Solvable quantum field theory models

Raimar Wulkenhaar

Mathematisches Institut, Westfälische Wilhelms-Universität Münster

Introduction

- Quantum field theory (QFT) is the theory that describes Nature at very high energy density.
- One famous such experiment measures the magnetic moment g of the electron: $\frac{g_{experiment}}{2} = 1.001\,159\,652\,180\,7$
- QFT predicts that number in terms of the electron charge e measured to $e^{-2} = 137.035\,999\,084$:

$$\begin{aligned}\frac{g_{QFT}}{2} &= 1 + \frac{1}{2}\frac{e^2}{\pi} + \Big\{\frac{197}{144} + \Big(\frac{1}{2} - 3\log 2\Big)\zeta(2) + \frac{3}{4}\zeta(3)\Big\}\Big(\frac{e^2}{\pi}\Big)^2 \\ &+ \Big\{\frac{28259}{5184} + \Big(\frac{17101}{235} - \frac{596}{2}\cdot\log 2\Big)\zeta(2) + \frac{139}{18}\zeta(3) + \frac{100}{3}\mathrm{Li}_4\Big(\frac{1}{2}\Big) \\ &\quad + \frac{25}{18}\Big(\log^4 2 - \pi^2\log^2 2\Big) - \frac{239\zeta(4) - 166\zeta(2)\cdot\zeta(3) + 215\zeta(5)}{24}\Big\}\Big(\frac{e^2}{\pi}\Big)^3 \\ &+ \Big\{\dots\Big\}\Big(\frac{e^2}{\pi}\Big)^4 \\ &= 1.001\,159\,652\,153\,5\end{aligned}$$

- The discipline that achieves this spectacular agreement with experiment is called perturbative quantum field theory.
- It starts from a classical field theory model and computes quantum corrections as series in quantities like $\frac{e^2}{\pi}$.
- Contributions to $(\frac{e^2}{\pi})^n$ are sums of integrals encoded by n-loop Feynman graphs.
- The problem is: The radius of convergence of the perturbation series is zero!

The success of perturbative QFT lacks a mathematical understanding.

Contents

→ In the talk I give an overview of mathematically rigorous formulations of QFT and their mutual relations. These have inspired many developments in mathematics.

→ Shortness of time does not allow to give any details. Some exactly solvable models will briefly be mentioned.

→ All this is far away from experiment.

Axiomatic QFT

A quantum field Φ is an operator-valued distribution:
For $f \in \mathcal{S}$ a test function on Minkowski space $\mathbb{R}^{1,d-1} \ni (t,x)$,
$\Phi(f)$ is an unbounded operator in a Hilbert space $\mathcal{H}$

Wightman axioms (1950s)

1. Covariance: There is a representation of the Poincaré group $\mathcal{P}_+^\uparrow \ni p$ by unitaries in $\mathcal{H}$ that implements the transformation $\Phi(f) \mapsto \Phi(p(f))$.
2. Vacuum: There is a unique Poincaré-invariant unit vector $\Omega \in \mathcal{H}$ such that $\{\text{Polynomials}(\Phi)\Omega\}$ is dense in $\mathcal{H}$.
3. Locality: For f, g causally independent, $[\Phi(f), \Phi(g)] = 0$.
4. Spectrum condition: The joint spectrum of the generators of translation subgroup of $\mathcal{P}_+^\uparrow$ lies in the forward lightcone.

Wightman functions

Vacuum expectation values of field operators

$$\mathcal{S}^N \ni (f_1, \dots, f_N) \mapsto W(f_1, \dots, f_N) := \langle \Omega, \Phi(f_1) \cdots \Phi(f_N) \Omega \rangle$$

- also called N-point functions or correlation functions
- have covariance, locality, positivity, spectrum and cluster properties inherited from Wightman axioms
- Wightman's reconstruction theorem: The $\{W(f_1, \dots, f_N)\}$ with these properties allow to reconstruct $\mathcal{H}$ and Φ.
- Scattering amplitudes are obtained from Wightman functions (Lehmann-Symanzik-Zimmermann formula).
- The $W(\delta_{(t_1,x_1)}, \dots, \delta_{(t_N,x_N)})$ are boundary values of holomorphic functions:
  They have an analytical continuation in time t_j and become at $t_j = \mathrm{i}\tau_j$ Schwinger functions in an Euclidean QFT.

