An overview of quantum field theory

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Classical field theory on $\Omega \subset \mathbb{R} \times \mathbb{R}^3$

- classical fields $\phi = \phi(t, \boldsymbol{x}) \in W_0^{k,p}(\Omega)$ (Sobolev space)
- classical action: non-linear functional $S : \mathcal{W}^{k,p}(\Omega) \to \mathbb{R}$
- main example: (free) scalar field theory

$$S[\phi] = \int_\Omega dt\, d\boldsymbol{x} \Big(\frac{1}{2}\Big(\frac{\partial \phi}{\partial t}\Big)^2 - \frac{1}{2}\sum_{i=1}^{3} \Big(\frac{\partial \phi}{\partial x^i}\Big)^2 + \frac{m^2}{2}\phi^2 \Big)$$

regard $\phi(t, \boldsymbol{x})$ as coordinates, define conjugate momenta by

functional derivative $\pi(\boldsymbol{x}, t) := \dfrac{\delta S}{\delta(\partial_t \phi(t, \boldsymbol{x}))}$

- Fréchet derivative $S[v+h] - S[v] = S'[v] \circ h + o(\|h\|)$
- identify linear functional $S'[v] \in (W_0^{k,p}(\Omega))^*$ with $\frac{\delta S}{\delta v} \in W_0^{k,q}(\Omega)$ through $S'[v] \circ h = \langle \frac{\delta S}{\delta v}, h \rangle$ for bilinear form $\langle v, w \rangle = \int_\Omega dt\, d^3\boldsymbol{x}\; v(t, \boldsymbol{x}) w(t, \boldsymbol{x})$ on $W_0^{k,p}(\Omega) \times W_0^{k,q}(\Omega)$

Quantisation and particle interpretation

We promote $\phi(t,\boldsymbol{x})$ and $\pi(t,\boldsymbol{x})$ to (unbounded) operators on Hilbert space and require canonical commutation relations.

Expansion w.r.t solutions $f_{\boldsymbol{k}}(t,\boldsymbol{x}) = e^{\pm \mathrm{i}(\omega_{\boldsymbol{k}} t - \boldsymbol{k}\cdot\boldsymbol{x})}$ of Klein-Gordon equation $\big(\frac{\partial^2}{\partial t^2} - \sum_{i=1}^3 \frac{\partial^2}{\partial x^i \partial x^i} + m^2\big) f_{\boldsymbol{k}}(t,\boldsymbol{x}) = 0$ yields

$$\phi(t,\boldsymbol{x}) = \underbrace{\int_{\mathbb{R}^3} d\mu_{\boldsymbol{k}}\, e^{-\mathrm{i}(\omega_{\boldsymbol{k}} t - \boldsymbol{k}\cdot\boldsymbol{x})} a(k)}_{\phi^-(t,\boldsymbol{x})} + \underbrace{\int_{\mathbb{R}^3} d\mu_{\boldsymbol{k}}\, e^{\mathrm{i}(\omega_{\boldsymbol{k}} t - \boldsymbol{k}\cdot\boldsymbol{x})} a^\dagger(k)}_{\phi^+(t,\boldsymbol{x})}$$

for measure $d\mu_{\boldsymbol{k}} = \dfrac{d\boldsymbol{k}}{2\omega_{\boldsymbol{k}}(2\pi)^3}$ and $\omega_{\boldsymbol{k}} = \sqrt{|\boldsymbol{k}|^2 + m^2}$

$\phi(t,\boldsymbol{x})$ and $\pi(t,\boldsymbol{x}) = \partial_t \phi(t,\boldsymbol{x})$ solve Klein-Gordon-equation

Creation and annihilation operators

$$[a_{\boldsymbol{k}}, a_{\boldsymbol{l}}] = 0 \quad [a^\dagger_{\boldsymbol{k}}, a^\dagger_{\boldsymbol{l}}] = 0 \quad [a_{\boldsymbol{k}}, a^\dagger_{\boldsymbol{l}}] = 2\omega_{\boldsymbol{k}}(2\pi)^3\delta(\boldsymbol{k}-\boldsymbol{l})$$

yields equal-time commutation relations

$[\phi(t,\boldsymbol{x}),\phi(t,\boldsymbol{y})] = 0$ $\qquad$ $[\pi(t,\boldsymbol{x}),\pi(t,\boldsymbol{y})] = 0$

$[\phi(t,\boldsymbol{x}),\pi(t,\boldsymbol{y})] = \delta(\boldsymbol{x}-\boldsymbol{y})$

Vacuum

declare vacuum $|0\rangle$ by $a_{\boldsymbol{k}}|0\rangle = 0$

$\phi^+(t,\boldsymbol{x})|0\rangle \equiv \phi(t,\boldsymbol{x})|0\rangle$ describes a state with a single particle created at time t at place $\boldsymbol{x}$ out of the vacuum

