QFT on NCG

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Introduction

Constructive renormalisation of quantum field theories was very successful in low dimensions but a complete failure in in four dimensions

- Yang-Mills theory is believed to exist due to asymptotic freedom, but too complicated
- treatable models such as QED or ϕ_4^4 do not exist due to the Landau ghost problem (triviality)

Observation

If we put the ϕ_4^4-model on a (particular) noncommutative Euclidean space, the β-fuction is modified so that the model should exist non-perturbatively. There is a realistic chance to prove this!

Similar as commutative QED $\Rightarrow$ noncommutative Yang-Mills:

Supersymmetric quantum mechanics

Let X be a d-dimensional smooth manifold, T^*X trivial

- $a_\mu = e^{-\omega h}\partial_\mu e^{\omega h} = \partial_\mu + W_\mu$, $\quad a_\mu^\dagger = -e^{\omega h}\partial_\mu e^{-\omega h} = -\partial_\mu + W_\mu$
  $h \in C^\infty(X)$ Morse function, $W_\mu = \omega\partial_\mu h$
- commutation relations:
  $[a_\mu, a_\nu] = [a_\mu^\dagger, a_\nu^\dagger] = 0$, $\qquad [a_\mu, a_\nu^\dagger] = 2\omega\partial_\mu\partial_\nu h$
- d fermionic ladder operators:
  $\{b_\mu, b_\nu\} = 0$, $\qquad \{b_\mu^\dagger, b_\nu^\dagger\} = 0$, $\qquad \{b_\mu, b_\nu^\dagger\} = \delta_{\mu\nu}$
- supercharges:
  $\mathfrak{Q} := \sum_{\mu=1}^d a_\mu \otimes b_\mu^\dagger$, $\qquad \mathfrak{Q}^\dagger := \sum_{\mu=1}^d a_\mu^\dagger \otimes b_\mu$
- supersymmetry algebra:
  $\{\mathfrak{Q}, \mathfrak{Q}^\dagger\} = \mathfrak{H} = \big(-\partial_\mu\partial^\mu + \omega^2(\partial_\mu h)(\partial^\mu h)\big)\otimes 1 + \omega(\partial^\mu\partial^\nu h)\otimes[b_\mu^\dagger, b_\nu]$
  $\{\mathfrak{Q}, \mathfrak{Q}\} = \{\mathfrak{Q}^\dagger, \mathfrak{Q}^\dagger\} = 0$, $\quad [\mathfrak{Q}, \mathfrak{H}] = [\mathfrak{Q}^\dagger, \mathfrak{H}] = 0$

cohomology of $\mathfrak{Q}$ related to Morse theory for h [Witten, 1982]

Harmonic oscillator spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D}_i)$

Morse function $h = \frac{1}{2}\|x\|^2$

implies constant $[a_\mu, a_\nu^\dagger] = 2\omega\delta_{\mu\nu}$

Hilbert space $\mathcal{H} = \ell^2(\mathbb{N}^d) \otimes \bigwedge(\mathbb{C}^d)$: declare ONB
$\Big\{(a_1^\dagger)^{n_1} \dots (a_d^\dagger)^{n_d} \otimes (b_1^\dagger)^{s_1} \dots (b_d^\dagger)^{s_d}|0\rangle \;:\; n_\mu \in \mathbb{N}\,, s_\mu \in \{0,1\}\Big\}$

TWO Dirac operators $\mathcal{D}_1 = \mathfrak{Q} + \mathfrak{Q}^\dagger\,, \qquad \mathcal{D}_2 = \mathrm{i}\mathfrak{Q} - \mathrm{i}\mathfrak{Q}^\dagger$

$$\mathcal{D}_1^2 = \mathcal{D}_2^2 = \mathfrak{H} = \big(a_\mu^\dagger a^\mu \otimes 1 + 2\omega \otimes b_\mu^\dagger b^\mu\big) = 2\omega(N_b + N_f) = H \otimes 1 + \omega \otimes \Sigma$$

where

$H = -\frac{\partial^2}{\partial x_\mu \partial x^\mu} + \omega^2 x_\mu x^\mu$ – harmonic oscillator hamiltonian