Algebraic (or local) QFT

Haag-Kastler axioms (1964)

Net of abstract C^*-algebras $\mathfrak{A}(\mathcal{O})$ assigned to bounded, open, causally complete regions $\mathcal{O}$ in Minkowski space, satisfying:

1. Isotony: $\mathcal{O}_1 \subset \mathcal{O}_2 \quad \Rightarrow \quad \mathfrak{A}(\mathcal{O}_1) \subset \mathfrak{A}(\mathcal{O}_2)$
2. Covariance: Poincaré group is realised as group of automorphisms of the net.
3. Locality: For $\mathcal{O}_1, \mathcal{O}_2$ causally independent, $[\mathfrak{A}(\mathcal{O}_1), \mathfrak{A}(\mathcal{O}_2)] = 0$
4. Time slice axiom: If $\mathcal{O}$ is a neighbourhood of a Cauchy hypersurface, then $\mathfrak{A}(\mathcal{O}) = \bigcup_{\mathcal{O}_i} \mathfrak{A}(\mathcal{O}_i)$

far more flexible than Wightman theory:
– allows to use deep results of theory of operator algebras,
– applies to curved space-time.

QFT models – quantisation of classical field theories

A classical field theory is defined by an action functional

$$S[\phi] = \int_{\mathbb{R}^{1,d-1}} d(t,x)\, \mathcal{L}\Big(\phi(t,x), \partial\phi(t,x), t, x\Big)$$

for a set of classical fields ϕ. This entails:

- equations of motion = Euler-Langrange equations

$$\frac{d}{dt}\frac{\delta\mathcal{L}}{\delta(\partial_t\phi(t,x))} + \sum_{i=1}^{d-1}\frac{d}{dx_i}\frac{\delta\mathcal{L}}{\delta(\partial_i\phi(t,x))} - \frac{\delta\mathcal{L}}{\delta\phi(t,x)} = 0$$

- Poisson structure on space of solutions
  $\{\phi(t,x), \pi(t,y)\}_{PB} = \delta(x-y)$ for $\pi(t,x) = \frac{\delta\mathcal{L}}{\delta(\partial_t\phi(t,x))}$
- conserved currents from symmetries of $\mathcal{L}$

Quantisation $\phi \mapsto \Phi$

try to etablish these properties for operators $\Phi(f)$ on $\mathcal{H}$:

- equations of motions for $\Phi(f)$ in distributional sense
- Canonical commutation relations $[\Phi(t,x), \Pi(t,y)] = \mathrm{i}\delta(x-y)$

Standard example: free scalar field

- Lagrangian $\mathcal{L} = \frac{1}{2}(\partial_t \phi)^2 - \frac{1}{2}(\mathrm{grad}\,\phi)^2 - \frac{m^2}{2}\phi^2$
- equations of motion satisfied for $(\omega_p^2 = p^2 + m^2)$

$$\Phi(t,x) = \int_{\mathbb{R}^{d-1}} \frac{dp}{(2\pi)^{d-1}\sqrt{2\omega_p}} \Big(a_p^- \, \mathrm{e}^{-\mathrm{i}(\omega_p t - \langle p,x\rangle)} + a_p^+ \, \mathrm{e}^{\mathrm{i}(\omega_p t - \langle p,x\rangle)} \Big)$$

- commutation relations satisfied for

$$[a_p^-, a_{p'}^-] = 0\,, \quad [a_p^+, a_{p'}^+] = 0\,, \quad [a_p^-, a_{p'}^+] = (2\pi)^3 \delta(p - p')$$

- Hilbert space is Fock space generated from vacuum Ω

$$\mathcal{H} = \overline{\mathrm{span}\big\{a_{p_1}^+ \cdots a_{p_N}^+ \Omega \;:\; N \in \mathbb{N}\,,\; p_i \in \mathbb{R}^{n-1}\big\}}^{\|\;\|}\,, \qquad a_p^- \Omega := 0$$

- N-point functions factorise into products of 2-point functions (Wick's theorem)

Interacting (i.e. non-linear) models cannot have a Fock space realisation (Haag's theorem).

