Progress in solving noncommutative ϕ^4-theory in four dimensions

Raimar Wulkenhaar

Mathematisches Institut der Westfälischen Wilhelms-Universität
Münster, Germany

(based on joint work with Harald Grosse)

Introduction

- The Standard Model is a perturbatively renormalisable quantum field theory.
- Scattering amplitudes can be computed as formal power series in coupling constants such as $e^2 \approx \frac{1}{137}$.
  The first terms agree to high precision with experiment.
- The radius of convergence in e^2 is zero!

- Refined summation techniques (e.g. Borel) may establish reasonable domains of analyticity.
- Unfortunately, this also fails for QED due to the Landau ghost problem.

  It is expected to work for non-Abelian gauge theories because of asymptotic freedom.

  But these theories are too complicated.

QFT's on noncommutative geometries may provide toy models for non-perturbative renormalisation in four dimensions.

Due to noncommutativity, these models may have an infinite number of Ward identities which could make them integrable.

General matrix models

- I – set of indices (finite, countable or continuous)
- $\mathcal{M} = \{M = (M_{ab})_{a,b\in I}\}$ space of matrices (with topology)

  product $(MN)_{ab} = \sum_{c\in I} M_{ac}N_{cb}$, trace $\mathrm{tr}(M) = \sum_{a\in I} M_{aa}$

  for continuous I take $(MN)_{ab} = \int_I d\mu(c)\; M_{ac}M_{cb}$

  $\mathcal{M}_* = \{M = M^* \in \mathcal{M}\}$, where $(M^*)_{ab} = \overline{M_{ba}}$ adjoint
- action = non-linear functional S on $\mathcal{M}_*$. We consider

$$S[\phi] = \mathrm{tr}(E\phi^2) + V[\phi]\;, \qquad V[\phi] = \mathrm{tr}(P[\phi])$$

  where $E \in \mathcal{M}_*$ is a positive external matrix and $P[\phi]$ an (e.g. even) polynomial in ϕ with scalar coefficients.

Euclidean quantum field theory

- action with source term $\longrightarrow$ partition function

$$\mathcal{Z}[J] = \int \mathcal{D}\phi \; \exp(-S[\phi] + \mathrm{tr}(\phi J))$$

where $\mathcal{D}\phi = \prod_{a,b \in I} d\phi_{ab}$

- connected correlation functions obtained from $\mathcal{W}[J] = \ln \mathcal{Z}[J]$ as

$$\langle \varphi_{a_1 b_1} \dots \varphi_{a_n b_n} \rangle = \frac{\partial^n \mathcal{W}[J]}{\partial J_{b_1 a_1} \dots \partial J_{b_n a_n}}\Big|_{J=0}$$

- unless I is finite, the resulting index sums may diverge and require a renormalisation

Ward identity

- unitary transformation $\phi \mapsto \tilde{\phi} = U\phi U^*$
  $U \in \mathcal{M}''$ with $UU^* = U^*U = \mathrm{id}$, leaves $\mathcal{M}_*$ invariant:

$$\int \mathcal{D}\phi \; \exp(-S[\phi] + \mathrm{tr}(\phi J)) = \int \mathcal{D}\tilde{\phi} \; \exp(-S[\tilde{\phi}] + \mathrm{tr}(\tilde{\phi} J))$$

- measure unitarily invariant: $\mathcal{D}\tilde{\phi} = \mathcal{D}\phi$:

$$0 = \int \mathcal{D}\phi \left[\exp(-S[\phi] + \mathrm{tr}(\phi J)) - \exp(-S[\tilde{\phi}] + \mathrm{tr}(\tilde{\phi} J)) \right]$$

  note: $[\ \ ] \neq 0$ because $\mathrm{tr}(E\phi^2)$, $\mathrm{tr}(\phi J)$ not unitarily invariant!

