Progress in solving a noncommutative quantum field theory in four dimensions

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(joint work with Harald Grosse, arxiv:0909.1389)

Abstract

We study self-dual ϕ^4-theory on 4D Moyal space.

1. model is perturbatively renormalisable [Grosse-W., 2004]
2. β-function vanishes to all orders [Disertori-Gurau-Magnen-Rivasseau, 2006]

The key for $\beta = 0$ are Ward identities and Schwinger-Dyson equations, but only the singular part is used.

- We extend this to a self-consistent non-perturbative integral equation for the renormalised two-point function alone.
- It can be solved perturbatively, without need of Feynman graphs, BPHZ renormalisation, forest formula etc.
- The solution takes values in a polynom ring. It is generated by iterated integrals labelled by rooted trees. The integrals evaluate to zeta functions and polylogarithms.

Standard Model

- As a classical field theory, it is a noncommutative geometry.
- Ignoring gravity, it gives rise to a perturbatively renormalised quantum field theory.
- Scattering amplitudes can be computed as formal power series in coupling constants such as $e^2 \approx \frac{1}{137}$. The first terms agree to high precision with very expensive experiments.
- The radius of convergence in e^2 is zero! We are far away from understanding the standard model (see e.g. confinement).

- Refined summation techniques (e.g. Borel) may establish reasonable domains of analyticity.
- Unfortunately, this also fails for QED due to the Landau ghost problem.

  It is expected to work for non-Abelian gauge theories because of asymptotic freedom, but these theories are too complicated.

QFT's on noncommutative geometries may provide toy models for non-perturbative renormalisation in four dimensions.

ϕ^4_4-theory on Moyal space with oscillator potential

action functional

$$S[\phi] = \int d^4x \Big(\frac{1}{2}\phi \star \big(-\Delta + \Omega^2 \tilde{x}^2 + \mu^2\big)\phi + \frac{\lambda}{4}\phi \star \phi \star \phi \star \phi\Big)(x)$$

Moyal product $\star$ defined by Θ and $\tilde{x} := 2\Theta^{-1} \cdot x$

parameters: $\mu^2, \lambda \in \mathbb{R}_+$, $\Omega \in [0,1]$, redef'n $\phi \mapsto Z^{\frac{1}{2}}\phi$, $Z \in \mathbb{R}_+$

- renormalisable as formal power series in λ [Grosse-W.]
  means: well-defined perturbative quantum field theory
- β-function vanishes to all orders in λ for $\Omega = 1$
  [Disertori-Gurau-Magnen-Rivasseau]
  means: model is believed to exist non-perturbatively

Up to the sign of μ^2, this model arises from a spectral triple

Harmonic oscillator spectral triple

$\mathcal{A}$ non-unital, but resolvent of $\mathcal{D}$ compact due to oscillator potential

- nilpotent supercharges $\mathcal{Q}, \mathcal{Q}^\dagger$ with
  $\mathcal{Q}\mathcal{Q}^\dagger + \mathcal{Q}^\dagger\mathcal{Q} = \Big(-\frac{\partial^2}{\partial x^2} + \omega^2 x^2\Big)1 + \omega\Sigma$
- two Dirac operators $\mathcal{D}_1 = \mathcal{Q} + \mathcal{Q}^\dagger$ and $\mathcal{D}_1 = \mathrm{i}\mathcal{Q} - \mathrm{i}\mathcal{Q}^\dagger$
- two representations $\gamma_1 = \pi_{\mathcal{D}_1}(\boldsymbol{c})$ and $\gamma_2 = \pi_{\mathcal{D}_2}(\boldsymbol{c})$ of volume form $\boldsymbol{c} \in Z_n(\mathcal{B}, \mathcal{B})$ on $\mathcal{H}$

  product $(-\mathrm{i})^n\gamma_1\gamma_2 = \mathrm{i}^n\gamma_2\gamma_1 = (-1)^{N_f}$ is $\mathbb{Z}_2$-grading!

index formula of [Elliott-Natsume-Nest, 1996]

$\mathcal{D}_i^+$ elliptic ΨDO in the sense of Shubin, of order 1 (in contrast to $\mathrm{i}\gamma^\mu\partial_\mu$)

$$\mathrm{index}\,(\mathcal{D}_i^+) = \frac{1}{(2\pi\mathrm{i})^n n!}\int_{T^*\mathbb{R}^n} \mathrm{tr}\big(\hat{e}_a(d\hat{e}_a)^{2n}\big)$$

e_a – graph projector of $\mathcal{D}_i^+$, $\hat{e}_a = e_a - \begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}$