To be precise, these are operator-valued distributions which have to be evaluated on test functions:

$$a_f = \int_{\mathbb{R}^3} d\mu_{\boldsymbol{p}} f(\boldsymbol{p}) a_{\boldsymbol{p}} \qquad a^\dagger_f = \int_{\mathbb{R}^3} d\mu_{\boldsymbol{p}} f(\boldsymbol{p}) a^\dagger_{\boldsymbol{p}}$$

Then: $[a_f, a_g] = 0 \qquad [a^\dagger_f, a^\dagger_g] = 0 \qquad [a_f, a^\dagger_g] = \int d\mu_{\boldsymbol{p}} f(\boldsymbol{p}) g(\boldsymbol{p})$

n-particle states

$$\begin{aligned}\phi^+(t_2,\boldsymbol{x}_2)\phi^+(t_1,\boldsymbol{x}_1)|0\rangle &= \phi(t_2,\boldsymbol{x}_2)\phi(t_1,\boldsymbol{x}_1)|0\rangle \\ &\quad -[\phi^-(t_2,\boldsymbol{x}_2),\phi^+(t_1,\boldsymbol{x}_1)]|0\rangle\end{aligned}$$

- state with two particles created first at $(t_1,\boldsymbol{x}_1)$, then at $(t_2,\boldsymbol{x}_2)$
- causality requires $t_1 < t_2$
- commutator (at non-equal time) arises:
$$\begin{aligned}\mathrm{i}\Delta(x_2-x_1) &:= [\phi^-(t_2,\boldsymbol{x}_2),\phi^+(t_1,\boldsymbol{x}_1)] \\ &= \int d\mu_{\boldsymbol{k}}\, \mathrm{e}^{-\mathrm{i}(\omega_{\boldsymbol{k}}(t_2-t_1)-\boldsymbol{k}\cdot(\boldsymbol{x}_2-\boldsymbol{x}_1))}\end{aligned}$$

A general n-particle state can be expressed as linear combination of $\phi(t_j,\boldsymbol{x}_j)\cdots\phi(t_2,\boldsymbol{x}_2)\phi(t_1,\boldsymbol{x}_1)|0\rangle$, with $t_1 < t_2 < \cdots < t_j$, with coefficients in polynomials of $\Delta(x-x')$

Transition amplitudes

Definition

The probability amplitude that a state
$|\psi_1\rangle = \phi^+(t_n, \boldsymbol{x}_n) \cdots \phi^+(t_1, \boldsymbol{x}_1)|0\rangle$ *evolves later into a state*
$|\psi_2\rangle = \phi^+(t'_m, \boldsymbol{x}'_m) \cdots \phi^+(t'_1, \boldsymbol{x}'_1)|0\rangle$ *is their scalar product*

$$\langle\psi_2|\psi_1\rangle = \langle 0|\phi^-(t'_1, \boldsymbol{x}'_1) \cdots \phi^-(t'_m, \boldsymbol{x}'_m)\phi^+(t_n, \boldsymbol{x}_n) \cdots \phi^+(t_1, \boldsymbol{x}_1)|0\rangle$$

- the ϕ^- can be rearranged so that field operators have increasing time argument from right to left
- interpretation: particles previously created via ϕ^+ are annihilated by ϕ^- in causal order, until vacuum is reached
- completion of $\phi^\pm$ to ϕ at expense of commutators
- $[\phi^-(t_2, \boldsymbol{x}_2), \phi^+(t_1, \boldsymbol{x}_1)] = \langle 0|\phi(t_2, \boldsymbol{x}_2)\phi(t_1, \boldsymbol{x}_1)|0\rangle$

Result: The theory is completely characterised by vacuum expectation values of time-ordered products of field operators

$$\langle 0|\phi(t_n, \boldsymbol{x}_n) \cdots \phi(t_1, \boldsymbol{x}_1)|0\rangle, \qquad t_1 < \cdots < t_n$$

Computation

1. write field operators in correct time order (*T*-symbol)
2. decompose $\phi = \phi^+ + \phi^-$
3. commute ϕ^- to the right, ϕ^+ to the left at expense of Δ's

Wick's theorem

e.g. $n = 4$:

$$\begin{aligned}\langle 0|T\phi_1\phi_2\phi_3\phi_4|0\rangle &= \langle 0|T\phi_1\phi_2|0\rangle\langle 0|T\phi_3\phi_4|0\rangle \\ &+ \langle 0|T\phi_1\phi_3|0\rangle\langle 0|T\phi_2\phi_4|0\rangle \\ &+ \langle 0|T\phi_1\phi_4|0\rangle\langle 0|T\phi_2\phi_3|0\rangle\end{aligned}$$

Consequence: It suffices to compute two-point functions!