- linearisation $\longrightarrow$ Ward identity (matricial equation!)

$$0 = \int \mathcal{D}\phi \left[E\phi\phi - \phi\phi E - J\phi + \phi J \right] \exp(-S[\phi] + \mathrm{tr}(\phi J))$$

- choose a reference frame where E is diagonal (but J is not)
- use functional derivative $\phi_{ab} = \frac{\partial}{\partial J_{ba}}$

Proposition

The partition function $\mathcal{Z}[J]$ of the matrix model defined by the external matrix E satisfies the $|I| \times |I|$ Ward identities

$$0 = \sum_{n\in I} \Big((E_a - E_p) \frac{\partial^2 \mathcal{Z}}{\partial_{an} \partial J_{np}} + J_{pn} \frac{\partial \mathcal{Z}}{\partial J_{an}} - J_{na} \frac{\partial \mathcal{Z}}{\partial J_{np}} \Big)$$

Main assumption

- $m \mapsto E_m > 0$ injective

Note: always the case if we pass to equivalence classes $[m]$ which have the same E_m; then $\sum_{m\in I} f(m) = \sum_{[m]\in[I]} \mu([m]) f([m])$

We will see: These Ward identities and the choice of $V[\phi]$ determine the QFT of the matrix model non-perturbatively!

Decomposition into cycles

From perturbative expansion into ribbon graphs: right index of J_{ab} is left index of another J_{bc}, or of the same J_{bb}.

Decomposition of $\mathcal{W}[J]$ for even $V[\phi]$ into J-cycles:

$$\begin{aligned}\mathcal{W}[J] = \mathcal{W}[0] &+ \frac{1}{2}\sum_{p,q\in I} G_{pq}(J_{pq}J_{qp}) + \frac{1}{2}\sum_{p,q\in I} G_{p|q}(J_{pp})(J_{qq}) \\ &+ \frac{1}{4}\sum_{p,q,r,s\in I} G_{pqrs}(J_{pq}J_{qr}J_{rs}J_{sp}) + \frac{1}{3}\sum_{p,q,r,s\in I} G_{pqr|s}(J_{pq}J_{qr}J_{rp})(J_{ss}) \\ &+ \frac{1}{8}\sum_{p,q,r,s\in I} G_{pq|rs}(J_{pq}J_{qp})(J_{rs}J_{sr}) + \frac{1}{4}\sum_{p,q,r,s\in I} G_{pq|r|s}(J_{pq}J_{qp})(J_{rr})(J_{ss}) \\ &+ \frac{1}{24}\sum_{p,q,r,s\in I} G_{p|q|r|s}(J_{pp})(J_{qq})(J_{rr})(J_{ss}) + \mathcal{O}(J^6)\end{aligned}$$

Attention: $G_{pp}J_{pp}J_{pp}$ is topologically different from $G_{p|p}J_{pp}J_{pp}$!

A new Ward identity for injective E

We turn the previous Ward identity

$$0 = \sum_{n \in I} \Big((E_a - E_p) \frac{\partial^2 \mathcal{Z}}{\partial_{an} \partial J_{np}} + J_{pn} \frac{\partial \mathcal{Z}}{\partial J_{an}} - J_{na} \frac{\partial \mathcal{Z}}{\partial J_{np}} \Big)$$

for injective E into a formula for

$$\sum_{n \in I} \frac{\partial^2 \mathcal{Z}[J]}{\partial J_{an} \partial J_{np}} = \sum_{n \in I} \Big(\frac{\partial^2 \mathcal{W}[J]}{\partial J_{an} \partial J_{np}} + \frac{\partial \mathcal{W}[J]}{\partial J_{an}} \frac{\partial \mathcal{W}[J]}{\partial J_{np}} \Big) \mathcal{Z}[J]$$

The J-cycle structure in $\mathcal{W}[J]$ creates

- singular contributions $\sim \delta_{ap}$
- regular contributions present for all a, p

Main Theorem (Ward identity for injective E)

$$\sum_{n\in I} \frac{\partial^2 \mathcal{Z}[J]}{\partial J_{an}\partial J_{np}}$$
$$= \delta_{ap}\Big(\sum_{n\in I} G_{an} + G_{a|a}$$
$$+ \sum_{r,s\in I} \Big\{ G_{a|r}G_{a|s} + \frac{1}{2}G_{a|a|r|s} + G_{rar|s} + \frac{1}{2}\sum_{n\in I} G_{r|s|an}\Big\} J_{rr}J_{ss}$$
$$+ \sum_{r,s\in I} \Big\{ \frac{1}{2}G_{a|a|rs} + G_{rars} + \frac{1}{2}\sum_{n\in I} G_{rs|an}\Big\} J_{rs}J_{sr} + \mathcal{O}(J^4)\Big)\mathcal{Z}[J]$$
$$-\frac{1}{E_a - E_p}\sum_{n\in I}\Big(J_{pn}\frac{\partial \mathcal{Z}[J]}{\partial J_{an}} - J_{na}\frac{\partial \mathcal{Z}[J]}{\partial J_{np}}\Big)$$

The $\mathcal{O}(J^4)$ terms are explicitly known. Injectivity of E is crucial!