$\Sigma = [b_\mu^\dagger, b^\mu]$ – spin matrix

algebra $\mathcal{A} = \mathcal{S}(\mathbb{R}^d)$ uniquely determined by smoothness

- $\mathcal{H}_\infty = \bigcap_{k\in\mathbb{N}} \mathrm{dom}(\mathcal{D}^{2k}) = \mathcal{S}(\mathbb{R}^d) \otimes \bigwedge(\mathbb{C}^d) \simeq \big(\mathcal{S}(\mathbb{R}^d)\big)^{2^d}$ (trivial projector of rank 2^d)

- Hermitian structure takes values in Schwartz class functions: We may choose

  1. the commutative algebra $\mathcal{A} = \mathcal{S}(\mathbb{R}^d)$
     $[\mathcal{D}_1, f] = \partial^\mu f \otimes (b_\mu^\dagger - b_\mu), \quad [\mathcal{D}_2, f] = \partial^\mu f \otimes (\mathrm{i}b_\mu^\dagger + \mathrm{i}b_\mu)$
     bounded & order-one

  2. the noncommutative Moyal algebra $\mathcal{A}_\Theta$ which provides an isospectral deformation
     $$[\mathcal{D}_1, L_\star(f)] = L_\star(\partial_\mu f) \otimes \big(\delta^{\mu\nu}(b_\nu^\dagger - b_\nu) + \tfrac{1}{2}\omega\Theta^{\mu\nu}(\mathrm{i}b_\nu^\dagger + \mathrm{i}b_\nu)\big)$$
     $$[\mathcal{D}_2, L_\star(f)] = L_\star(\partial_\mu f) \otimes \big(\delta^{\mu\nu}(\mathrm{i}b_\nu^\dagger + \mathrm{i}b_\nu) + \tfrac{1}{2}\omega\Theta^{\mu\nu}(b_\nu^\dagger - b_\nu)\big)$$

All axioms of spectral triples satisfied, with minor adaptation:

1. metric dimension is obtained from the dimension spectrum, which is the subset $d - \mathbb{N} \subset \mathbb{C}$

   The asymptotics of eigenvalues of $\langle\mathcal{D}\rangle^{-1} = (\mathcal{D}^2+1)^{-\frac{1}{2}}$ gives the wrong dimension $2d$.

2. Hochschild cycle $\boldsymbol{c}$ takes values in unitisation $\mathcal{B} = \{b \in \mathcal{A}'' \,:\, b, [\mathcal{D}, b] \in \bigcap_{k\in\mathbb{N}} \mathrm{dom}(\delta^k)\} \ni u_\mu = e^{\mathrm{i}x_\mu}$

$$\boldsymbol{c} = \sum_{\sigma\in S_d} \epsilon(\sigma) \frac{\mathrm{i}^{\frac{d(d-1)}{2}}}{d!} (u_1\cdots u_d)^{-1} \otimes u_{\sigma(1)} \otimes \cdots \otimes u_{\sigma(d)} \in Z_d(\mathcal{B},\mathcal{B})$$

3. There are TWO images of $\boldsymbol{c}$ in $\mathcal{B}(\mathcal{H})$:

$$\gamma_1 := \pi_{\mathcal{D}_1}(\boldsymbol{c}) = \mathrm{i}^{\frac{d(d+1)}{2}} (b_1^\dagger - b_1)\cdots(b_d^\dagger - b_d)\;,$$
$$\gamma_2 := \pi_{\mathcal{D}_2}(\boldsymbol{c}) = \mathrm{i}^{\frac{d(d+3)}{2}} (b_1^\dagger + b_1)\cdots(b_d^\dagger + b_d)$$

   - satisfy $\gamma_i^2 = 1$, $\gamma_i^* = \gamma_i$, but not $\mathcal{D}_i\gamma_i + (-1)^d \gamma_i\mathcal{D}_i = 0$!
   - product $(-\mathrm{i})^d\gamma_1\gamma_2 = \mathrm{i}^d\gamma_2\gamma_1 = (-1)^{N_f}$ is $\mathbb{Z}_2$-grading!

