

The two-point function of noncommutative ϕ_4^4 -theory

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and work in progress with Harald Grosse and Vincent Rivasseau)

Introduction

- The **Standard Model** is a **perturbatively renormalisable quantum field theory**.
- Scattering amplitudes can be computed as **formal power series in coupling constants such as $e^2 \approx \frac{1}{137}$** .
The first terms agree to high precision with experiment.
- The **radius of convergence in e^2 is zero!**
We are far away from understanding the Standard Model (see e.g. confinement).

- Refined summation techniques (e.g. Borel) may establish **reasonable domains of analyticity**.
- Unfortunately, this also fails for QED due to the **Landau ghost problem**.

It is expected to work for **non-Abelian gauge theories** because of **asymptotic freedom**.

But these theories are too complicated.

QFT's on **noncommutative geometries** may provide toy models for **non-perturbative renormalisation** in four dimensions.

ϕ_4^4 -theory on Moyal space with oscillator potential

action functional

$$S[\phi] = \int d^4x \left(\frac{1}{2} \phi \star (-\Delta + \Omega^2 \tilde{x}^2 + \mu^2) \phi + \frac{\lambda}{4} \phi \star \phi \star \phi \star \phi \right) (x)$$

Moyal product \star defined by Θ and $\tilde{x} := 2\Theta^{-1} \cdot x$

parameters: $\mu^2, \lambda \in \mathbb{R}_+$, $\Omega \in [0, 1]$, redef'n $\phi \mapsto Z^{\frac{1}{2}} \phi$, $Z \in \mathbb{R}_+$

- **renormalisable as formal power series** in λ [Grosse-W.]
means: well-defined **perturbative** quantum field theory
- **β -function vanishes to all orders** in λ for $\Omega = 1$
[Disertori-Gurau-Magnen-Rivasseau]
means: model is believed to exist **non-perturbatively**

Up to the sign of μ^2 , this model arises from a **spectral triple**.

The matrix basis of noncommutative Moyal space

$$(f \star g)(x) = \int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{dy dk}{(2\pi)^d} f(x + \frac{1}{2} \Theta \cdot k) g(x+y) e^{iky}$$

- central observation (in 2D):

$$f_{00} := 2e^{-\frac{1}{\theta}(x_1^2 + x_2^2)} \Rightarrow f_{00} \star f_{00} = f_{00}$$

- left and right creation operators applied to f_{00} lead to

$$f_{mn}(\rho, \varphi) = 2(-1)^m \sqrt{\frac{m!}{n!}} e^{i\varphi(n-m)} \left(\sqrt{\frac{2}{\theta}} \rho\right)^{n-m} e^{-\frac{\rho^2}{\theta}} L_m^{n-m}\left(\frac{2}{\theta} \rho^2\right)$$

- \star -product becomes simple matrix product:

$$f_{mn} \star f_{kl} = \delta_{nk} f_{ml}, \quad \int d^2x f_{mn}(x) = \sqrt{\det(2\pi\Theta)} \delta_{mn}$$

- $(-\Delta + \Omega^2 \tilde{x}^2 + \mu^2) f_{mn}$
 $= (\mu^2 + \frac{2}{\theta}(1 + \Omega^2)(m+n+1)) f_{mn}(x)$
 $-\frac{2}{\theta}(1 - \Omega^2)(\sqrt{mn} f_{m-1, n-1} + \sqrt{(m+1)(n+1)} f_{m+1, n+1})$

The 4D-action functional for $\Omega = 1$

expand $\phi(x) = \sum_{m_1, m_2, n_1, n_2} \phi_{mn} f_{m_1 n_1}(x_1, x_2) f_{m_2 n_2}(x_3, x_4)$

- matrices $(\phi_{mn})_{m, n \in \mathbb{N}_\Lambda^2} \in M_\Lambda$ with **cut-off Λ** in matrix size
- correlation functions generated by **partition function**

$$\mathcal{Z}[J] = N \int \left(\prod_{m, n \in \mathbb{N}_\Lambda^2} d\phi_{mn} \right) \exp(-S[\phi] + \text{tr}(\phi J))$$

We are interested in $\mathbb{N}_\Lambda^2 \rightarrow \mathbb{N}^2$. Correlation functions ill-defined unless $S[\phi]$ is a suitably divergent function of Λ :

$$S[\phi] = \sum_{m, n \in \mathbb{N}_\Lambda^2} \frac{1}{2} \phi_{mn} H_{mn} \phi_{nm} + V(\phi)$$

$$H_{mn} = Z(\mu_{bare}^2 + |m| + |n|), \quad V(\phi) = \frac{Z^2 \lambda}{4} \sum_{m, n, k, l \in \mathbb{N}_\Lambda^2} \phi_{mn} \phi_{nk} \phi_{kl} \phi_{lm}$$

with $|m| = m_1 + m_2$ and divergent $\mu_{bare}[\Lambda, \lambda], Z[\Lambda, \lambda]$.