How to use the Ward identity

Functional integration yields, up to irrelevant constant,

$$\mathcal{Z}[J] = e^{-V[\frac{\partial}{\partial J}]} e^{\frac{1}{2}\langle J,J\rangle_E} , \qquad \langle J, J\rangle_E := \sum_{m,n\in I} \frac{J_{mn}J_{nm}}{E_m + E_n}$$

Example: G_{ab} (for $a \neq b$)

$$G_{ab} = \frac{1}{\mathcal{Z}[0]} \frac{\partial^2 \mathcal{Z}[J]}{\partial J_{ba}\partial J_{ab}}\Big|_{J=0} = \frac{1}{\mathcal{Z}[0]} \Big\{ \frac{\partial}{\partial J_{ba}} e^{-V\left[\frac{\partial}{\partial J}\right]} \frac{\partial}{\partial J_{ab}} e^{\frac{1}{2}\langle J,J\rangle_E} \Big\}_{J=0}$$

$$= \frac{1}{E_a + E_b} + \frac{1}{(E_a + E_b)\mathcal{Z}[0]} \Big\{ \Big(\phi_{ab} \frac{\partial(-V)}{\partial \phi_{ab}} \Big) \Big[\frac{\partial}{\partial J}\Big] \Big\} \mathcal{Z}[J]\Big|_{J=0}$$

$\frac{\partial(-V)}{\partial \phi_{ab}}$ contains, for any V, the twofold derivative $\sum_n \frac{\partial^2}{\partial J_{an}\partial J_{np}}$

Results for $V[\phi] = \frac{\lambda_4}{4}\mathrm{tr}(\phi^4)$

$$G_{ab} = \frac{1}{E_a+E_b} - \frac{\lambda_4}{E_a+E_b}\Big(G_{ab}\sum_{p\in I} G_{ap} + G_{ab}G_{a|a} + G_{aaab} + G_{abab} + \sum_{p\in I} G_{ap|ab} + G_{a|a|ab}\Big) + \frac{\lambda_4}{E_a+E_b}\Big(\sum_{p\in I}\frac{G_{pb}-G_{ab}}{E_p-E_a} + \frac{G_{b|b}-G_{a|b}}{E_b-E_a}\Big)$$

$$G_{a|b} = -\frac{\lambda_4}{E_a+E_a}\Big(G_{a|b}\sum_{p\in I} G_{ap} + 3G_{a|b}G_{a|a} + G_{b|aaa} + G_{a|bab} + \sum_{p\in I} G_{a|b|ap} + G_{a|a|a|b}\Big) + \frac{\lambda_4}{E_a+E_a}\Big(\frac{G_{bb}-G_{ab}}{E_b-E_a} + \sum_{p\in I}\frac{G_{p|b}-G_{a|b}}{E_p-E_a}\Big)$$

Genus expansion

- Perturbatively, a ribbon graph is a simplicial complex which topologically describes a genus-g Riemann surface on which it can be drawn.
- Our connected functions $G_{ab}, G_{a|b}, G_{abcd}$, etc., involve all these topologies. Accordingly, we expand $G_{ab} = \sum_{g=0}^{\infty} G^g_{ab}$, $G_{a|b} = \sum_{g=0}^{\infty} G^g_{a|b}$, $G_{abcd} = \sum_{g=0}^{\infty} G^g_{abcd}$, etc.
- The operations leading to the Ward identities increase the genus by 1 whenever two J-cycles are connected by a bridge.