U(1)-Higgs model

tensor $(\mathcal{A},\mathcal{H},\mathcal{D}_1)$ with $(\mathbb{C}\oplus\mathbb{C},\mathbb{C}^2,M\sigma_1,\sigma_3)$ [Connes+Lott]

- $\mathcal{D}=\mathcal{D}_1\otimes\sigma_3+1\otimes\sigma_1 M=\begin{pmatrix}\mathcal{D}_1 & M\\ M & -\mathcal{D}_1\end{pmatrix}$ $\qquad \begin{pmatrix} f & 0\\ 0 & g\end{pmatrix}\in\mathcal{A}_{tot}$
- selfadjoint fluctuated Dirac operators $\mathcal{D}_A:=\mathcal{D}+\sum_i a_i[\mathcal{D},b_i]$, $a_i,b_i\in\mathcal{A}_{tot}=\mathcal{A}\oplus\mathcal{A}$, are of the form
  $$\mathcal{D}_A=\begin{pmatrix}\mathcal{D}_1+\mathrm{i}A^\mu\otimes(b^\dagger_\mu-b_\mu) & \phi\otimes 1\\ \overline{\phi}\otimes 1 & -(\mathcal{D}_1+\mathrm{i}B^\mu\otimes(b^\dagger_\mu-b_\mu))\end{pmatrix}$$
  for $A_\mu=\overline{A_\mu},\ B_\mu=\overline{B_\mu},\ \phi\in\mathcal{A}$

Spectral action principle

most general form of bosonic action is $S(\mathcal{D}_A)=Tr\big(\chi(\mathcal{D}_A^2)\big)$

computable thanks to Mehler kernel

The spectral action

$$S(\mathcal{D}_A) = \frac{2\chi_{-4}}{\omega^4} + \frac{8\chi_{-2}}{3\omega^2} + \frac{52\chi_0}{45} + \frac{\chi_0}{\pi^2}\int d^4x \Big\{ D^\mu\phi\overline{D_\mu\phi} + \frac{5}{12}(F^A_{\mu\nu}F_A^{\mu\nu} + F^B_{\mu\nu}F_B^{\mu\nu}) + \Big((|\phi|^2)^2 - \frac{2\chi_{-1}}{\chi_0}|\phi|^2 + 2\omega^2\|x\|^2|\phi|^2\Big)\Big\} + \mathcal{O}(\chi_1)$$

- spectral action is finite
- only difference in field equations to infinite volume is additional harmonic oscillator potential for the Higgs
- Yang-Mills is unchanged (in contrast to Moyal)
- vacuum is at $A_\mu = B_\mu = 0$ and (after gauge transformation) $\phi \in \mathbb{R}$, rotationally invariant

rescale $r = 2^{\frac{1}{4}}\sqrt{\omega}\|x\|$, $\phi = \frac{\pi}{\sqrt{2\chi_0}}\varphi$, $\mu^2 = \frac{\chi_{-1}}{\sqrt{8}\,\omega\chi_0}$, $\lambda = \frac{\pi^2}{\sqrt{2}\,\omega\chi_0}$

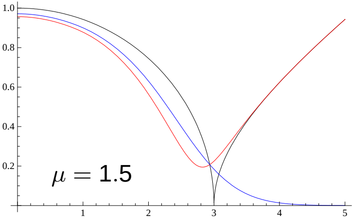
Field equation

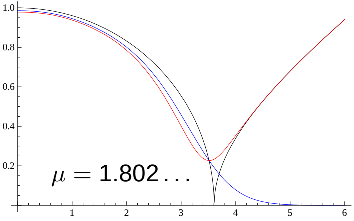
$$-\varphi''(r) - \frac{3}{r}\varphi'(r) + (r^2 - 4\mu^2)\varphi(r) = -\lambda\varphi^3(r)$$

numerical solution for ϕ_{vac}:

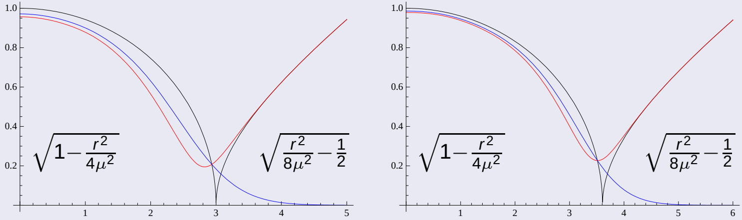
- scale of bare fermion and gauge field masses given by vacuum expectation value $\sqrt{\frac{4\mu^2}{\lambda}}\frac{\varphi_{vac}}{\mu} = \sqrt{\frac{2\chi_{-1}}{\pi^2}}\frac{\varphi_{vac}}{\mu}$
- bare Higgs mass given by difference function

$$\sqrt{\sqrt{2}\omega((r^2 - 4\mu^2) + 12\mu^2\frac{\varphi_{vac}^2}{\mu^2})} = \sqrt{\frac{4\chi_{-1}}{\chi_0}}\frac{\sqrt{\frac{3}{2}\varphi_{vac}^2 - \frac{1}{2}\mu^2 + \frac{1}{8}r^2}}{\mu}$$





Smooth transition between two phases



1. *Spontaneously broken phase* $\omega^2\|x\|^2 < \frac{\chi_{-1}}{\chi_0}$
   fermions, gauge fields and Higgs are massive
2. *Unbroken phase* $\omega^2\|x\|^2 \geq \frac{\chi_{-1}}{\chi_0}$
   fermions + gauge fields massless, Higgs remains massive

For $\omega \to 0$ we recover the usual Higgs scenario.

In QFT the limit $\omega \to 0$ must be taken with care …

The spectral action: noncommutative case

$$
\begin{aligned}
S(\mathcal{D}_A) = {} & \frac{\theta^4\chi_{-4}}{8\Omega^4} + \frac{2\theta^2\chi_{-2}}{3\Omega^2} + \frac{52\chi_0}{45} + \frac{\chi_0}{2\pi^2(1+\Omega^2)^2}\int d^4x\,\Big\{2D_\mu\phi\star\overline{D_\mu\phi} \\
& + \Big(\frac{(1-\Omega^2)^2}{2} - \frac{(1-\Omega^2)^4}{3(1+\Omega^2)^2}\Big)\big(F^A_{\mu\nu}\star F_A^{\mu\nu} + F^B_{\mu\nu}\star F_B^{\mu\nu}\big) \\
& + \Big(\phi\star\bar\phi + \frac{4\Omega^2}{1+\Omega^2}X_A^\mu\star X_{A\mu} - \frac{\chi_{-1}}{\chi_0}\Big)^2 \\
& + \Big(\bar\phi\star\phi + \frac{4\Omega^2}{1+\Omega^2}X_B^\mu\star X_{B\mu} - \frac{\chi_{-1}}{\chi_0}\Big)^2 \\
& - 2\Big(\frac{4\Omega^2}{1+\Omega^2}X_0^\mu\star X_{0\mu} - \frac{\chi_{-1}}{\chi_0}\Big)^2\Big\}(x) + \mathcal{O}(\chi_1)
\end{aligned}
$$

$$
\Theta = \begin{pmatrix} 0 & \theta & 0 & 0 \\ -\theta & 0 & 0 & 0 \\ 0 & 0 & 0 & \theta \\ 0 & 0 & -\theta & 0 \end{pmatrix} \qquad \omega = \frac{2\Omega}{\theta}
$$

deeper entanglement of gauge and Higgs fields

covariant coordinates $X_{A\mu}(x) = (\Theta^{-1})_{\mu\nu}x^\nu + A_\mu(x)$ appear with Higgs field ϕ in unified potential; vacuum is non-trivial!