The Thirring model (1958)

For 2D spinor $\psi(t,x)=\binom{\psi_+(t,x)}{\psi_-(t,x)}$, with $t,x\in\mathbb{R}$, and $\sigma_3=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$, let

$$\mathcal{L}=\mathrm{i}\psi^*\partial_t\psi+\mathrm{i}\psi^*\sigma_3\partial_x\psi+\frac{\lambda}{2}\big((\psi^*\psi)^2-(\psi^*\sigma_3\psi)^2\big)$$

- regularised currents [Johnson, 1961]

$$j^\epsilon(t,x):=\begin{pmatrix}\psi^*(t+\epsilon_1,x+\epsilon_2)\psi(t,x)\\ \psi^*(t+\epsilon_1,x+\epsilon_2)\sigma_3\psi(t,x)\end{pmatrix},\quad \tilde{j}^{\tilde{\epsilon}}(t,x):=\begin{pmatrix}\psi^*(t+\tilde{\epsilon}_1,x+\tilde{\epsilon}_2)\sigma_3\psi(t,x)\\ \psi^*(t+\tilde{\epsilon}_1,x+\tilde{\epsilon}_2)\psi(t,x)\end{pmatrix}$$

- postulate commutation relations

$$[\psi(t,x),j_1^\epsilon(t,y)]=a\delta(x-y)\psi(t,x),\quad [\psi(t,x),\tilde{j}_1^{\tilde{\epsilon}}(t,y)]=\tilde{a}\sigma_3\delta(x-y)\psi(t,x)$$

- quantum equations of motions imply $a=\frac{1}{1-\frac{\lambda}{2\pi}}$, $\tilde{a}=\frac{1}{1+\frac{\lambda}{2\pi}}$
- solution of 2-point function ($t>0$, appropriate $\lim_{\epsilon,\tilde{\epsilon}\to 0}$)

$$\langle\Omega,\psi(t,x),\psi(0,0)\Omega\rangle=\exp\left(-\frac{\mathrm{i}\frac{\lambda^2}{\pi^2}}{1-\frac{\lambda^2}{4\pi^2}}D_+(t,x)\right)\langle\Omega,\Phi(t,x),\Phi(0,0)\Omega\rangle_{\text{free scalar}}$$

- general case by [Klaiber, 1967], [Hagen, 1967]

The Schwinger model (1962), or QED_2

$$\mathcal{L} = \mathrm{i}\psi^*(\partial_t - \mathrm{i}eA_1)\psi + \mathrm{i}\psi^*\sigma_3(\partial_x - \mathrm{i}eA_2)\psi + \frac{1}{2}(\partial_t A_2 - \partial_x A_1)^2$$

- gauge invariance expresses ψ, A in terms of free fields:

$$\psi = \mathrm{e}^{\mathrm{i}\sqrt{\pi}\sigma_3(\eta^+ + \Sigma^+)}\chi \mathrm{e}^{\mathrm{i}\sqrt{\pi}\sigma_3(\eta^- + \Sigma^-)}$$

$$A_1 = -\frac{\sqrt{\pi}}{e}\big(\partial_t\eta - \partial_x\Sigma\big)\;, \qquad A_2 = -\frac{\sqrt{\pi}}{e}\big(\partial_x\eta + \partial_t\Sigma\big)$$

  where

  χ – free spinor of mass zero

  η – free scalar of mass zero

  Σ – free scalar of mass $\frac{e}{\sqrt{\pi}}$ (anomalous mass generation)