We find:

$$G^g_{ab} = \frac{\delta_{g0}}{E_a+E_b} - \frac{\lambda_4}{E_a+E_b}\Big(\sum_{g'+g''=g} G^{g'}_{ab}\sum_{p\in I} G^{g''}_{ap} + \sum_{g'+g''=g-1} G^{g'}_{ab}G^{g''}_{a|a}$$
$$+G^{g-1}_{aaab} + G^{g-1}_{abab} + \sum_{p\in I} G^{g-1}_{ap|ab} + G^{g-1}_{a|a|ab}\Big)$$
$$+\frac{\lambda_4}{E_a+E_b}\Big(\sum_{p\in I}\frac{G^g_{pb}-G^g_{ab}}{E_p-E_a} + \frac{G^{g-1}_{b|b}-G^{g-1}_{a|b}}{E_b-E_a}\Big)$$

Theorem

The (unrenormalised!) planar regular two-point function of the (E,ϕ^4)-QFT satisfies, and is determined by, the non-linear self-consistency equation

$$G^0_{ab} = \frac{1}{E_a+E_b} - \frac{\lambda_4}{E_a+E_b}\sum_{p\in I}\Big(G^0_{ab}G^0_{ap} - \frac{G^0_{pb}-G^0_{ab}}{E_p-E_a}\Big)$$

- There is a hierarchy such that all other equations are affine in the top degree function.
- Some of them need renormalisation of E, ϕ, and λ_n.

The renormalised noncommutative ϕ^4_4-theory

Example: ϕ^4 on $4D$-Moyal space w/ harmonic oscillator potential

$$S[\phi] = \int d^4x \Big(\frac{Z}{2} \phi \star \big(-\Delta + \Omega^2 \tilde{x}^2 + \mu_0^2\big)\phi + \frac{\lambda_0 Z^2}{4} \phi \star \phi \star \phi \star \phi \Big)(x)$$

Moyal product $\star$ defined by Θ and $\tilde{x} := 2\Theta^{-1} \cdot x$
parameters: $\mu_0^2, \lambda_0, Z \in \mathbb{R}_+$ and $\Omega \in [0,1]$

- renormalisable as formal power series in λ [Grosse-W.]
  means: well-defined perturbative quantum field theory
- β-function vanishes to all orders in λ for $\Omega = 1$
  [Disertori-Gurau-Magnen-Rivasseau]
  conjecture: model is integrable

Up to the sign of μ_0^2, this model arises from a spectral triple.

Rewriting as a matrix model

- Moyal algebra has a basis in which $\star$-product becomes matrix product for $I = \mathbb{N}^2$ (in d dimensions: $I = \mathbb{N}^{\frac{d}{2}}$)
- The kinetic term is of non-local form $\mathrm{Tr}(\mathcal{E}(\phi \otimes \phi))$. For $\Omega = 1$ it reduces to $\mathrm{tr}(E\phi\phi)$, with

$$E_{mn} = \delta_{m_1 n_1}\delta_{m_2 n_2} Z(|m| + \tfrac{1}{2}\mu_0^2), \qquad \begin{cases} m = (m_1, m_2) \in I \\ |m| := m_1 + m_2 \end{cases}$$

  only $|m| \mapsto E_m$ is injective, not $(m_1, m_2) \mapsto E_m$

- But G_{mn}, etc. only depend on $|m|$, and each sum over m_1, m_2 with $|m| = m_1 + m_2$ yields a measure factor $|m|+1$.
  $\Big[$In d dimensions, the measure is $\binom{|m| + \frac{d}{2} - 1}{\frac{d}{2} - 1}\Big]$

With cut-off N and $a \equiv |a|$, the unrenormalised equation becomes

$$G^0_{ab} = \frac{1}{Z(a+b+\mu_0^2)} - \frac{\lambda}{(a+b+\mu_0^2)} \sum_{p=0}^{N} (p+1)\Big(Z G^0_{ab} G^0_{ap} - \frac{G^0_{pb} - G^0_{ab}}{p-a}\Big)$$

- We pass to 1PI functions $G^0_{ab} =: (Z(a+b+\mu_0^2) - \Gamma_{ab})^{-1}$ and Taylor-expand $\Gamma_{ab} = Z\mu_0^2 - \mu^2 + (Z-1)(a+b) + \Gamma^{ren}_{ab}$
- normalisation conditions: $\Gamma^{ren}_{00} = 0$ and $(\partial\Gamma^{ren})_{00} = 0$ determine μ_0^2 and Z
- no renormalisation of $\lambda = \lambda_0$ because of $\beta_\lambda = 0$