potential cannot be restricted to Higgs part if distinction into discrete and continuous geometries no longer possible

The vacuum

vacuum field equations $(\phi^{vac} = \overline{\phi^{vac}}, \quad A_\mu^{vac} = B_\mu^{vac})$

$$\frac{1}{g^2}[X_{A\nu},[X_A^\mu,X_A^\nu]_\star]_\star + 2[\phi,[X_A^\mu,\phi]_\star]_\star = \frac{4\Omega^2}{1+\Omega^2}\Big\{\phi\star\phi+\frac{4\Omega^2}{1+\Omega^2}X_A^\nu\star X_{A\nu}-\mu^2, X_A^\mu\Big\}_\star$$

$$2[X_{A\nu},[\phi,X_A^\nu]_\star]_\star = \Big\{\phi\star\phi+\frac{4\Omega^2}{1+\Omega^2}X_A^\nu\star X_{A\nu}-\mu^2,\phi\Big\}_\star$$

$$\Big(\text{with } \frac{1}{4g^2} = \frac{(1-\Omega^2)^2}{2} - \frac{(1-\Omega^2)^4}{3(1+\Omega^2)^2} \qquad \mu^2 = \frac{\chi_{-1}}{\chi_0}\Big)$$

spirit of emerging geometry through phase transitions

- $\Omega = 0 \quad \Rightarrow$ solution: $\phi=\mu 1$, $\quad [X_\mu,X_\nu] = \begin{cases} \Theta_{\mu\nu} \\ 0 \end{cases}$

  $\Omega \neq 0$ gives some dynamical geometry
- analytical solution seems impossible

  $\Rightarrow$ need numerical simulations

ϕ_4^4-theory on Moyal space with oscillator potential

We consider the previous model for vanishing Yang-Mills fields $A = B = 0$ and reversed sign in Higgs mass

action functional

$$S[\phi] = \int d^4x \Big(\frac{1}{2}\phi \star \big(-\Delta + \Omega^2 \tilde{x}^2 + \mu^2\big)\phi + \frac{\lambda}{4}\phi \star \phi \star \phi \star \phi \Big)(x)$$

Moyal product $\star$ defined by Θ and $\tilde{x} := 2\Theta^{-1} \cdot x$

parameters: $\mu^2, \lambda \in \mathbb{R}_+$, $\Omega \in [0,1]$, redef'n $\phi \mapsto Z^{\frac{1}{2}}\phi$, $Z \in \mathbb{R}_+$

- renormalisable as formal power series in λ [Grosse-W.]
  means: well-defined perturbative quantum field theory
- β-function vanishes to all orders in λ for $\Omega = 1$
  [Disertori-Gurau-Magnen-Rivasseau]
  means: model is believed to exist non-perturbatively

The action functional for $\Omega = 1$ in matrix basis

- ensemble of selfadjoint large matrices $(\phi_{mn})_{m,n\in\mathbb{N}^2_\Lambda} \in M_\Lambda$ with cut-off Λ in the matrix size
- correlation functions generated by partition function
  $\mathcal{Z}[J] = N\int \Big(\prod_{m,n\in\mathbb{N}^2_\Lambda} d\phi_{mn}\Big)\ \exp\big(-S[\phi]+\mathrm{tr}(\phi J)\big)$

We are interested in $\mathbb{N}^2_\Lambda \to \mathbb{N}^2$. Correlation functions ill-defined unless $S[\phi]$ is a suitably divergent function of Λ:

$$S[\phi] = \sum_{m,n\in\mathbb{N}^2_\Lambda} \frac{1}{2}\phi_{mn}H_{mn}\phi_{nm} + V(\phi)$$