The action functional for $\Omega = 1$ in matrix basis

- ensemble of selfadjoint large matrices $(\phi_{mn})_{m,n\in\mathbb{N}^2_\Lambda} \in M_\Lambda$ with cut-off Λ in the matrix size
- correlation functions generated by partition function
  $\mathcal{Z}[J] = N \int \Big(\prod_{m,n\in\mathbb{N}^2_\Lambda} d\phi_{mn} \Big) \; \exp\big(- S[\phi] + \mathrm{tr}(\phi J)\big)$

We are interested in $\mathbb{N}^2_\Lambda \to \mathbb{N}^2$. Correlation functions ill-defined unless $S[\phi]$ is a suitably divergent function of Λ:

$$S[\phi] = \sum_{m,n\in\mathbb{N}^2_\Lambda} \frac{1}{2}\phi_{mn}H_{mn}\phi_{nm} + V(\phi)$$

$$H_{mn} = Z\left(\mu^2_{bare} + |m| + |n|\right) \quad V(\phi) = \frac{Z^2\lambda}{4} \sum_{m,n,k,l\in\mathbb{N}^2_\Lambda} \phi_{mn}\phi_{nk}\phi_{kl}\phi_{lm}$$

with $|m| = m_1 + m_2$ and divergent $\mu_{bare}[\Lambda,\lambda], Z[\Lambda,\lambda]$
There is no separate Λ-dependence in λ!

Ward identity

- inner automorphism $\phi \mapsto U\phi U^\dagger$ of M_Λ, infinitesimally $\phi_{mn} \mapsto \phi_{mn} + \mathrm{i} \sum_{k\in\mathbb{N}^2_\Lambda} (B_{mk}\phi_{kn} - \phi_{mk}B_{kn})$
- not a symmetry of the action, but translation invariance of the measure $\mathcal{D}\phi = \prod_{m,n\in\mathbb{N}^2_\Lambda} d\phi_{mn}$ gives

$$0 = \frac{\delta W}{\mathrm{i}\delta B_{ab}} = \frac{1}{\mathcal{Z}} \int \mathcal{D}\phi \Big(- \frac{\delta S}{\mathrm{i}\delta B_{ab}} + \frac{\delta}{\mathrm{i}\delta B_{ab}} (\mathrm{tr}(\phi J)) \Big) e^{-S+\mathrm{tr}(\phi J)}$$

$$= \frac{1}{\mathcal{Z}} \int \mathcal{D}\phi \sum_n \Big((H_{nb} - H_{an})\phi_{bn}\phi_{na} + (\phi_{bn}J_{na} - J_{bn}\phi_{na}) \Big) e^{-S+\mathrm{tr}(\phi J)}$$

where $W[J] = \ln \mathcal{Z}[J]$ generates connected functions

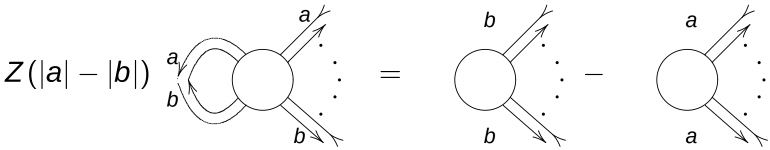
perturbation trick $\phi_{mn} \mapsto \frac{\partial}{\partial J_{nm}}$

$$0 = \Big\{ \sum_n \Big((H_{nb} - H_{an}) \frac{\delta^2}{\delta J_{nb}\, \delta J_{an}} + \Big(J_{na} \frac{\delta}{\delta J_{nb}} - J_{bn} \frac{\delta}{\delta J_{an}} \Big) \Big) \times \exp\Big(- V\big(\tfrac{\delta}{\delta J}\big) \Big) e^{\frac{1}{2} \sum_{p,q} J_{pq} H^{-1}_{pq} J_{qp}} \Big\}_c$$

Interpretation

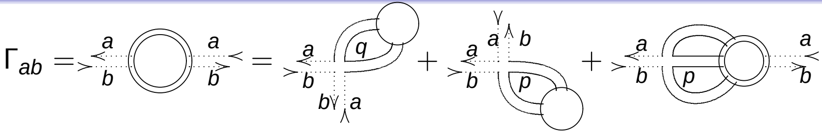
The insertion of a special vertex $V_{ab}^{ins} := \sum_n (H_{an} - H_{nb})\phi_{bn}\phi_{na}$

into an external face of a ribbon graph is the same as the difference between the exchanges of external sources $J_{nb} \mapsto J_{na}$ and $J_{an} \mapsto J_{bn}$