For free scalar field they are given by Feynman propagator

$$\langle 0|T\phi(t,\boldsymbol{x})\phi(s,\boldsymbol{y})|0\rangle = \lim_{\epsilon\to 0}\int \frac{dk_0\,d\boldsymbol{k}}{(2\pi)^4}\,\frac{\mathrm{e}^{-\mathrm{i}(k_0(t-s)-\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{y}))}}{k_0^2-\|\boldsymbol{k}\|^2-m^2+\mathrm{i}\epsilon}$$

Wightman axioms

1. Unitary representation U of (cover of) Poincaré group G in Hilbert space $\mathcal{H}$, and vacuum state $|0\rangle \in \mathcal{H}$ fixed by U
2. spectrum condition $P_0 \geq 0$ and $P_0^2 - \sum_{j=1}^3 P_j^2 \geq 0$
3. quantum fields $\phi_i(f)$ as operator-valued tempered distributions defined on dense invariant domain $D \subset \mathcal{H}$, and $|0\rangle \in D$ is cyclic
4. transformation law $U(g)\phi_j(f)U(g^{-1}) = \sum_i R_{ji}(g^{-1})\phi_i(gf)$, with $(gf)(x) = f(g^{-1}x)$ and R_{ij} a finite-dimensional repr.
5. locality and micro-causality, i.e. fields localised in space-like separated regions (anti-)commute
6. scattering and asymptotic completeness $\mathcal{H} = \mathcal{H}_{in} = \mathcal{H}_{out}$ (existence of S-matrix)

Wightman reconstruction theorem

Wightman functions $\langle 0|\phi(f_1)\dots\phi(f_n)|0\rangle$ satisfying

- regularity,
- spectrum condition,
- relativistic transformation law,
- local commutativity,
- cluster property

suffice to reconstruct the theory

Remarks:

- time-ordered vacuum expectation values also follow
- PCT-theorem and spin-statistics theorem are proven in this setting

BUT

In 4 dimensions, no realisation of these axioms is known apart from free fields!

Interacting QFTs

$$S[\phi] = \int dt\, d\boldsymbol{x} \Big(\frac{1}{2}\Big(\frac{\partial\phi}{\partial t}\Big)^2 - \frac{1}{2}\sum_{i=1}^{3}\Big(\frac{\partial\phi}{\partial x^i}\Big)^2 + \frac{m^2}{2}\phi^2 + \frac{\lambda}{4}\phi^4\Big)$$

Key assumption of perturbative QFT

There exists a unitary transformation between field operators of the free and the interacting theory $\phi^{\mathrm{i}}(t,\boldsymbol{x}) = U^{-1}(t)\phi^{\mathrm{f}}(t,\boldsymbol{x})U(t)$

This assumption is false [Haag], but let's ignore this...

- time evolution $\mathrm{i}\frac{\partial U(t)}{\partial t} = H_I(t)U(t)$ governed by interaction Hamiltonian $H_I(t) = \int d\boldsymbol{x}\, \big(-\frac{\lambda}{4!}\phi^4(t,\boldsymbol{x})\big)$
- integral representation
  $$U(t,t') \equiv U(t)U^{-1}(t') = 1 - \mathrm{i}\int_{t'}^{t} dt_1\, H_I(t_1)U(t_1,t')$$
- solution: $U(t,t') = T\exp\Big(-\mathrm{i}\int_{t'}^{t} dt_1\, H_I(t_1)\Big)$

Gell-Mann-Low formula

Insert $\phi^{\mathrm{i}}(t,\boldsymbol{x}) = U^{-1}(t)\phi^{\mathrm{f}}(t,\boldsymbol{x})U(t)$ into $\langle 0^{\mathrm{i}}|T\phi^{\mathrm{i}}\cdots\phi^{\mathrm{i}}|0^{\mathrm{i}}\rangle$

Gell-Mann-Low formula

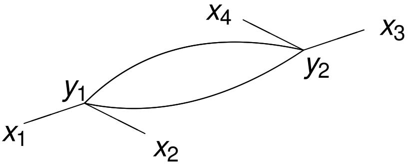
$$\left\langle 0^{\mathrm{i}}\right|T\phi^{\mathrm{i}}(x_1)\dots\phi^{\mathrm{i}}(x_n)\left|0^{\mathrm{i}}\right\rangle = \frac{\left\langle 0^{\mathrm{f}}\right|T\phi^{\mathrm{f}}(x_1)\dots\phi^{\mathrm{f}}(x_n)\,\mathrm{e}^{\mathrm{i}S_{\mathrm{int}}[\phi^{\mathrm{f}}]}\left|0^{\mathrm{f}}\right\rangle}{\left\langle 0^{\mathrm{f}}\right|T\mathrm{e}^{\mathrm{i}S_{\mathrm{int}}[\phi^{\mathrm{f}}]}\left|0^{\mathrm{f}}\right\rangle}$$

expand

$$\mathrm{e}^{\mathrm{i}S_{\mathrm{int}}[\phi^{\mathrm{f}}]} = \sum_{n=0}^{\infty}\frac{\mathrm{i}^n\lambda^n}{(4!)^n n!}\int_{(\mathbb{R}^4)^n} dx_1\dots dx_n\,(\phi(x_1))^4\cdots(\phi(x_n))^4$$