After introduction of continuous variables $a = \mu^2 \frac{\alpha}{1-\alpha}$ and elimination of Z,μ_0, the limit $N \to \infty$ exists:

Theorem [Grosse-W., 2009]

The renormalised planar connected two-point function $G_{\alpha\beta}$ of self-dual n.c. ϕ^4_4-theory satisfies (and is determined by)

$$G_{\alpha\beta} = 1 - \lambda\Big(\frac{1-\alpha}{1-\alpha\beta}(\mathcal{M}_\beta - \mathcal{L}_\beta - \beta\mathcal{Y}) + \frac{1-\beta}{1-\alpha\beta}(\mathcal{M}_\alpha - \mathcal{L}_\alpha - \alpha\mathcal{Y})$$
$$+ \frac{(1-\alpha)(1-\beta)}{1-\alpha\beta}(G_{\alpha\beta} - 1)\mathcal{Y} - \frac{\alpha(1-\beta)}{1-\alpha\beta}(\mathcal{L}_\beta + \mathcal{N}_{\alpha\beta} - \mathcal{N}_{\alpha 0})$$
$$+ \frac{1-\beta}{1-\alpha\beta}(G_{\alpha\beta} - G_{\alpha 0})\frac{\mathcal{M}_\alpha - \mathcal{L}_\alpha + \alpha\mathcal{N}_{\alpha 0}}{G_{\alpha 0}}\Big)$$

with $\alpha, \beta \in [0,1)$ and

$$\mathcal{L}_\alpha := \int_0^1 d\rho\, \frac{G_{\alpha\rho} - G_{0\rho}}{1-\rho} \qquad \mathcal{M}_\alpha := \int_0^1 d\rho\, \frac{\alpha\, G_{\alpha\rho}}{1-\alpha\rho}$$

$$\mathcal{N}_{\alpha\beta} := \int_0^1 d\rho\, \frac{G_{\rho\beta} - G_{\alpha\beta}}{\rho - \alpha} \qquad \mathcal{Y} = \lim_{\alpha\to 0} \frac{\mathcal{M}_\alpha - \mathcal{L}_\alpha}{\alpha}$$

Theorem

The renormalised planar 1PI four-point function $\Gamma_{\alpha\beta\gamma\delta}$ of self-dual n.c. ϕ^4_4-theory satisfies (and is determined by)

$$\Gamma_{\alpha\beta\gamma\delta} = \lambda \cdot \frac{\Big(1 - \dfrac{(1-\alpha)(1-\gamma\delta)(G_{\alpha\delta}-G_{\gamma\delta})}{G_{\gamma\delta}(1-\delta)(\alpha-\gamma)} + \displaystyle\int_0^1 \rho\, d\rho \frac{(1-\beta)(1-\alpha\delta)G_{\beta\rho}G_{\delta\rho}}{(1-\beta\rho)(1-\delta\rho)} \frac{\Gamma_{\rho\beta\gamma\delta}-\Gamma_{\alpha\beta\gamma\delta}}{\rho-\alpha}\Big)}{G_{\alpha\delta}+\lambda\Big((\mathcal{M}_\beta-\mathcal{L}_\beta-\mathcal{Y})G_{\alpha\delta}+\displaystyle\int_0^1 d\rho \frac{G_{\alpha\delta}G_{\beta\rho}(1-\beta)}{(1-\delta\rho)(1-\beta\rho)} + \int_0^1 \rho\, d\rho \frac{(1-\beta)(1-\alpha\delta)G_{\beta\rho}}{(1-\beta\rho)(1-\delta\rho)} \frac{(G_{\rho\delta}-G_{\alpha\delta})}{(\rho-\alpha)}\Big)}$$

This equation is linear in the top degree function. There are standard methods to solve it (see later).

Progress in solving our equation

- These integral equations determine the $\phi^{\star 4}$-Euclidean QFT non-perturbatively. The main difficulty is the non-linearity of the equation for $G_{\alpha\beta}$.
- We were able to solve the non-linearity perturbatively up to order λ^3. There appear iterated integrals which evaluate to polylogarithms and ζ-functions.
- At order n there are $(\log(1-\alpha))^n$ contributions and a huge number of terms. A resummation will not work: We have to study the equation non-perturbatively.