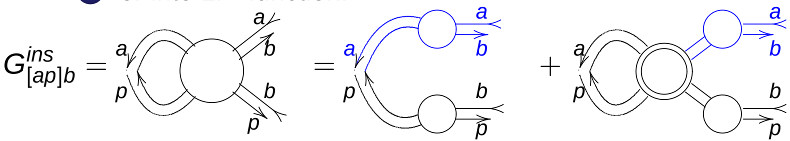


$$Z(|a|-|b|)G^{ins}_{[ab]\dots} = G_{b\dots} - G_{a\dots}$$

Two-point Schwinger-Dyson equation



- vertex is $Z^2\lambda$, connected two-point function is G_{ab}:
  first graph equals $Z^2\lambda \sum_q G_{aq}$
- in other two graphs we open the p-face and compare with insertion into connected two-point function; it inserts
  1. either into one-particle reducible line
  2. or into 1PI function:



- amputation of G_{ab}:
  last two graphs together equal $Z^2\lambda \sum_p G^{-1}_{ab} G^{ins}_{[ap]b}$

Result (using $G^{-1}_{ab} = H_{ab} - \Gamma_{ab}$):

$$\Gamma_{ab} = Z^2\lambda \sum_p \Big(G_{ap} + G^{-1}_{ab} G^{ins}_{[ap]b} \Big) = Z^2\lambda \sum_p \Big(G_{ap} - G^{-1}_{ab} \frac{G_{bp} - G_{ba}}{Z(|p| - |a|)} \Big)$$

$$= Z^2\lambda \sum_p \Big(\frac{1}{H_{bp} - \Gamma_{bp}} + \frac{1}{H_{ap} - \Gamma_{ap}} - \frac{1}{H_{bp} - \Gamma_{bp}} \frac{(\Gamma_{bp} - \Gamma_{ab})}{Z(|p| - |a|)} \Big)$$

- A sort of this formula ($a, b = 0$, $Z = 1$) already appeared in [Disertori-Gurau-Magnen-Rivasseau]. They were interested in the singular part only.
- We focus on the regular part and perform the renormalisation directly in the Schwinger-Dyson equation for Γ_{ab}. The Z-factors are essential for that.
- This is elementary calculus. The difficulty was to identify the optimal quantities in terms of which the equation is simpler. Guiding principle is perturbation theory.

Renormalisation

$$\Gamma_{ab} = Z^2\lambda \sum_p \Big(\frac{1}{H_{bp}-\Gamma_{bp}} + \frac{1}{H_{ap}-\Gamma_{ap}} - \frac{1}{H_{bp}-\Gamma_{bp}} \frac{(\Gamma_{bp}-\Gamma_{ab})}{Z(|p|-|a|)}\Big)$$

- Taylor: $\Gamma_{ab} = Z\mu^2_{bare} - \mu^2 + (Z-1)(|a|+|b|) + \Gamma^{ren}_{ab}$ with $\Gamma^{ren}_{00} = 0$ and $(\partial\Gamma^{ren})_{00} = 0$
- yields $G^{-1}_{ab} = H_{ab} - \Gamma_{ab} = |a| + |b| + \mu^2 - \Gamma^{ren}_{ab}$

mass renormalisation $\Gamma_{ab} - \Gamma_{00}$

$$\begin{aligned}(Z-1)(|a|+|b|) + \Gamma^{ren}_{ab} &= \lambda \sum_p \Big(\frac{Z}{|b|+|p|+\mu^2-\Gamma^{ren}_{bp}} + \frac{Z^2}{|a|+|p|+\mu^2-\Gamma^{ren}_{ap}} - \frac{Z^2+Z}{|p|+\mu^2-\Gamma^{ren}_{0p}} \\ &\quad - \frac{Z}{|b|+|p|+\mu^2-\Gamma^{ren}_{bp}} \frac{\Gamma^{ren}_{bp}-\Gamma^{ren}_{ab}}{|p|-|a|} + \frac{Z}{p+\mu^2-\Gamma^{ren}_{0p}} \frac{\Gamma^{ren}_{0p}}{|p|}\Big)\end{aligned}$$

Integral representation

- We replace discrete indices $a, b, p \in \mathbb{N}^2$ by continuous indices $a, b, p \in (\mathbb{R}_+)^2$, and sums by integrals.
- This captures the behaviour at $\Lambda \to \infty$ of the discrete version, or alternatively defines another field theory which is interesting, too.
- The mass-renormalised Schwinger-Dyson equation depends only on the length $|a| = a_1 + a_2$. Partial derivatives $\frac{\partial}{\partial a_i}$ needed to extract Z are equal. Therefore, Γ^{ren}_{ab} depends only on $|a|$ and $|b|$.
- Hence, $\displaystyle\int_{(\mathbb{R}_+)^2}^{(\Lambda)} dp_1\, dp_2\, f(|p|) = \int_0^\Lambda |p|\, d|p|\, f(|p|)$