  [$F^{\pm}(t,x)$ refers to Fock space operators]

- bosonisation: chiral components of operator product $\psi^*(x)\psi(x)$ related to scalar field (only possible in 2D)

- model shows confinement (Coulomb potential $\propto$ distance)

Statistical Physics

consider spin maps $\sigma : \mathbb{Z}^d \to M$ (discrete space)

- assign probability $p[\sigma] = \frac{1}{\mathcal{Z}} \exp(-\beta H[\sigma])$ to Hamiltonian $H[\sigma]$, where $\mathcal{Z} := \sum_\sigma \exp(-\beta H[\sigma])$ is partition function
- gives rise to expectation value of observables $\mathcal{O}[\sigma]$ by $\langle\mathcal{O}\rangle = \sum_\sigma p[\sigma]\mathcal{O}[\sigma] = \frac{1}{\mathcal{Z}} \sum_\sigma \mathcal{O}[\sigma] \exp(-\beta H[\sigma])$
- often critical behaviour at certain inverse temperature β_c, i.e. the correlation length diverges
- power-law behaviour of physical quantities near the critical point ($\to$ critical exponents)
- relations between and universality of the critical exponents expained by renormalisation group [Wilson, 1971]

Solvable statistical physics models

- Ising model (1925), spins $\sigma : \mathbb{Z}^d \to \pm 1$
  2D critical model solved by [Onsager, 1944]
- Potts model (1952), spins $\sigma : \mathbb{Z}^d \to \{e^{\frac{2\pi i k}{q}}\}$,
  in 2D solvable for $q = 3$ and $q = \infty$ (XY-model)
- 6-vertex model (or ice-model), solved by [Lieb, 1967]
- 8-vertex model, solved by [Baxter, 1971]
- Hard hexagon model, solved by [Baxter, 1980]
- many 1D-models (1D Potts model, Heisenberg model, Hubbard model, Luttinger model, Toda lattice, . . .)

Methods: transfer matrix, Bogoliubov transformation, quantum inverse scattering method, renormalisation group, Yang-Baxter equation, Rogers-Ramanujan identity

Euclidean QFT

make Feynman's path integral rigorous (Feynman-Kac formula) [Symanzik, 1964]

- starting point is action $S[\phi] = \int_{\mathbb{R}^d} dx\,\mathcal{L}[\phi]$ of classical field theory on Euclidean space
- assign measure $p[\phi]\mathcal{D}[\phi] = \frac{\exp(-S[\phi])}{\mathcal{Z}}\mathcal{D}[\phi]$, where $\mathcal{Z} := \int \mathcal{D}[\phi]\ \exp(-S[\phi])$ is partition function
- main difficulty is to make sense of the measure $p[\phi]\mathcal{D}[\phi]$ (renormalisation)
- gives candidates for Schwinger functions $S(x_1,\dots,x_N) = \frac{1}{\mathcal{Z}} \int \mathcal{D}[\phi]\ \phi(x_1)\cdots\phi(x_N)\exp(-S[\phi])$

Osterwalder-Schrader reconstruction theorem

Distinguish a vector $v \in \mathbb{R}^d$ (which becomes time) and let $x \mapsto \bar{x}$ be the reflection on the hyperplane $v^\perp$.

Theorem [Osterwalder-Schrader, 1973–1975]

Assume for $S(x_1, \dots, x_N)$:

1. symmetry and analyticity
2. Euclidean covariance
3. reflection positivity: for each $N \leq N_0$ test function $f_N \in \mathcal{S}^N$,
$$\sum_{M,N} \int dxdy\; S(x_1, \dots, x_N, y_1, \dots y_M)\overline{f_N(\bar{x}_1, \dots, \bar{x}_N)}f_M(y_1, \dots, y_M) \geq 0$$
4. cluster property

Then the analytical continuation of $S(x_1, \dots, x_N)$ provides Wightman functions of a true relativistic QFT.

This is the main road to construct non-trivial QFT models!

