2}\sin(\frac{\pi}{2}\frac{k}{N}))$, $N = 2000$





eff. coupling: $\Gamma_{0000} = 0.536041$

$Z^{-1} = -0.24$

$\|G - F\|_2 = 1 * 10^{-2}$,

$F = (1 - \beta^p)^{\frac{1}{p}}$, $p = 2.7455$

asymmetry

$\max_{\alpha,\beta} |G_{\alpha\beta} - G_{\beta\alpha}| = 2.5$

What happens at $\lambda^* = \frac{1}{\pi}$?

- evidence that $G = TG$ does not solve ▸ original equation for $\lambda > \frac{1}{\pi}$
- probable reason is ignorance of non-trivial solution of homogeneous ▸ Carleman equation
- More precise treatment [Muskhelishvili] relates that solution to index / winding number of complex logarithm

We think this sets in at $\lambda = \frac{1}{\pi}$ and produces $G_{\beta 0} = 0$ for $\beta < 1$.
Each jump in the index gives another zero.

- Perturbatively one has $G_{\alpha 0} + \lambda \int_0^1 d\rho \frac{G_{\rho 0}}{1 - \alpha\rho} = 1 + \mathcal{O}(\lambda)$
- For $\mathcal{O}(\lambda) \mapsto 0$, this is solved by Mehler-Fock transform.
- Solution is $G_{\alpha 0} \propto (1 - \alpha)^{\frac{\arcsin(\lambda\pi)}{\pi}}$ for $|\lambda\pi| \leq 1$.
Analytic continuation to $\lambda\pi > 1$ oscillates.

An analogy

2D Ising model	4D nc ϕ^4-theory
temperature T, $K = \frac{J}{k_B T}$	frequency Ω
Kramers-Wannier duality $\sinh(2K)\sinh(2K^*) = 1$	Langmann-Szabo duality $\Omega\Omega^* = 1$
solvable at $K = K^*$ scale-invariant	solvable at $\Omega = \Omega^*$ almost scale-invariant
CFT minimal model ($m = 3$)	matrix model
operator product expansion Virasoro constraints	Schwinger-Dyson equation Ward identities
critical exponents $G^{\sigma\sigma}_{n0} \propto \frac{1}{n^{d-2+\eta}}$, $\eta = \frac{1}{4}$ $G^{\sigma\sigma\sigma\sigma}_{nn00} \propto \frac{1}{n^{2(d-1/\nu)}}$, $\nu = 1$	critical exponents $G^{\phi\phi}_{n0} \propto \frac{1}{n^{1+\lambda}}$, $\lambda \in]0, \frac{1}{\pi}]$ $G^{\phi\phi\phi\phi}_{nn00}$ (next week)
Virasoro algebra, CFT, subfactors, ...	???