There is no separate Λ -dependence in λ !

Ward identity

- inner automorphism $\phi \mapsto U\phi U^\dagger$ of M_Λ
infinitesimally $\phi_{mn} \mapsto \phi_{mn} + i \sum_{k \in \mathbb{N}_\Lambda^2} (B_{mk} \phi_{kn} - \phi_{mk} B_{kn})$
- not a symmetry of the action, but translation invariance of the measure $\mathcal{D}\phi = \prod_{m,n \in \mathbb{N}_\Lambda^2} d\phi_{mn}$ gives

$$\begin{aligned}
 0 &= \frac{\delta W}{i\delta B_{ab}} = \frac{1}{\mathcal{Z}} \int \mathcal{D}\phi \left(-\frac{\delta S}{i\delta B_{ab}} + \frac{\delta}{i\delta B_{ab}} (\text{tr}(\phi J)) \right) e^{-S + \text{tr}(\phi J)} \\
 &= \frac{1}{\mathcal{Z}} \int \mathcal{D}\phi \sum_n \left((H_{nb} - H_{an}) \phi_{bn} \phi_{na} + (\phi_{bn} J_{na} - J_{bn} \phi_{na}) \right) e^{-S + \text{tr}(\phi J)}
 \end{aligned}$$

where $W[J] = \ln \mathcal{Z}[J]$ generates **connected** functions

perturbation trick $\phi_{mn} \mapsto \frac{\delta}{\delta J_{nm}}$

$$\begin{aligned}
 0 &= \left\{ \sum_n \left((H_{nb} - H_{an}) \frac{\delta^2}{\delta J_{nb} \delta J_{an}} + \left(J_{na} \frac{\delta}{\delta J_{nb}} - J_{bn} \frac{\delta}{\delta J_{an}} \right) \right) \right. \\
 &\quad \left. \times \exp \left(-V \left(\frac{\delta}{\delta J} \right) \right) e^{\frac{1}{2} \sum_{p,q} J_{pq} H_{pq}^{-1} J_{qp}} \right\}_c
 \end{aligned}$$

Interpretation

The insertion of a special vertex $V_{ab}^{ins} := \sum_n (H_{an} - H_{nb}) \phi_{bn} \phi_{na}$

into an **external face of a ribbon graph** is the same as the difference between the exchanges of external sources

$J_{nb} \mapsto J_{na}$ and $J_{an} \mapsto J_{bn}$

$$\begin{aligned}
 Z(|a| - |b|) \text{ [diagram with loops]} &= \text{[diagram with } b \text{ labels]} - \text{[diagram with } a \text{ labels]} \\
 Z(|a| - |b|) G_{[ab]...}^{ins} &= G_{b...} - G_{a...}
 \end{aligned}$$

Two-point Schwinger-Dyson equation

$$\Gamma_{ab} = \text{graph} = \text{graph}_q + \text{graph}_p + \text{graph}_p$$

- vertex is $Z^2\lambda$, connected two-point function is G_{ab} :
first graph equals $Z^2\lambda \sum_q G_{aq}$
- in other two graphs we open the p -face and compare with insertion into connected two-point function; it inserts
 - either into one-particle reducible line
 - or into 1PI function:

$$G_{[ap]b}^{ins} = \text{graph}_p = \text{graph}_p + \text{graph}_p$$

- amputation of G_{ab} :
last two graphs together equal $Z^2\lambda \sum_p G_{ab}^{-1} G_{[ap]b}^{ins}$