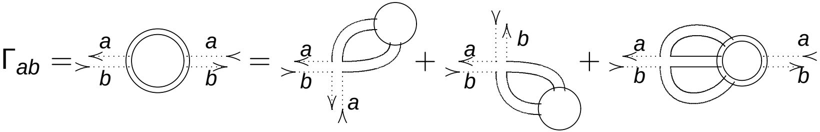
$$H_{mn} = Z\left(\mu^2_{bare}+|m|+|n|\right) \qquad V(\phi) = \frac{Z^2\lambda}{4}\sum_{m,n,k,l\in\mathbb{N}^2_\Lambda}\phi_{mn}\phi_{nk}\phi_{kl}\phi_{lm}$$

with $|m| = m_1 + m_2$ and divergent $\mu_{bare}[\Lambda,\lambda], Z[\Lambda,\lambda]$

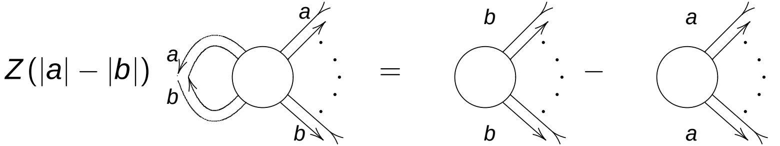
There is no separate Λ-dependence in λ!

Schwinger-Dyson equations and Ward identities

1. Schwinger-Dyson equations express n-point function in terms of $(m > n)$-point functions:



2. The model admits a Ward identity which expresses n-point functions in terms of $(n-2)$-point functions:



Result: closed formula for 2-point function alone

$$\Gamma_{ab} = Z^2 \lambda \sum_p \Big(\frac{1}{H_{bp} - \Gamma_{bp}} + \frac{1}{H_{ap} - \Gamma_{ap}} - \frac{1}{H_{bp} - \Gamma_{bp}} \frac{(\Gamma_{bp} - \Gamma_{ab})}{Z(|p| - |a|)} \Big)$$

Direct renormalisation of SD-equation for Γ_{ab}

- Taylor: $\Gamma_{ab} = Z\mu_{bare}^2 - \mu^2 + (Z-1)(|a|+|b|) + \Gamma_{ab}^{ren}$ with $\Gamma_{00}^{ren} = 0$ and $(\partial\Gamma^{ren})_{00} = 0$

  $\Rightarrow \qquad G_{ab}^{-1} = H_{ab} - \Gamma_{ab} = |a| + |b| + \mu^2 - \Gamma_{ab}^{ren}$

- We replace discrete indices $a, b, p \in \mathbb{N}^2$ by continuous indices $a, b, p \in (\mathbb{R}_+)^2$, and sums by integrals.

- This captures the behaviour at $\Lambda \to \infty$ of the discrete version, or defines another interesting field theory.

- perturbative calculation suggests change of variables

  $$|a| = \frac{\alpha\mu^2}{1-\alpha}\,, \qquad \Gamma_{ab}^{ren} = \frac{\Gamma_{\alpha\beta}\mu^2}{(1-\alpha)(1-\beta)}\,, \qquad \Lambda = \frac{\xi\mu^2}{1-\xi}$$

Elimination of Z shows that limit $\xi \to 1$ exists

Theorem [Grosse-W., 2009]

The renormalised planar connected two-point function $G_{\alpha\beta}$ of Moyal-harmonic-oscillator ϕ_4^4-theory satisfies

$$\begin{aligned} G_{\alpha\beta} = 1 + \lambda\Big(& \frac{1-\alpha}{1-\alpha\beta}(\mathcal{M}_\beta - \mathcal{L}_\beta - \beta\mathcal{Y}) + \frac{1-\beta}{1-\alpha\beta}(\mathcal{M}_\alpha - \mathcal{L}_\alpha - \alpha\mathcal{Y}) \\ & + \frac{(1-\alpha)(1-\beta)}{1-\alpha\beta}(G_{\alpha\beta}-1)\mathcal{Y} - \frac{\alpha(1-\beta)}{1-\alpha\beta}(\mathcal{L}_\beta + \mathcal{N}_{\alpha\beta} - \mathcal{N}_{\alpha 0}) \\ & + \frac{1-\beta}{1-\alpha\beta}\Big(\frac{G_{\alpha\beta}}{G_{0\alpha}} - 1\Big)(\mathcal{M}_\alpha - \mathcal{L}_\alpha + \alpha\mathcal{N}_{\alpha 0})\Big) \end{aligned}$$