- computation reduced to free functions $\langle 0^{\mathrm{f}}|T\phi^{\mathrm{f}}\cdots\phi^{\mathrm{f}}|0^{\mathrm{f}}\rangle$
- space-time integrals over products of free two-point functions

This corresponds to Feynman graphs with edges labelled by Feynman propagator $\Delta_F(x,y) = \langle 0^{\mathrm{f}} | T\phi^{\mathrm{f}}(x)\phi^{\mathrm{f}}(y) | 0^{\mathrm{f}} \rangle$, e.g.



$$= \Delta_F(x_1,y_1)\Delta_F(x_2,y_1)\Delta_F(x_3,y_2)\Delta_F(x_4,y_2)\cdot\big(\Delta_F(y_1,y_2)\big)^2$$

Divergences

Feynman propagator is a distribution, its product is ill-defined (such as $(\Delta_F(y_1,y_2))^2$)

Epstein-Glaser method to extend the product again to a distribution, respecting locality

Attention: These graphs are not computed in momentum space! The "usual" rules rely on the Euclidean trick!

Euclidean quantum field theory

Wick rotation

time t and energy k_0 turned into complex variables, analytic continuation to imaginary axis $\mathrm{i}\mathbb{R}$ to achieve positivity

Wick rotation of Gell-Mann–Low leads to Schwinger functions

$$S(x_1,\dots,x_n)=\frac{1}{Z}\int d\mu_C(\phi)\;\phi(x_1)\dots\phi(x_n)e^{-S_{int}(\phi)}$$

$$Z=\int d\mu_C(\phi)\;e^{-S_{int}(\phi)}\qquad \text{(partition function)}$$

$$d\mu_C(\phi)=\prod_{x\in\mathbb{R}^4} d\phi(x)\;e^{-\int_{\mathbb{R}^4\times\mathbb{R}^4} dx\,dy\,\frac{1}{2}\phi(x)C^{-1}(x,y)\phi(y)}$$

Euclidean measure $d\mu_C(\phi)$ rigorously defined due to Minlos' theorem for positive covariances C; for scalar fields this is

$$C(x,y)=\int_{\mathbb{R}^4}\frac{dp}{(2\pi)^4}\,\frac{e^{-ip(x-y)}}{p^2+m^2}=\int_0^\infty\frac{d\alpha}{(4\pi\alpha)^2}\,e^{-\alpha m^2-\frac{\|x-y\|^2}{4\alpha}}$$

Connected functions and 1PI functions

- $Z(J) = \int d\mu_C(\phi)\; e^{-S_{int}(\phi)+\int_{\mathbb{R}^4} dx\, \phi(x)J(x))}$
- $S(x_1,\dots,x_n) = \dfrac{1}{Z(0)}\left(\dfrac{\delta^n Z(J)}{\delta J(x_1)\dots\delta J(x_n)}\right)_{J=0}$

Starting point for perturbation theory and Feynman graphs:

$$Z(J) = e^{-S_{int}(\frac{\delta}{\delta J})} \int d\mu_C(\phi)\; e^{\int_{\mathbb{R}^4} dx\, \phi(x)J(x)}$$

$$= N\, e^{-S_{int}(\frac{\delta}{\delta J})} e^{-\frac{1}{2}\int_{\mathbb{R}^4\times\mathbb{R}^4} dx\, dy\, J(x)C(x,y)J(y)}$$

connected graphs: $W(J) = -\log Z(J)$

one-particle irreducible (1PI) graphs

- define classical fields $\phi_{c\ell}(x) = \frac{\delta W(J)}{\delta J(x)}$
- Legendre transformation

$$\Gamma(\phi_{c\ell}) = \int_{\mathbb{R}^4} dx\; \phi_{c\ell}(x)J(x) - W(J)\Big|_{J=J(\phi_{c\ell})}$$

Osterwalder-Schrader axioms

deduce required properties of Wightman functions from Schwinger functions

1. Regularity
2. Euclidean covariance
3. Reflection positivity: The expectation value of $F(\phi)\overline{F^r(\phi)}$ is positive for any function $F(\phi)$, where $F^r(\phi)$ is the reflection on any hyperplane.
   (This guarantees the locality axiom.)
4. Symmetry
5. Cluster property. Asymptotic factorisation of Schwinger functions if arguments are far away.
   (This guarantees uniqueness of the vacuum.)

Regularisation

Constructing the complete measure $\frac{1}{Z}e^{-S_{int}(\phi)}d\mu_C(\phi)$ would establish Wightman axioms. This fails due to divergences.

Need two regularisations

1. UV-cutoff $C(x,y) = \int_{1/\Lambda}^{\infty} d\alpha \dots$
2. IR-cutoff $S_{int} = \frac{\lambda}{4!}\int_\Omega dx\, \phi^4(x)$

All regularised Schwinger functions $S_{\Lambda,\Omega}$ exist

At the end, we have to send for physically meaningful quantities $\Omega \to \mathbb{R}^4$ (thermodynamic limit) and $\Lambda \to \infty$ (continuum limit).