Constructed quantum field theories

- 2D ϕ^4-model [Glimm-Jaffe, 1968–1972]
- $P(\phi)_2$, i.e. 2D scalar field with polynomial interaction [Glimm-Jaffe-Spencer, 1974], [Simon, 1974]
- 3D ϕ^4-model [Glimm-Jaffe, 1973], [Feldman-Osterwalder, 1976]
- Gross-Neveu model (1974), constructed in 2D by [Gawędzki-Kupiainen, 1985] and [Feldman-Magnen-Rivasseau-Séneor, 1986]
- non-example: ϕ^4 in $D = 4+\epsilon$ is trivial [Aizenman, 1981], [Fröhlich, 1982]

Methods: random walks, cluster expansion, correlation inequalities, Borel resummation

Models in 2D conformal field theory

- Infinitesimal 2D conformal transformations form the infinite-dimensional Witt algebra.
- Most conformal quantum field theories have a conformal anomaly that leads to a central extension of the Witt algebra.
- The resulting Virasoro algebra has generators L_n, $n \in \mathbb{Z}$,

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3-m)\delta_{m,-n}$$

- Highest-weight represesentations of the Virasoro algebra define a family of solvable "Minimal Models" at central charge

$$c = 1 - \frac{6}{m(m+1)}, \quad m = 3, 4, 5, \ldots$$

  [Belavin-Polyakov-Zamolodchikov, Friedan-Qiu-Schenker, 1984]
  $m = 3$ is critical Ising model, $m = 5$ is 3-state Potts model,...
- Solvable model of different type: Wess-Zumino-Witten model. The solution is realised by affine Kac-Moody algebras.

Matrix models

1. 2D quantum gravity is the enumeration of random triangulations of surfaces.
   - Its asymptotic behaviour is captured by the matrix model partition function

$$\mathcal{Z} = \int dM \ \exp\Big(-\mathcal{N}\sum_n t_n \operatorname{tr}(M^n)\Big)\ , \qquad M = M^* \in M_{\mathcal{N}}(\mathbb{C})$$

   - For $\mathcal{N} \to \infty$, this series in (t_n) is evaluated in terms of the τ-function for the Korteweg-de Vries (KdV) hierarchy.
2. 2D topological quantum gravity has correlation functions which are intersection numbers of complex curves.
   - They can be arranged into a generating functional with series parameters (t_n).

[Witten, 1990] conjectured that both (t_n)-series are the same.

The Kontsevich model

- [Kontsevich, 1992] computed the intersection numbers in terms of weighted sums over ribbon graphs.
- He proved these graphs to be generated from the Airy function matrix model (Kontsevich model)

$$\mathcal{Z}[E] = \frac{\displaystyle\int dM\ \exp\big(-\tfrac{1}{2}\mathrm{tr}(EM^2) + \tfrac{\mathrm{i}}{6}\mathrm{tr}(M^3)\big)}{\displaystyle\int dM\ \exp\big(-\tfrac{1}{2}\mathrm{tr}(EM^2)\big)}\,, \qquad M{=}M^*{\in}M_{\mathcal{N}}(\mathbb{C})$$

for $E = E^* > 0$ and $t_n = (2n{-}1)!!\mathrm{tr}(E^{-(2n-1)})$.

- Limit $\mathcal{N} \to \infty$ of $\mathcal{Z}[E]$ gives the KdV evolution equation, thus proving Witten's conjecture.

A matrix model inspired by noncommutative QFT

- The simplest QFT on a 4D noncommutative manifold can be written as a matrix model

$$\mathcal{Z}[E,J,\lambda] = \frac{\displaystyle\int dM\ \exp\Big(-\mathrm{tr}(EM^2)+\mathrm{tr}(JM)-\tfrac{\lambda}{4}\mathrm{tr}(M^4)\Big)}{\displaystyle\int dM\ \exp\Big(-\mathrm{tr}(EM^2)-\tfrac{\lambda}{4}\mathrm{tr}(M^4)\Big)}\,,$$

  where $E = E^* \in M_{\mathcal{N}}(\mathbb{C})$ is the 4D Laplacian, $\lambda \geq 0$ and $J \in M_{\mathcal{N}}(\mathbb{C})$ generates correlation functions.
- In joint work with Harald Grosse [arXiv:1205.0465v2] we achieved the exact solution of $\mathcal{Z}[E,J,\lambda]$ for $\mathcal{N}\to\infty$ and after renormalisation of E,λ.
- This defines a QFT toy model in four dimensions, which is non-trivial with coupling constant $0 \leq \lambda \leq \frac{1}{\pi}$.