Main step

Three coupled equations for $G_{\alpha 0}$, $G_{\alpha\beta} - G_{\alpha 0}$ and $\mathcal{Y}$.
The equation for $G_{\alpha\beta} - G_{\alpha 0}$ is a linear singular integral equation of Carleman type. It is solved by the finite Hilbert transform.

The finite Hilbert transform

$$\mathcal{H}[f](x) = \frac{1}{\pi}\mathcal{P}\int_{-1}^{1} dt\, \frac{f(t)}{t-x} \qquad \text{(Cauchy principal value)}$$

- $\mathcal{H} : L^p([-1,1]) \to L^p([-1,1])$ linear+continuous for $p>1$ (but not for $p=1$) [Riesz]
- $\mathcal{H} : C([-1,1]) \to C([-1,1])$ not continuous
- $\mathcal{H} : L^p \cap H^\gamma_{loc} \to L^p \cap H^\gamma_{loc}$ lin+cont's for $p>1$, $0<\gamma<1$ [Okada-Elliott]

$H^\gamma([a,b])$ – Hölder space, $|f(x)-f(y)| \leq C|x-y|^\gamma$
$H^\gamma_{loc}(a,b)$ – Hölder on every compact subinterval of (a,b)

The Carleman equation

Theorem [Carleman, Tricomi]

The singular linear integral equation

$$a(x)y(x) + \lambda\pi\mathcal{H}[y](x) = f(x)\,, \qquad x \in [-1,1]$$

is for $a(x)$ continuous + Hölder near ± 1 and $f \in L^q$ solved by

$$y(x) = \frac{a(x)f(x)}{a^2(x)+\lambda^2\pi^2} + \lambda\pi A(x)\mathcal{H}[A^* f](x) + C\frac{A(x)}{1-x} \in L^p$$

where $A(x) = \dfrac{\exp\big(\mathcal{H}[\arctan\frac{\lambda\pi}{a}](x)\big)}{\sqrt{a^2(x)+\lambda^2\pi^2}}$, $\quad A^*(x) = \dfrac{\exp\big(-\mathcal{H}[\arctan\frac{\lambda\pi}{a}](x)\big)}{\sqrt{a^2(x)+\lambda^2\pi^2}}$

- Our equations for $G_{\alpha\beta} - G_{\alpha 0}$ and $G_{\alpha\beta\gamma\delta}$ are of this type. $C = 0$ matches the perturbative solution.
- The solution expresses:
  – $G_{\alpha\beta}$ in terms of $G_{\alpha 0}$ and $G_{\beta 0}$
  – $G_{\alpha\beta\gamma\delta}$ in terms of $G_{\alpha\beta}$

Consequence: Solving the equation for the two-point function (in some $L^p \cap H^{\gamma}_{loc}$-space) amounts to solve the entire model!

The equation for $G_{\alpha 0}$

Main challenge is to solve

$$G_{\alpha 0} + \lambda \int_0^1 d\rho \, \frac{G_{\rho 0}}{1-\alpha\rho} = 1 + \lambda \Psi_\alpha(G_{\bullet 0}, G_{\bullet\bullet}[G_{\bullet 0}, \mathcal{Y}], \mathcal{Y}, \lambda)$$

where Ψ_α is non-linear but bounded for bounded $G_{\alpha 0}, \; G_{\alpha\beta}$

- The integral kernel $\frac{1}{1-\alpha\rho}$ is not square-integrable so that perturbative iteration does not converge.
- equation transformed into

$$y(x) + \lambda \int_1^\infty dt \, \frac{y(t)}{t+x} = f(x)$$

- solved by Mehler-Fock transform

Mehler-Fock transform

- for appropriate function f on $[1,\infty)$, define
  $$\mathcal{F}[f](\tau) := \int_1^\infty dx\; P_{-\frac{1}{2}+\mathrm{i}\tau}(x) f(x)\;, \qquad t \in [0,\infty)$$
  $P_{-\frac{1}{2}+\mathrm{i}\tau}(x)$ – Legendre spherical function of first kind
- $\mathcal{F}: L^p([1,\infty)) \to L^q(e^{-\alpha\tau};\mathbb{R}_+)$ bounded [Yakubovich-Saigo]
- $\mathcal{F}^{-1}[\tilde{f}](x) := \int_0^\infty d\tau\; \tau \tanh(\pi\tau) P_{-\frac{1}{2}+\mathrm{i}\tau}(x)\tilde{f}(\tau)$