Result (using $G_{ab}^{-1} = H_{ab} - \Gamma_{ab}$):

$$\begin{aligned}\Gamma_{ab} &= Z^2 \lambda \sum_p \left(G_{ap} + G_{ab}^{-1} G_{[ap]b}^{ins} \right) = Z^2 \lambda \sum_p \left(G_{ap} - G_{ab}^{-1} \frac{G_{bp} - G_{ba}}{Z(|p| - |a|)} \right) \\ &= Z^2 \lambda \sum_p \left(\frac{1}{H_{bp} - \Gamma_{bp}} + \frac{1}{H_{ap} - \Gamma_{ap}} - \frac{1}{H_{bp} - \Gamma_{bp}} \frac{(\Gamma_{bp} - \Gamma_{ab})}{Z(|p| - |a|)} \right)\end{aligned}$$

- This is a self-consistent functional equation for Γ_{ab} . It is non-linear and singular. Its singular part at $(a, b = 0)$ already appeared in [Disertori-Gurau-Magnen-Rivasseau].
- We perform the renormalisation directly in the SD-equation for Γ_{ab} . The Z-factors are essential for that.
- Taylor: $\Gamma_{ab} = Z\mu_{bare}^2 - \mu^2 + (Z-1)(|a|+|b|) + \Gamma_{ab}^{ren}$
 $\Rightarrow G_{ab}^{-1} = H_{ab} - \Gamma_{ab} = |a| + |b| + \mu^2 - \Gamma_{ab}^{ren}$
- $\Gamma_{00}^{ren} = 0$ and $(\partial\Gamma^{ren})_{00} = 0$ determine μ_{bare}^2 and Z .

Integral representation

- We replace discrete indices $a, b, p \in \mathbb{N}^2$ by **continuous indices** $a, b, p \in (\mathbb{R}_+)^2$, and sums by integrals.
- This **captures the $\Lambda \rightarrow \infty$ behaviour** of the discrete version (or defines another interesting field theory).
- The mass-renormalised Schwinger-Dyson equation depends only on the length $|a| = a_1 + a_2$. **Partial derivatives $\frac{\partial}{\partial a_i}$ needed to extract Z are equal.** Therefore, Γ_{ab}^{ren} **depends only on $|a|$ and $|b|$.**

- Hence,
$$\int_{(\mathbb{R}_+)^2}^{(\Lambda)} dp_1 dp_2 f(|p|) = \int_0^\Lambda |p| d|p| f(|p|)$$

Mass renormalisation = subtraction at 0:

$$\begin{aligned}
 & (Z - 1)(|a| + |b|) + \Gamma_{ab}^{ren} \\
 &= \lambda \int_0^\Lambda |p| d|p| \left(\frac{Z}{|b| + |p| + \mu^2 - \Gamma_{bp}^{ren}} + \frac{Z^2}{|a| + |p| + \mu^2 - \Gamma_{ap}^{ren}} - \frac{Z^2 + Z}{|p| + \mu^2 - \Gamma_{0p}^{ren}} \right. \\
 &\quad \left. - \frac{Z}{|b| + |p| + \mu^2 - \Gamma_{bp}^{ren}} \frac{\Gamma_{bp}^{ren} - \Gamma_{ab}^{ren}}{|p| - |a|} + \frac{Z}{p + \mu^2 - \Gamma_{0p}^{ren}} \frac{\Gamma_{0p}^{ren}}{|p|} \right)
 \end{aligned}$$

- perturbative solution depends on combination $\frac{a}{1+a}$ and $\frac{\Lambda}{1+\Lambda}$
- change of variables

$$\begin{aligned}
 |a| &=: \mu^2 \frac{\alpha}{1 - \alpha}, & |b| &=: \mu^2 \frac{\beta}{1 - \beta}, & |p| &=: \mu^2 \frac{\rho}{1 - \rho}, \\
 \Gamma_{ab}^{ren} &=: \mu^2 \frac{1 - \alpha\beta}{(1 - \alpha)(1 - \beta)} \left(1 - \frac{1}{G_{\alpha\beta}} \right), & \Lambda &=: \mu^2 \frac{\xi}{1 - \xi}
 \end{aligned}$$

- $\frac{\partial}{\partial a_i} \Big|_{a=0} = \frac{\partial}{\partial \alpha} \Big|_{\alpha=0}$ to extract Z

Theorem [Grosse-W., 2009]