with $\alpha, \beta \in [0,1)$ and

$$\mathcal{L}_\alpha := \int_0^1 d\rho\, \frac{G_{\alpha\rho} - G_{0\rho}}{1-\rho} \qquad \mathcal{M}_\alpha := \int_0^1 d\rho\, \frac{\alpha\, G_{\alpha\rho}}{1-\alpha\rho}$$

$$\mathcal{N}_{\alpha\beta} := \int_0^1 d\rho\, \frac{G_{\rho\beta} - G_{\alpha\beta}}{\rho-\alpha} \qquad \mathcal{Y} = \lim_{\alpha\to 0} \frac{\mathcal{M}_\alpha - \mathcal{L}_\alpha}{\alpha}$$

- Integral equation for $G_{\alpha\beta}$ is non-perturbatively defined.
- We can nevertheless look for a perturbative solution, which computes $G_{\alpha\beta} = \sum_{n=0}^{\infty} \lambda^n G^{(n)}_{\alpha\beta}$ without passing through Feynman graphs and further renormalisation.

This involves iterated integrals labelled by rooted trees. Up to $\mathcal{O}(\lambda^3)$ we need

$$I_\alpha := \int_0^1 dx\, \frac{\alpha}{1-\alpha x} = -\ln(1-\alpha)\;,$$

$$I_{\stackrel{\alpha}{\bullet}} := \int_0^1 dx\, \frac{\alpha\, I_x}{1-\alpha x} = \mathrm{Li}_2(\alpha) + \frac{1}{2}\big(\ln(1-\alpha)\big)^2$$

$$I_{\stackrel{\alpha}{\bullet\;\bullet}} = \int_0^1 dx\, \frac{\alpha\, I_x \cdot I_x}{1-\alpha x} = -2\,\mathrm{Li}_3\Big(-\frac{\alpha}{1-\alpha}\Big)$$

$$I_{\stackrel{\alpha}{\vdots}} = \int_0^1 dx\, \frac{\alpha\, I_{\stackrel{x}{\bullet}}}{1-\alpha x} = -2\mathrm{Li}_3\Big(-\frac{\alpha}{1-\alpha}\Big) - 2\mathrm{Li}_3(\alpha) - \ln(1-\alpha)\zeta(2) + \ln(1-\alpha)\mathrm{Li}_2(\alpha) + \frac{1}{6}\big(\ln(1-\alpha)\big)^3$$

In terms of I_t and $A := \frac{1-\alpha}{1-\alpha\beta}$, $B := \frac{1-\beta}{1-\alpha\beta}$:

$$
\begin{aligned}
G_{\alpha\beta} = 1 + \lambda\Big\{ & A(I_\beta - \beta) + B(I_\alpha - \alpha)\Big\} \\
+ \lambda^2\Big\{ & A\big(\beta I_{\underset{\bullet}{\beta}} - \beta I_\beta\big) - \alpha AB\big((I_\beta)^2 - 2\beta I_\beta + I_\beta\big) \\
& + B\big(\alpha I_{\underset{\bullet}{\alpha}} - \alpha I_\alpha\big) - \beta BA\big((I_\alpha)^2 - 2\alpha I_\alpha + I_\alpha\big) \\
& + AB\big((I_{\underset{\bullet}{\alpha}} - \alpha) + (I_{\underset{\bullet}{\beta}} - \beta) + (I_\alpha - \alpha)(I_\beta - \beta) + \alpha\beta(\zeta(2) + 1)\big)\Big\} \\
+ \lambda^3\Big\{ & A\mathcal{W}_\beta + \alpha AB\big(-\mathcal{U}_\beta + I_\alpha I_\beta + I_{\underset{\bullet}{\alpha}} I_\beta\big) + \alpha A^2 B\mathcal{V}_\beta \\
& + B\mathcal{W}_\alpha + \beta BA\big(-\mathcal{U}_\alpha + I_\beta I_\alpha + I_{\underset{\bullet}{\beta}} I_\alpha\big) + \beta B^2 A\mathcal{V}_\alpha \\
& + AB\big(\mathcal{T}_\beta + \mathcal{T}_\alpha - I_\beta (I_\alpha)^2 - I_\alpha (I_\beta)^2 - 6 I_\alpha I_\beta\big) \\
& + AB^2\big((1-\alpha)(I_{\underset{\bullet}{\alpha}} - \alpha) + 3 I_\alpha I_\beta + I_{\underset{\bullet}{\beta}} I_\alpha + I_\beta (I_\alpha)^2\big) \\
& + BA^2\big((1-\beta)(I_{\underset{\bullet}{\beta}} - \beta) + 3 I_\alpha I_\beta + I_{\underset{\bullet}{\alpha}} I_\beta + I_\alpha (I_\beta)^2\big)\Big\} + \mathcal{O}(\lambda^4)
\end{aligned}
$$

where

$$
\begin{aligned}
\mathcal{T}_\beta &:= \beta I_{\underset{\bullet}{\beta}} - \beta I_\beta + (I_\beta - \beta)\,, \\
\mathcal{U}_\beta &:= -\beta I_{\underset{\bullet\,\bullet}{\beta}} - (I_\beta)^3 + \beta I_{\underset{\bullet}{\beta}} I_\beta + 2 I_{\underset{\bullet}{\beta}} I_\beta + \beta\zeta(2) I_\beta - 2\beta\zeta(3) \\
&\quad - 2(I_\beta)^2 + \beta(I_\beta)^2 + I_{\underset{\bullet}{\beta}} + \beta I_{\underset{\bullet}{\beta}} + 2 I_\beta - \beta^2\,, \\
\mathcal{V}_\beta &:= \beta I_{\underset{\substack{\bullet\\\bullet}}{\beta}} - \beta^2 I_{\underset{\bullet\,\bullet}{\beta}} - 2\beta^2\zeta(3) + 2\beta I_{\underset{\bullet}{\beta}} I_\beta - I_\beta^3 + 2\beta I_\beta \zeta(2) - 3\beta^2\zeta(2) \\
&\quad + (1-\beta)\big(2\beta I_{\underset{\bullet}{\beta}} - 3 I_\beta^2 + 3\beta I_\beta - 3 I_\beta + \beta\big)\,, \\
\mathcal{W}_\beta &:= (I_{\underset{\substack{\bullet\\\bullet}}{\beta}} - \beta\zeta(2)) - \frac{1}{2} I_\beta \frac{I_\beta - \beta}{\beta} + \frac{1}{2}(I_\beta)^2 - (I_{\underset{\bullet}{\beta}} - \beta) - \frac{1}{2}(I_\beta - \beta) - \frac{1}{2}\beta^2
\end{aligned}
$$

Remark: $\frac{I_\beta - \beta}{\beta} = \int_0^1 dx\, \frac{\beta x}{1 - \beta x}$

(optimal family of iterated integrals not yet determined)

Observations

- Polylogarithms and multiple zeta values appear in singular part of individual graphs of e.g. ϕ^4-theory [Broadhurst-Kreimer]
- We encounter them for regular part of all graphs together

Conjecture

$\Gamma_{\alpha\beta}$ is analytic (at least smooth) in neighbourhood of $\lambda = 0$

1. study the recursive generation of these iterated integrals
2. use implicit function theorem in an appropriate Banach or Fréchet space

- SD-equations for all higher n-point functions G^n (planar / non-planar) are expected to be linear in G^n with inhomogeneity given by $G^{<n}$ (verified for planar $n = 4$)

From the known planar 2-point function we could compute or estimate the whole theory – which is an interacting QFT in 4D!