Proposition [Lebedev]

$$y(x) + \lambda \int_1^\infty dt\, \frac{y(t)}{t+x} = f(x)$$ has the unique solution

$$y(x) = f(x) - \lambda\pi\, \mathcal{F}^{-1}\Big[\frac{\mathcal{F}[f](\bullet)}{\lambda\pi + \cosh(\pi\bullet)}\Big](x)\;, \qquad \lambda\pi \geq -1$$

Proposition

The integral equation $G_{\alpha 0} + \lambda \int_0^1 d\rho \frac{G_{\rho 0}}{1-\alpha\rho} = 1$ can be solved for $\lambda\pi \geq -1$ with unique solution

$$G_{\alpha 0} = \frac{\arcsin(\lambda\pi)}{\lambda\pi}\Big(1-\frac{\arcsin(\lambda\pi)}{\pi}\Big)(1-\alpha)^{\frac{\arcsin(\lambda\pi)}{\pi}}\,{}_2F_1\Big(\genfrac{}{}{0pt}{}{\frac{\arcsin(\lambda\pi)}{\pi},\,1+\frac{\arcsin(\lambda\pi)}{\pi}}{2}\Big|\alpha\Big)$$
$$= \frac{\pi^2+4(\operatorname{arcosh}(\lambda\pi))^2}{4\lambda\pi^2}\,{}_2F_1\Big(\genfrac{}{}{0pt}{}{\frac{1}{2}-\mathrm{i}\frac{\operatorname{arcosh}(\lambda\pi)}{\pi},\,\frac{1}{2}+\mathrm{i}\frac{\operatorname{arcosh}(\lambda\pi)}{\pi}}{2}\Big|-\frac{\alpha}{1-\alpha}\Big)$$
$$= 1 - \frac{\log(1-\alpha)}{\alpha} + \mathcal{O}(\lambda^2)$$

- $-1\leq\lambda\pi<0$: $G_{\alpha 0} \in L^{\frac{\pi}{\arcsin|\lambda\pi|}-\epsilon}$ but unbounded at $\alpha \to 1$, $\mathcal{N}_{\alpha 0} = \int_0^1 d\rho\, \frac{G_{\rho 0}-G_{\alpha 0}}{\rho-\alpha} \in L^{\frac{\pi}{\arcsin|\lambda\pi|}-\epsilon}$ unbounded
- $0<\lambda\pi<1$: $G_{\alpha 0} \in H^{\frac{\arcsin(\lambda\pi)}{\pi}}([0,1])$, is positive, vanishes for $\alpha\to 1$, $\mathcal{N}_{\alpha 0} \in H^{\frac{\arcsin(\lambda\pi)}{\pi}}([0,1])$
- $1\leq\lambda\pi$: $G_{\alpha 0} \in H^{\frac{1}{2}-\epsilon}([0,1])$, oscillating, vanishes for $\alpha\to 1$

Conclusion

The cases $\lambda > 0$ and $\lambda < 0$ differ drastically:

- For $\lambda > 0$ the solution is Hölder-continuous on $[0, 1]$, i.e. we are in Banach space setting.
- For $\lambda < 0$ solution unbounded but in $(L^p \cap H^\gamma_{loc})([0, 1))$ for some $p{>}1$ and $\gamma{>}0$, i.e. we are in Fréchet space setting.

Main problem

Prove that $\mathcal{F}^{-1} \circ \frac{\mathcal{F}}{\lambda\pi+\coth(\pi\bullet)}$ (Mehler-Fock) is continuous endomorphism of $H^\gamma([0, 1])$ resp. $(L^p \cap H^\gamma_{loc})([0, 1))$.

- If so, and if $G_{\alpha 0} \mapsto G_{\alpha\beta}$ (Carleman) is continuous, then existence of Hölder-continuous solution $G_{\alpha\beta}$ (and $G_{\alpha\beta\gamma\delta}$) is for $0{\leq}\lambda{\leq}\lambda_0$ achieved by Banach fixed-point theorem!
- No substitute in the Fréchet setting $\lambda < 0$! The seminorms grow to ∞; iteration is unlikely to converge for some $\lambda < 0$!