- easy for Ω: connected functions
- difficult for Λ: needs renormalisation (enough for 1PI functions)
  This can be carried out in perturbation theory only.

Renormalisation

Theory depends on parameters m (covariance) and λ (interaction)

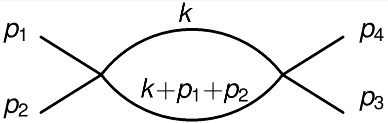
renormalisation of parameters in vertex functions

1. find relevant functions $\Gamma_m(m,\lambda,\Lambda)$, $\Gamma_\lambda(m,\lambda,\Lambda)$
2. solve (m,λ) for $(\Gamma_m,\Gamma_\lambda)$ at given Λ
3. prove for fixed $\Gamma_m = m_r, \Gamma_\lambda = \lambda_r$ existence of
$$\lim_{\Lambda\to\infty} S_{\{m(\Gamma_m,\Gamma_\lambda,\Lambda),\lambda(\Gamma_m,\Gamma_\lambda,\Lambda),\Lambda\}}(x_1,\dots,x_n)\Big|_{\Gamma_m=m_r,\Gamma_\lambda=\lambda_r}$$

This can be done by perturbative expansion in λ:

- all $S_{\{m,\lambda,\Lambda\}}(x_1,\dots,x_n)$ including Γ_m,Γ_r become formal power series in λ, which can be inverted
- recursive existence of renormalised functions established by Hepp's theorem
- global forest formula for recursion [Zimmermann]
- combinatorics of recursion encoded in Hopf algebra [CK]

Example



$$\Gamma_{m,\lambda,\Lambda}(p_1,\dots p_4)$$

$$= \lambda - \frac{\lambda^2}{2}\int_{\mathbb{R}^4_\Lambda} \frac{dk}{(2\pi)^4}\frac{1}{((k+p_1+p_2)^2+m^2)(k^2+m^2)} + \begin{Bmatrix} p_2 \mapsto p_3 \\ p_2 \mapsto p_4 \end{Bmatrix} + \mathcal{O}(\lambda^3)$$

Fact: Relevant function is $\Gamma_\lambda = \Gamma_{m,\lambda,\Lambda}(0,\dots,0)$.

Its limit for $\mathbb{R}^4_\Lambda \to \mathbb{R}^4$ does not exist, but by fixing normalisation condition $\Gamma_{m,\lambda,\Lambda}(0,\dots,0) = \lambda_r$ we have

$$\lambda_r = \lambda - \frac{3\lambda^2}{2}\int_{\mathbb{R}^4_\Lambda} \frac{dk}{(2\pi)^4}\,\frac{1}{(k^2+m^2)^2} + \mathcal{O}(\lambda^3)$$

The perturbative inversion is

$$\lambda = \lambda_r + \frac{3\lambda_r^2}{2}\int_{\mathbb{R}^4_\Lambda} \frac{dk}{(2\pi)^4}\,\frac{1}{(k^2+m^2)^2} + \mathcal{O}(\lambda_r^3)$$

Inserted back:

$$\Gamma_{m,\lambda,\Lambda}(p_1,\dots p_4)$$
$$= \lambda_r - \frac{\lambda_r^2}{2}\int_{\mathbb{R}^4_\Lambda} \frac{dk}{(2\pi)^4}\Big(\frac{1}{((k+p_1+p_2)^2+m^2)(k^2+m^2)} - \frac{1}{(k^2+m^2)^2}\Big)$$
$$+ (p_2 \mapsto p_3) + (p_2 \mapsto p_4) + \mathcal{O}(\lambda_r^3)$$

- Save removal of cutoff: The integral exists for $\mathbb{R}^4_\Lambda \to \mathbb{R}^4$ (Zimmermann's Taylor subtraction under the integral.)
- $\lim_{\Lambda\to\infty} \lambda(\lambda_r, \Lambda)$ remains ill-defined, but λ not observable.
- Perturbatively renormalisable models require a finite number of such re-normalisations
- Any renormalisation scheme must for same set of normalisation conditions give the same value for observables

Parametric representation

propagator with momentum k_ℓ is $C(k_\ell) = \int_{1/\Lambda}^{\infty} d\alpha_\ell e^{-(k^2+m^2)\alpha_\ell}$

factor at vertex v is momentum conservation:

$$\delta\Big(\sum_\ell \epsilon_{v\ell} p_\ell + \sum_e \eta_{ve} p_e\Big) = \int_{\mathbb{R}^4} \frac{d\xi_v}{(2\pi^4)} e^{\mathrm{i}\sum_\ell \epsilon_{v\ell}p_\ell + \mathrm{i}\sum_e \eta_{ve}p_e}$$

resulting exponent is positive quadratic form in (p_ℓ, ξ_v); Gauß:

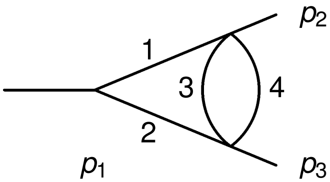
Kirchhoff-Symanzik formula

$$A_G = \int_{1/\Lambda}^{\infty} \frac{d\alpha_1 \dots d\alpha_I}{U(\alpha)^{\frac{4}{2}}} e^{-\frac{V(\alpha,p)}{U_\alpha} - m^2 \sum_\ell \alpha_\ell}$$

$$U(\alpha) = \sum_{\text{s-trees } T\in G} \Big(\prod_{\ell\notin T} \alpha_\ell\Big) \qquad V(\alpha,p) = \Big(\sum_{\text{2-trees } T_2\in G} \Big(\prod_{\ell\notin T_2} \alpha_\ell\Big)\Big)\Big(\sum_{e\in T_1} p_e\Big)^2$$

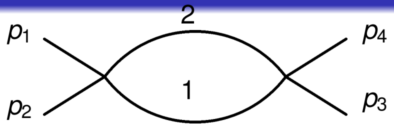
The zero-locus $U(\alpha) = 0$ contains number-theoretical data!

Example



- spanning trees: $(13),(14),(23),(24),(12)$. not: (34)
  $U(\alpha)=(\alpha_1+\alpha_2)(\alpha_3+\alpha_4)+\alpha_3\alpha_4$
- 2-trees: $1,2,3,4$
  $V(p,\alpha)=\alpha_2\alpha_3\alpha_4 p_3^2+\alpha_1\alpha_3\alpha_4 p_2^2+\alpha_1\alpha_2(\alpha_3+\alpha_4)p_1^2$

The other example



- spanning trees: 1, 2 $\Rightarrow$ $U(\alpha) = \alpha_1 + \alpha_2$
- 2-trees: $\varnothing$ $\Rightarrow$ $V(p,\alpha) = \alpha_1\alpha_2 q^2$, $q := p_1 + p_2$

$$\Gamma = \int_{\mathbb{R}^4_\Lambda} \frac{dk}{(2\pi)^4}\Big(\frac{1}{((k+q)^2+m^2)(k^2+m^2)} - \frac{1}{(k^2+m^2)^2}\Big)$$

$$= -\frac{1}{16\pi^2}\int_0^\infty \frac{d\alpha_1\, d\alpha_2}{(\alpha_1+\alpha_2)^2}\Big(1 - e^{-\frac{\alpha_1\alpha_2 q^2}{\alpha_1+\alpha_2}}\Big) e^{-m^2(\alpha_1+\alpha_2)}$$

from $0 < 1 - e^{-x} < x$ for $x > 0$: $-\frac{1}{96\pi^2}\frac{q^2}{m^2} < \Gamma < 0$

The result is finite for fixed q, but might be problematic as a subgraph for large q. More precisely: $\Gamma \sim -\text{const}\cdot\log\frac{p^2+m^2}{m^2}$

Results in QED

Most famous is the magnetic moment of the electron:

$$\frac{g}{2} = 1 + \frac{1}{2}\frac{\alpha}{\pi} + \Big\{\frac{197}{144} + \Big(\frac{1}{2} - 3\log 2\Big)\zeta(2) + \frac{3}{4}\zeta(3)\Big\}\Big(\frac{\alpha}{\pi}\Big)^2$$
$$+ \Big\{\frac{83}{72}\pi^2\zeta(3) - \frac{215}{24}\zeta(5) + \frac{100}{3}\Big(\mathrm{Li}_4(\tfrac{1}{2}) - \frac{\pi^2 - 1}{24}\log^2(2)\Big)$$
$$- \frac{239}{2160}\pi^4 + \frac{139}{18}\zeta(3) - \frac{298}{9}\pi^2\log(2) + \frac{17101}{810}\pi^2 + \frac{28259}{5184}\Big\}\Big(\frac{\alpha}{\pi}\Big)^3$$
$$+ \mathcal{O}(\alpha^4)$$

The (squared) electron charge α has to be determined by experiment.
Quantum Hall effect yields $\alpha^{-1} = 137.036003\dots$

(At higher order: mass dependence and QCD corrections)

Up to $\mathcal{O}(\alpha^5)$ one has

$$\frac{g}{2} = \begin{cases} 1.001\,159\,652\,146\,5 & (\textit{theory}) \\ 1.001\,159\,652\,188\,3 & (\textit{experiment}) \end{cases}$$

Remarkable agreement between theory and experiment, but. . .

Fact

It is useless to compute g to much higher order because the perturbation series necessarily has zero radius of convergence!

First terms of asymptotic series give reasonable approximation, but at some order it gets worse

QED is not capable to describe Nature!