We have no idea what mathematical structure made this possible.

Topological expansion

- Connected Feynman graphs in matrix models are ribbon graphs.
- Viewed as simplicial complexes, they encode the topology (B,g) of a genus-g Riemann surface with B boundary components (or punctures, marked points, holes, faces).
- The k^{th} boundary component carries a cycle $J^{N_k}_{p_1\dots p_{N_k}} := \prod_{j=1}^{N_k} J_{p_j p_{j+1}}$ of N_k external sources, $N_k+1\equiv 1$.
- We expand $\log \mathcal{Z}[J] = \sum \frac{1}{S} G_{|p_1\dots p_{N_1}|\dots|q_1\dots q_{N_B}|} J^{N_1}_{p_1\dots p_{N_1}} \cdots J^{N_B}_{q_1\dots q_{N_B}}$ according to the cycle structure.

Ward identity

$E = (E_a\delta_{ab})$ – unbounded, positive, selfadjoint, compact resolvent

Unitary transformation $M \mapsto UMU^*$ leads to Ward identity

$$0 = \sum_{n\in I}\Big((E_a - E_p)\frac{\partial^2 \mathcal{Z}}{\partial J_{an}\partial J_{np}} + J_{pn}\frac{\partial \mathcal{Z}}{\partial J_{an}} - J_{na}\frac{\partial \mathcal{Z}}{\partial J_{np}}\Big)$$

Theorem

$$\sum_{n\in I}\frac{\partial^2 \mathcal{Z}[J]}{\partial J_{an}\partial J_{np}} = \delta_{ap}\Big\{\sum_{(K)}\frac{J_{P_1}\cdots J_{P_K}}{S_K}\Big(\sum_{n\in I} G_{|an|P_1|\dots|P_K|} + G_{|a|a|P_1|\dots|P_K|}$$
$$+\sum_{r\geq 1}\sum_{q_1,\dots,q_r\in I} G_{|q_1aq_1\dots q_r|P_1|\dots|P_K|}J^r_{q_1\dots q_r}\Big)$$
$$+\sum_{(K),(K')}\frac{J_{P_1}\cdots J_{P_K}J_{Q_1}\cdots J_{Q_{K'}}}{S_K S_{K'}}G_{|a|P_1|\dots|P_K|}G_{|a|Q_1|\dots|Q_{K'}|}\Big\}\mathcal{Z}[J]$$
$$-\frac{1}{E_a - E_p}\sum_{n\in I}\Big(J_{pn}\frac{\partial \mathcal{Z}[J]}{\partial J_{an}} - J_{na}\frac{\partial \mathcal{Z}[J]}{\partial J_{np}}\Big)$$

Schwinger-Dyson equations (for $V[M] = \frac{\lambda_4}{4}\mathrm{tr}(M^4)$)

The previous formula lets the usually infinite tower of Schwinger-Dyson equations collapse:

after genus expansion $G_{\dots} = \sum_{g=0}^{\infty} G^{(g)}_{\dots}$:

1. A closed non-linear equation for $G^{(0)}_{ab}$ (planar+regular):

$$G^{(0)}_{|ab|} = \frac{1}{E_a + E_b} - \frac{\lambda_4}{E_a + E_b} \sum_{p \in I} \Big(G^{(0)}_{|ab|} G^{(0)}_{|ap|} - \frac{G^{(0)}_{|pb|} - G^{(0)}_{|ab|}}{E_p - E_a} \Big)$$

2. For every other $G^{(g)}_{a_1 \dots a_N}$ an equation which only depends on

- $G^{(g)}_{a_1 \dots a_k}$ for $k \leq N$,
- $G^{(h)}_{a_1 \dots a_k}$ with $h < g$ and $k \leq N+2$;

this dependence is linear in the top degree (N, g)

Some $G_{\dots}$ need renormalisation of E, M, and λ_4!