The renormalised planar connected two-point function $G_{\alpha\beta}$ of self-dual n.c. ϕ_4^4 -theory satisfies (and is determined by)

$$\begin{aligned}
 G_{\alpha\beta} = & 1 + \lambda \left(\frac{1-\alpha}{1-\alpha\beta} (\mathcal{M}_\beta - \mathcal{L}_\beta - \beta\mathcal{Y}) + \frac{1-\beta}{1-\alpha\beta} (\mathcal{M}_\alpha - \mathcal{L}_\alpha - \alpha\mathcal{Y}) \right. \\
 & + \frac{(1-\alpha)(1-\beta)}{1-\alpha\beta} (G_{\alpha\beta} - 1)\mathcal{Y} - \frac{\alpha(1-\beta)}{1-\alpha\beta} (\mathcal{L}_\beta + \mathcal{N}_{\alpha\beta} - \mathcal{N}_{\alpha 0}) \\
 & \left. + \frac{1-\beta}{1-\alpha\beta} \left(\frac{G_{\alpha\beta}}{G_{0\alpha}} - 1 \right) (\mathcal{M}_\alpha - \mathcal{L}_\alpha + \alpha\mathcal{N}_{\alpha 0}) \right)
 \end{aligned}$$

with $\alpha, \beta \in [0, 1)$ and

$$\begin{aligned}
 \mathcal{L}_\alpha &:= \int_0^1 d\rho \frac{G_{\alpha\rho} - G_{0\rho}}{1-\rho} & \mathcal{M}_\alpha &:= \int_0^1 d\rho \frac{\alpha G_{\alpha\rho}}{1-\alpha\rho} \\
 \mathcal{N}_{\alpha\beta} &:= \int_0^1 d\rho \frac{G_{\rho\beta} - G_{\alpha\beta}}{\rho-\alpha} & \mathcal{Y} &= \lim_{\alpha \rightarrow 0} \frac{\mathcal{M}_\alpha - \mathcal{L}_\alpha}{\alpha}
 \end{aligned}$$

Discussion

- The previous integral equation for $G_{\alpha\beta}$ is **non-perturbatively** defined. It is **non-linear** and **singular** (at 1).
- Nonlinearity and singularity can be resolved in perturbation theory. Then: **Is the perturbation series analytic at $\lambda = 0$?**
- Non-perturbative approach:
 - There are methods for **singular but linear integral equations** (Riemann-Hilbert problem).
 - Non-linearity treatable by **implicit function theorem** (or Nash-Moser theorem), but singularity is problematic.

If we could solve the equation for $G_{\alpha\beta}$, then **all other n -point function should result from a hierarchy of Ward-identities and Schwinger-Dyson equations** which are **linear (and inhomogeneous)** in the highest-order function.

Theorem

The renormalised planar 1PI four-point function $\Gamma_{\alpha\beta\gamma\delta}$ of self-dual n.c. ϕ_4^4 -theory satisfies (and is determined by)

$$\Gamma_{\alpha\beta\gamma\delta} = \lambda \cdot \frac{\left(1 - \frac{(1-\alpha)(1-\gamma\delta)(G_{\alpha\delta} - G_{\gamma\delta})}{G_{\gamma\delta}(1-\delta)(\alpha-\gamma)} + \int_0^1 \rho d\rho \frac{(1-\beta)(1-\alpha\delta)G_{\beta\rho}G_{\delta\rho}}{(1-\beta\rho)(1-\delta\rho)} \frac{\Gamma_{\rho\beta\gamma\delta} - \Gamma_{\alpha\beta\gamma\delta}}{\rho-\alpha}\right)}{G_{\alpha\delta} + \lambda \left((\mathcal{M}_\beta - \mathcal{L}_\beta - \mathcal{Y})G_{\alpha\delta} + \int_0^1 d\rho \frac{G_{\alpha\delta}G_{\beta\rho}(1-\beta)}{(1-\delta\rho)(1-\beta\rho)} + \int_0^1 \rho d\rho \frac{(1-\beta)(1-\alpha\delta)G_{\beta\rho}(G_{\rho\delta} - G_{\alpha\delta})}{(1-\beta\rho)(1-\delta\rho)(\rho-\alpha)} \right)}$$

Corollary

$\Gamma_{\alpha\beta\gamma\delta} = 0$ is not a solution!

We have a non-trivial (interacting) QFT in four dimensions!