Constructive renormalisation theory

Why the perturbative expansion cannot converge:

toy model: ϕ^4 in dimension zero

$$y_{2n}(\lambda) := \int_{\mathbb{R}} dx\; x^{2n}\, e^{-x^2-\frac{\lambda}{4}x^4} = \sum_{k=0}^{\infty} \frac{(-1)^k \lambda^k}{k! 4^k} \int_{-\infty}^{\infty} dx\; x^{2n+4k} e^{-x^2}$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k 2^n \Gamma(k+\frac{n}{2}+\frac{1}{4})\Gamma(k+\frac{n}{2}+\frac{3}{4})}{\sqrt{2\pi}\,k!}\lambda^k$$

radius of convergence $= 0$; typical situation in QFT

- If radius of convergence was non-zero, then series convergent for some $\lambda < 0$, contradiction.
- Of course, the integral exists for all $\lambda > 0$:

$$0 < y_{2n}(\lambda) < \frac{2^n \Gamma(\frac{n}{2}+\frac{1}{4})\Gamma(\frac{n}{2}+\frac{3}{4})}{\sqrt{2\pi}} \qquad (k=0)$$

Can one give a sense to the perturbation series?

$$y(\lambda)=\sum_{k=0}^{\infty}\frac{(-1)^k 2^n \Gamma(k+\frac{n}{2}+\frac{1}{4})\Gamma(k+\frac{n}{2}+\frac{3}{4})}{\sqrt{2\pi}k!}\lambda^k=\sum_{k=0}^{\infty}y_k\lambda^k$$

Borel transformation

$$(\mathcal{B}y)(z)=\sum_{k=1}^{\infty}\frac{y_k}{(k-1)!}z^{k-1}=-\frac{1}{4}\Gamma(\tfrac{5}{2}+n)\,{}_2F_1(\tfrac{5}{4}+\tfrac{n}{2},\tfrac{7}{4}+\tfrac{n}{2};2;-z)$$

1. $\mathcal{B}y$ has non-zero radius of convergence (here: 1)
2. $\mathcal{B}y$ has analytic continuation in strip containing $\mathbb{R}^+$ to
$$\hat{y}(z)=-\tfrac{1}{4}\Gamma(\tfrac{5}{2}+n)\,{}_2F_1(\tfrac{5}{4}+\tfrac{n}{2},\tfrac{7}{4}+\tfrac{n}{2};2;-z)$$
$$=-\frac{\Gamma(\frac{5}{2}+n)}{4(1+z)^{\frac{5}{4}+\frac{n}{2}}}\,{}_2F_1(\tfrac{5}{4}+\tfrac{n}{2},\tfrac{1}{4}-\tfrac{n}{2};2;\tfrac{z}{1+z})$$
3. $\hat{y}$ is exponentially bounded on $\mathbb{R}^+$ (here: even bounded)

Laplace transform $y_B(\lambda)=y(0)+\int_0^\infty dt\; e^{-\frac{t}{\lambda}}\hat{y}(t)$ exists

Nevanlinna-Sokal theorem

If a function y_B is

1. analytic in a ball tangent in 0 to imaginary axis and
2. possesses in this region an asymptotic expansion $\sum_{k=0}^{\infty} y_k \lambda^k$ with $|y_B(\lambda) - \sum_{k=0}^{N-1} y_k \lambda^k| \leq C\,\sigma^N \lambda^N N!$

then y_B is the unique function with these properties and given by the absolutely convergent Laplace integral of its Borel transform.

- This uniqueness makes Borel summability is as good as standard summability.
  It is the best one can hope for bosonic quantum field theories.
- This was indeed achieved for ϕ^4-theory in 2 and 3 dimensions, but not in 4D!

Constructive renormalisation

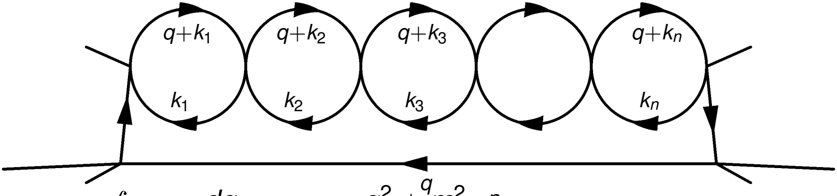
…give a meaning to the asymptotic expansion of QFT by Borel resummation

- establish factorial error bounds
  problem: too many Feynman graphs with n vertices
  – fermions OK due to Pauli principle
  – trees are fine (Cayley: n^{n-2} trees for n vertices)
  – loops are bad

  strategy: only expand trees, hide loops in funcional integral (cluster and Mayer expansion)
- Further problem: renormalons, depends on signs
  – bad for ϕ^4 or QED in 4 dimensions
  – good for non-Abelian Yang-Mills theory (but too difficult) and for some scalar QFT on noncommutative Moyal space

The renormalon problem

renormalised amplitudes are finite order by order in perturbation theory, but also increasingly larger with the order:



$$\sim \lambda^{n+3} \int_{\mathbb{R}^4} \frac{dq}{(q^2+m^2)^3} \Big(\log \frac{q^2+m^2}{m^2} \Big)^n \quad \overset{n\to\infty}{\sim} \quad \lambda^{n+3} c^n n!$$