Integral equations

We choose $E_a = \mathcal{N}^2 Z(a\frac{\Lambda^2}{\mathcal{N}} + \mu_{bare}^2)$, for $a \in \mathbb{N}$ and constants Λ^2, Z, μ_{bare}, and with multiplicity of the 4D Laplacian.

- double scaling limit $\mathcal{N} \to \infty$ makes matrix indices continuous: $\frac{a\Lambda^2}{\mathcal{N}} \in [0, \Lambda^2]$.
- continuum limit $\Lambda \to \infty$ requires suitable $Z[\Lambda], \mu_{bare}^2[\Lambda]$.

Integral equation for Hölder-continuous $G_{ab} = \lim_{\mathcal{N}\to\infty} \mathcal{N}^2 \frac{\partial^2 \log \mathcal{Z}}{\partial J_{ab}\partial J_{ba}}$

$$\Big(\frac{b}{a} + \frac{1+\mathcal{Y}+\lambda\pi a\mathcal{H}_a\big[G_{\bullet 0}\big]}{aG_{a0}}\Big)D_{ab} - \lambda\pi\mathcal{H}_a\big[D_{\bullet b}\big] = -G_{a0}$$

where

- $D_{ab} := \frac{a}{b}(G_{ab} - G_{a0})$
- $\mathcal{Y}$ determined by $\frac{d}{da}G_{a0}\big|_{a=0} = -1$
- Hilbert transform $\quad \mathcal{H}_a[f(\bullet)] := \frac{1}{\pi}\lim_{\epsilon\to 0}\Big(\int_0^{a-\epsilon} + \int_{a+\epsilon}^{\infty}\Big)\frac{f(q)\,dq}{q-a}$

The Carleman equation

- This is a singular integral equation of Carleman type.
- Its solution theory [Carleman 1922, Tricomi 1957] gives the entire two-point function G_{ab} in terms of its boundary G_{a0}:

Theorem

$$G_{ab} = (1+\mathcal{Y})\frac{\sin(\theta_b(a))}{|\lambda|\pi a} e^{\mathcal{H}_a[\theta_b(\bullet)]-\mathcal{H}_0[\theta_0(\bullet)]}$$

$$\frac{\mathcal{Y}}{1+\mathcal{Y}} = \lambda\int_0^{\Lambda^2} dp\, \frac{\sin^2(\theta_0(p))}{(\lambda\pi p)^2}$$

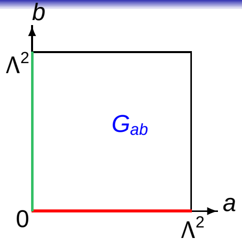
$$\theta_b(a) := \underset{[0,\pi]}{\arctan}\left(\frac{\lambda\pi a}{b+\frac{1+\mathcal{Y}+\lambda\pi a\mathcal{H}_a[G_{\bullet 0}]}{G_{a0}}}\right)$$

Consequence: $G_{ab} \geq 0$!

The self-consistency equation

Given boundary value G_{a0},
Carleman computes G_{ab},
in particular G_{0b}

symmetry forces $G_{b0} = G_{0b}$



Master equation

The theory is completely determined by the solution of the fixed point equation (with $\mathcal{Y}$ determined by $\frac{dG_{b0}}{db}\big|_{b=0} = -1$)

$$G_{b0} = \frac{1+\mathcal{Y}}{1+b+\mathcal{Y}} \exp\left(-\lambda \int_0^b dt \int_0^\infty \frac{dp}{(\lambda\pi p)^2 + \left(t + \frac{1+\mathcal{Y}+\lambda\pi p \mathcal{H}_p[G_{\bullet 0}]}{G_{p0}}\right)^2}\right)$$

Problem 1 (Analysis): Rigorously prove existence and uniqueness of solution G_{b0} in Hölder space!