Perturbative solution

- We look for an iterative solution $G_{\alpha\beta} = \sum_{n=0}^{\infty} \lambda^n G_{\alpha\beta}^{(n)}$.
- This involves **iterated integrals labelled by rooted trees**.

Up to $\mathcal{O}(\lambda^3)$ we need

$$I_{\alpha} := \int_0^1 dx \frac{\alpha}{1 - \alpha x} = -\ln(1 - \alpha),$$

$$I_{\bullet}^{\alpha} := \int_0^1 dx \frac{\alpha I_x}{1 - \alpha x} = \text{Li}_2(\alpha) + \frac{1}{2} (\ln(1 - \alpha))^2$$

$$I_{\bullet\bullet}^{\alpha} = \int_0^1 dx \frac{\alpha I_x \cdot I_x}{1 - \alpha x} = -2 \text{Li}_3\left(-\frac{\alpha}{1 - \alpha}\right)$$

$$I_{\bullet\bullet\bullet}^{\alpha} = \int_0^1 dx \frac{\alpha I_x \cdot \bullet}{1 - \alpha x} = -2 \text{Li}_3\left(-\frac{\alpha}{1 - \alpha}\right) - 2 \text{Li}_3(\alpha) - \ln(1 - \alpha) \zeta(2) \\ + \ln(1 - \alpha) \text{Li}_2(\alpha) + \frac{1}{6} (\ln(1 - \alpha))^3$$

In terms of I_t and $A := \frac{1-\alpha}{1-\alpha\beta}$, $B := \frac{1-\beta}{1-\alpha\beta}$:

$$\begin{aligned}
 G_{\alpha\beta} = & 1 + \lambda \left\{ A(I_\beta - \beta) + B(I_\alpha - \alpha) \right\} \\
 & + \lambda^2 \left\{ A(\beta I_\beta - \beta I_\beta) - \alpha AB((I_\beta)^2 - 2\beta I_\beta + I_\beta) \right. \\
 & \quad + B(\alpha I_\alpha - \alpha I_\alpha) - \beta BA((I_\alpha)^2 - 2\alpha I_\alpha + I_\alpha) \\
 & \quad \left. + AB((I_\alpha - \alpha) + (I_\beta - \beta) + (I_\alpha - \alpha)(I_\beta - \beta) + \alpha\beta(\zeta(2) + 1)) \right\} \\
 & + \lambda^3 \left\{ AW_\beta + \alpha AB(-\mathcal{U}_\beta + I_\alpha I_\beta + I_\alpha I_\beta) + \alpha A^2 B \mathcal{V}_\beta \right. \\
 & \quad + BW_\alpha + \beta BA(-\mathcal{U}_\alpha + I_\beta I_\alpha + I_\beta I_\alpha) + \beta B^2 A \mathcal{V}_\alpha \\
 & \quad + AB(\mathcal{T}_\beta + \mathcal{T}_\alpha - I_\beta(I_\alpha)^2 - I_\alpha(I_\beta)^2 - 6I_\alpha I_\beta) \\
 & \quad + AB^2((1-\alpha)(I_\alpha - \alpha) + 3I_\alpha I_\beta + I_\beta I_\alpha + I_\beta(I_\alpha)^2) \\
 & \quad \left. + BA^2((1-\beta)(I_\beta - \beta) + 3I_\alpha I_\beta + I_\alpha I_\beta + I_\alpha(I_\beta)^2) \right\} + \mathcal{O}(\lambda^4)
 \end{aligned}$$

where

$$\mathcal{T}_\beta := \beta \dot{I}_\beta - \beta I_\beta + (I_\beta - \beta),$$

$$\begin{aligned} \mathcal{U}_\beta := & -\beta \dot{I}_\beta - (I_\beta)^3 + \beta \dot{I}_\beta I_\beta + 2 \dot{I}_\beta I_\beta + \beta \zeta(2) I_\beta - 2\beta \zeta(3) \\ & - 2(I_\beta)^2 + \beta (I_\beta)^2 + \dot{I}_\beta + \beta \dot{I}_\beta + 2I_\beta - \beta^2, \end{aligned}$$

$$\begin{aligned} \mathcal{V}_\beta := & \beta \dot{I}_\beta - \beta^2 \dot{I}_\beta - 2\beta^2 \zeta(3) + 2\beta \dot{I}_\beta I_\beta - I_\beta^3 + 2\beta I_\beta \zeta(2) - 3\beta^2 \zeta(2) \\ & + (1 - \beta)(2\beta \dot{I}_\beta - 3I_\beta^2 + 3\beta I_\beta - 3I_\beta + \beta), \end{aligned}$$

$$\mathcal{W}_\beta := (\dot{I}_\beta - \beta \zeta(2)) - \frac{1}{2} I_\beta \frac{I_\beta - \beta}{\beta} + \frac{1}{2} (I_\beta)^2 - (I_\beta - \beta) - \frac{1}{2} (I_\beta - \beta) - \frac{1}{2} \beta^2$$