This is the renormalon problem, it destroys Borel summability of the amplitudes

It is solved for some good models by the renormalisation group

The renormalisation group

The problem: $Z(J) = e^{-S_0(J)} = \int d\mu_C(\phi)\, e^{-S_{int}(\phi,J)}$

Key property of Gaußian random process

If covariance has decomposition $C = \sum_{i=1}^{\rho} C_i$, there is an associated decomposition of the field $\phi = \sum_{i=1}^{\rho} \phi_i$ and of the measure $d\mu_C(\phi) = \bigotimes_{i=1}^{\rho} d\mu_{C_i}(\phi_i)$ into independent pieces

Wilson's idea

Instead of integrating in one huge step over $d\mu_C(\phi)$ we can iteratively integrate over the $d\mu_{C_i}(\phi_i)$

Connected parts define sequence (S_i) of effective actions:

$$e^{-S_{i-1}(\phi_1+\cdots+\phi_{i-1},J)} = \int d\mu_{C_i}(\phi_i)\, e^{-S_i(\phi_1+\cdots+\phi_{i-1}+\phi_i,J)}$$

$$S_\rho = S_{int}$$

Relevant and irrelevant functions

- Perturbatively there is an (infinite) basis (S^b) for these actions by n-point functions with external momenta.
- This leads to sequences (λ_i^b) of effective couplings.
- The step $\lambda_i^b \mapsto \lambda_{i-1}^b$ is controllable by scaling laws.

Result

1. Some initial $\lambda_\rho^{rel,b}$ influence S_0, these are called relevant/marginal couplings.

   The λ_i^{rel} are precisely those which are re-normalised, they must be finitely many.
2. All other $\lambda_\rho^{irrel,b}$ die out, they are called irrelevant.

Invisibility of $\lambda_\rho^{irrel,b}$ in S_0 is the reason why only renormalisable theories are realised in Nature and why gravity is weak.
Existence of gravity indicates new geometry at $\ell_{Planck} = 10^{-33cm}$

Importance of the RG

View this operation as $S_{j-i} = \mathcal{R}_i(S_j)$

- gives semi-group law $\mathcal{R}_{i+i'} = \mathcal{R}_i \circ \mathcal{R}_{i'}$, the renormalisation *group* (RG)
- no inverse because of kernel of irrelevant functions

Also appears in statistical physics: explains why a system with 10^{23} degrees of freedom is described by few thermodynamic quantities such as pressure and temperature

- a rescaling is put into $\mathcal{R}$ so that $S_{-\infty}$ becomes a fixed point of the RG
- fixed point characterised by few relevant parameters
- behaviour near fixed point characterised by critical exponents, fall into universality classes

The β-function

We compute the sequence λ_i for the ϕ^4-coupling constant

decomposition $C(p) = \sum_{i=0}^{\rho} C_i(p)$, with geometric progression

$$C_i(p) = \int_{M^{-i}}^{M^{1-i}} d\alpha\; e^{-\alpha(p^2+m^2)} \qquad \text{for } i = 1, \dots, \rho,\ M > 1$$
$$C_0(p) = \int_1^{\infty} d\alpha\; e^{-\alpha(p^2+m^2)}$$

step from i to $i{-}1$:

$$\lambda_{i-1} = \lambda_i - \beta\lambda_i^2\,, \qquad \beta = \frac{3}{32\pi^2} \sum_{\inf(j,k)=i}^{\rho} \int_{M^{-j}}^{M^{1-j}} \int_{M^{-k}}^{M^{1-k}} \frac{d\alpha_1 d\alpha_2}{(\alpha_1+\alpha_2)^2} > 0$$

- Inversion: $\lambda_{i+1} = \lambda_i + \beta\lambda_i^2$
- For small steps this is differential equation $\frac{\lambda'(i)}{(\lambda(i))^2} = \beta(i)$ with solution

$$\lambda(i) = \frac{\lambda_0}{1 - \lambda_0 \int_0^i dj\ \beta(j)} \overset{\beta=\text{const}}{\longmapsto} \frac{\lambda_0}{1 - \lambda_0\beta i}$$

- $\lambda(i)$ develops a pole for finite i (Landau ghost) unless $\lambda_0 = 0$ (triviality of ϕ^4-theory)

Same problem arises in QED

This running of couplings is experimentally confirmed; electron charge increases with energy ($\alpha^{-1} \approx 127$ at 91 GeV)!

QCD

For QCD one has $\beta = \frac{1}{\pi}\left(-\frac{11\cdot 3}{6} + \frac{N_f}{3}\right) < 0$ for $N_f \leq 16$ flavours due to gluon self-coupling:

$$\lambda(i) = \frac{\lambda_0}{1 + |\beta| i \lambda_0}$$

This means $\lim_{i\to\infty} \lambda(i) = 0$ (asymptotic freedom)

Asymptotic freedom solves the renormalon problem:
Contribution becomes

$$\int \frac{dq}{(2\pi)^4} \frac{1}{(q^2+m^2)^3} (\lambda(q))^{n+3} \log^n\left(\frac{q^2}{m^2}\right)$$

which is uniformly bounded for $\lambda(q) = \mathcal{O}((\log q^2)^{-1})$

There is no rigorous proof so far!