Correlation functions for $B = 1$ punctures

Schwinger-Dyson equation for $G_{ab_1\dots b_{N-1}}$

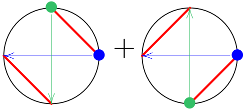
$$\Big(\frac{b_1}{a}+\frac{1+\mathcal{Y}+\lambda\pi a\mathcal{H}_a[G_{\bullet 0}]}{aG_{a0}}\Big)\cdot(aG_{ab_1\dots b_{N-1}})-\lambda\pi\mathcal{H}_a\big[\bullet G_{\bullet b_1\dots b_{N-1}}\big]$$
$$=\lambda\sum_{l=1}^{\frac{N-2}{2}} G_{b_1\dots b_{2l}}\frac{G_{b_{2l}b_{2l+1}\dots b_{N-1}}-G_{ab_{2l+1}\dots b_{N-1}}}{b_{2l}-a}$$

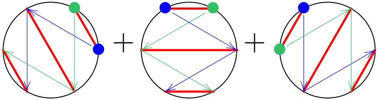
- This is again a Carleman equation, with identical linear part as for ▸ two-point function.
- Reality $\mathcal{Z}=\overline{\mathcal{Z}}$ implies invariance under orientation reversal $G_{ab_1\dots b_{N-1}}=G_{b_{N-1}\dots b_1 a}=G_{ab_{N-1}\dots b_1}$

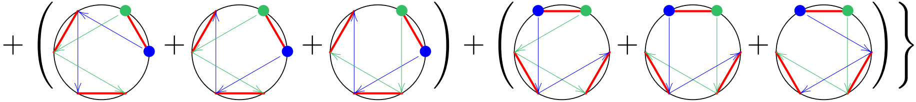
Theorem (algebraic recursion formula for N-point function)

$$G_{b_0b_1\dots b_{N-1}}=(-\lambda)\sum_{l=1}^{\frac{N-2}{2}}\frac{G_{b_0b_1\dots b_{2l-1}}G_{b_{2l}b_{2l+1}\dots b_{N-1}}-G_{b_{2l}b_1\dots b_{2l-1}}G_{b_0b_{2l+1}\dots b_{N-1}}}{(b_0-b_{2l})(b_1-b_{N-1})}$$

Graphical realisation

$$G_{b_0b_1b_2b_3} = (-\lambda)\frac{G_{b_0b_1}G_{b_2b_3} - G_{b_0b_3}G_{b_2b_1}}{(b_0-b_2)(b_1-b_3)} = -\lambda\Big\{$$  $$\Big\}$$

$$G_{b_0\dots b_5} = \lambda^2\Big\{$$ 

 $$\Big\}$$

 $= G_{b_ib_j}$ leads to non-crossing chord diagrams; these are counted by the Catalan number $C_{\frac{N}{2}} = \frac{N!}{(\frac{N}{2}+1)!\frac{N}{2}!}$

 $= \frac{1}{b_i - b_j}$ leads to rooted trees connecting the even or odd vertices, intersecting the chords only at vertices

Problem 2 (Combinatorics): Which trees arise for a given chord diagram?

More open problems

Problem 3 (Analysis): The homogeneous Carleman equation has non-trivial solutions not taken into account. They arise from a winding number and seem to be relevant for $\lambda > \frac{1}{\pi}$.

Problem 4 (Physics): So far this is a Euclidean quantum field theory (no time). Is there an analytic continuation to a true relativistic quantum field theory?

Problem 5 (Integrability): Is there a known integrable model which explains these results, in analogy to the KdV equation for the Kontsevich model?

Problem 6 (Algebraic geometry): What topic in algebraic geometry does the M^4-matrix model compute, in analogy to the intersection numbers for the Kontsevich model?