Remark: $\frac{I_\beta - \beta}{\beta} = \int_0^1 dx \frac{\beta x}{1 - \beta x}$

(optimal family of iterated integrals not yet determined)

Beyond perturbation

Ansatz (suggested by perturbation, but consistent in general)

$$\mathcal{G}_{\alpha\beta} = \mathbf{1} + \left(\frac{1-\alpha}{1-\alpha\beta}\right)\beta^2\mathcal{G}_\beta + \left(\frac{1-\beta}{1-\alpha\beta}\right)\alpha^2\mathcal{G}_\alpha + \left(\frac{1-\alpha}{1-\alpha\beta}\right)\left(\frac{1-\beta}{1-\alpha\beta}\right)\alpha\beta\mathcal{G}_{\alpha\beta}$$

- coupled system of integral equations for $\mathcal{G}_\alpha, \mathcal{G}_{\alpha\beta}$
- The $\mathbf{1}$ inserted into \mathcal{M}_α produces $\lambda \ln(1 - \alpha)$ in \mathcal{G}_α which spreads everywhere
- $\frac{1}{\mathcal{G}_{0\alpha}} = \frac{1}{1 + \alpha^2 \mathcal{G}_\alpha}$ becomes singular at some $0 < \alpha(\lambda) < 1$ for any $\lambda < 0$.

This could be cancelled by a **common zero** of $\mathcal{M}_\alpha - \mathcal{L}_\alpha + \alpha \mathcal{N}_{\alpha 0}$, which is **hard to control**.

New strategy

- To avoid $\ln(1-\alpha)$ we need $\mathcal{G}_\alpha = -1 + \mathcal{S}_\alpha$ with $\lim_{\alpha \rightarrow 1} \mathcal{S}_\alpha = 0$
- This additional condition distinguishes one special value of $\lambda = \lambda_0$ at which we want to prove existence of the theory.

ansatz $\mathcal{G}_{\alpha\beta} = -2 - \alpha\mathcal{G}_\alpha - \beta\mathcal{G}_\beta + \mathcal{T}_{\alpha\beta}$ with $\lim_{\alpha \rightarrow 1} \mathcal{T}_{\alpha\beta} = 0$

$\lim_{\alpha \rightarrow 1} \mathcal{S}_\alpha = 0$ equivalent to

$$-1 = \frac{\lambda}{1 + \frac{\lambda}{2}} \left(\int_0^1 d\rho \frac{\rho^2 \mathcal{S}_\rho}{1 - \rho} - 3 \int_0^1 d\rho \rho^2 \mathcal{S}_\rho + \int_0^1 d\rho \rho \mathcal{T}_{0\rho} \right) \quad (*)$$

- (*) is intrinsically non-perturbative
- insert (*) back into equation for $\mathcal{G}_\alpha = -1 + \mathcal{S}_\alpha$:

$$\begin{aligned}
& \mathcal{S}_\alpha + \frac{\lambda}{1 + \frac{\lambda}{2}} \int_0^1 d\rho K(\alpha, \rho) \mathcal{S}_\rho \\
&= \frac{\lambda}{1 + \frac{\lambda}{2}} \left(-(1 - \alpha) \mathcal{Y} + (1 - \alpha) \mathcal{S}_\alpha \mathcal{Y} \right. \\
&\quad \left. - \int_0^1 d\rho \frac{1}{\alpha} \left(\frac{(1 - \alpha)^2}{(1 - \alpha\rho)^3} \rho \mathcal{T}_{\alpha\rho} - \rho(1 - \alpha) \mathcal{T}_{0\rho} \right) \right)
\end{aligned}$$

with

$$\begin{aligned}
K(\alpha, \rho) &= \frac{1}{1 - \rho} \frac{\rho^5 (1 - \alpha)^3}{(1 - \alpha\rho)^3} + \frac{(1 - \alpha)(1 - \rho)^2}{(1 - \alpha\rho)^3} \rho^2 - 3 \frac{(1 - \alpha)}{1 - \alpha\rho} \rho^2 \\
\mathcal{Y} &= 1 + 3 \int_0^1 d\rho \rho^2 \mathcal{S}_\rho - \int_0^1 d\rho \rho \mathcal{T}_{0\rho}
\end{aligned}$$

- integral operator K is unbounded, rhs non-linear
- but K is bounded on functions vanishing polynomially at 1:

$$\left| \int_0^1 d\rho K(\alpha, \rho) (1 - \rho)^\nu \right| \leq K_\nu (1 - \alpha)^\nu, \quad 0 < \nu \leq 1$$

We put $\mathcal{S}_\alpha = (1 - \alpha)g(\alpha)$ and $\tilde{K}(\alpha, \rho) = \frac{(1-\rho)}{(1-\alpha)}K(\alpha, \rho)$

Lemma

$(\text{id} + \frac{\lambda}{1+\frac{\lambda}{2}}\tilde{K}) : C([0, 1]) \rightarrow C([0, 1])$ is invertible for $|\lambda| < \frac{3}{7}$, with

$$\left\| \frac{\lambda}{1 + \frac{\lambda}{2}} \left(\text{id} + \frac{\lambda}{1 + \frac{\lambda}{2}} \tilde{K} \right)^{-1} \right\| \leq \frac{|\lambda|}{1 - \frac{7}{3}|\lambda|}$$

We put $\mathcal{T}_{\alpha\rho} = 0$ and define recursively $g_0 = 0$

$$g_{n+1} = \frac{\lambda}{1+\frac{\lambda}{2}} \left(\text{id} + \frac{\lambda\tilde{K}}{1+\frac{\lambda}{2}} \right)^{-1} (\ell g_n - 1) \left(1 + 3 \int_0^1 d\rho \rho^2 (1 - \rho) g_n(\rho) \right)$$

with $\ell(\alpha) = 1 - \alpha$

Proposition

Let $\mathcal{T}_{\alpha\rho} = 0$ and $|\lambda| < \frac{12}{55} = 0.218$. Then:

- 1 The sequence (g_n) is uniformly convergent to $g = \lim_{n \rightarrow \infty} g_n \in C([0, 1])$.
- 2 $\mathcal{S}_\alpha = (1 - \alpha)g(\alpha) \in C_0([0, 1])$ is the unique solution of our integral equation, with

$$|\mathcal{S}_\alpha| \leq \frac{12 - 43|\lambda| - \sqrt{(12 - 55|\lambda|)(12 - 31|\lambda|)}}{6|\lambda|} (1 - \alpha) \leq \frac{55}{6} |\lambda| (1 - \alpha)$$

- 3 $G_{0\alpha} \geq (1 - \alpha)(1 + \alpha - \frac{55}{6} |\lambda| \alpha^2) > 0$ for all $\alpha \in [0, 1[$.

Equation for $\mathcal{T}_{\alpha\beta}$ is regular in first approximation!

With careful discussion of signs, this extends to $-\frac{6}{17} < \lambda \leq \frac{6}{5}$.

The next steps

- 1 establish differentiability of \mathcal{S}_α to control $\int_0^1 d\rho \frac{\mathcal{S}_\rho - \mathcal{S}_\alpha}{\rho - \alpha}$
- 2 interpret equation for \mathcal{T} as recursion $\mathcal{T}^{n+1}(\mathcal{T}^n, \mathcal{S}^n)$ with $\mathcal{T}^0 = 0$ and \mathcal{S}^0 from Proposition
- 3 compute \mathcal{T}^1 and re-iterate (g_n) for smaller $|\lambda|$
- 4 iterate the procedure

Vision

The resulting function $G_{\alpha\beta}$ solves the original problem only for

$$\frac{1}{\lambda} = -\frac{1}{2} - \int_0^1 d\rho \frac{\rho^2 \mathcal{S}_\rho}{1 - \rho} + 3 \int_0^1 d\rho \rho^2 \mathcal{S}_\rho - \int_0^1 d\rho \rho \mathcal{T}_{0\rho}$$

This equation defines the value of λ at which there is a realistic chance to prove **non-perturbative existence of the theory**.