$$(Z-1)(|a|+|b|)+\Gamma^{ren}_{ab}$$
$$=\lambda\int_0^\Lambda |p|\,d|p|\Big(\frac{Z}{|b|+|p|+\mu^2-\Gamma^{ren}_{bp}}+\frac{Z^2}{|a|+|p|+\mu^2-\Gamma^{ren}_{ap}}-\frac{Z^2+Z}{|p|+\mu^2-\Gamma^{ren}_{0p}}$$
$$-\frac{Z}{|b|+|p|+\mu^2-\Gamma^{ren}_{bp}}\frac{\Gamma^{ren}_{bp}-\Gamma^{ren}_{ab}}{|p|-|a|}+\frac{Z}{p+\mu^2-\Gamma^{ren}_{0p}}\frac{\Gamma^{ren}_{0p}}{|p|}\Big)$$

- perturbative solution depends on combination $\frac{a}{1+a}$ and $\frac{\Lambda}{1+\Lambda}$
- change of variables

$$|a|=:\mu^2\frac{\alpha}{1-\alpha}\,,\quad |b|=:\mu^2\frac{\beta}{1-\beta}\,,\quad |p|=:\mu^2\frac{\rho}{1-\rho}\,,$$
$$\Gamma^{ren}_{ab}=:\mu^2\frac{\Gamma_{\alpha\beta}}{(1-\alpha)(1-\beta)}\,,\quad \Lambda=:\mu^2\frac{\xi}{1-\xi}$$

- $\frac{\partial}{\partial a_i}\Big|_{a=0}=\frac{\partial}{\partial\alpha}\Big|_{\alpha=0}$ to extract Z

$$(Z-1)\Big(\frac{\alpha}{1-\alpha}+\frac{\beta}{1-\beta}\Big)+\frac{\Gamma_{\alpha\beta}}{(1-\alpha)(1-\beta)}$$
$$=\lambda\int_0^\xi \frac{\rho\, d\rho}{(1-\rho)^2}\Big(\frac{Z^2(1-\alpha)}{1-\alpha\rho-\Gamma_{\alpha\rho}}-\frac{Z^2}{1-\Gamma_{0\rho}}\Big)$$
$$-\lambda\int_0^\xi \frac{d\rho}{(1-\rho)}\Big(\frac{Z(1-\Gamma_{\beta\alpha})}{1-\beta\rho-\Gamma_{\beta\rho}}+\frac{Z\alpha}{1-\beta\rho-\Gamma_{\beta\rho}}\frac{\Gamma_{\beta\rho}-\Gamma_{\beta\alpha}}{\rho-\alpha}-\frac{Z}{1-\Gamma_{0\rho}}\Big)$$

and from β-derivative

$$Z-1=-Z\lambda\int_0^\xi \frac{d\rho}{(1-\rho)}\frac{(\rho+\Gamma'_{0\rho})}{(1-\Gamma_{0\rho})^2}$$

where $\Gamma'_{0\rho}:=\lim_{\alpha\to 0}\frac{\Gamma_{\alpha\rho}-\Gamma'_{0\rho}}{\alpha}$

- suggests to introduce $1-\alpha\beta-\Gamma_{\alpha\beta}=\frac{1-\alpha\beta}{G_{\alpha\beta}}$
- insertion of Z^{-1} and rational fraction expansion

$$G_{\alpha\beta} = 1 + \lambda\Big\{ G_{\alpha\beta}\frac{(1-\beta)}{1-\alpha\beta}\Big(\frac{(1-\alpha)\mathcal{K}^\xi_\alpha - \alpha\mathcal{X}^\xi + \mathcal{M}^\xi_\alpha - \mathcal{L}^\xi_\alpha}{1+\lambda(\mathcal{X}^\xi - \mathcal{Y}^\xi)} - \alpha\ln(1-\xi)\Big)$$
$$+ \Big(\frac{(1-\alpha)(1-\beta)}{1-\alpha\beta}G_{\alpha\beta} - 1\Big)\mathcal{Y}^\xi$$
$$+ \frac{(1-\alpha)}{1-\alpha\beta}(\mathcal{M}^\xi_\beta - \mathcal{L}^\xi_\beta) - \frac{\alpha(1-\beta)}{1-\alpha\beta}(\mathcal{L}^\xi_\beta + \mathcal{N}^\xi_{\alpha\beta})\Big\}$$

with

$$\mathcal{K}^\xi_\alpha := \int_0^\xi d\rho\,\frac{G_{\alpha\rho} - G_{0\rho}}{(1-\rho)^2} \qquad \mathcal{X}^\xi := \int_0^\xi d\rho\,\frac{G_{0\rho}}{(1-\rho)}$$

$$\mathcal{M}^\xi_\alpha := \int_0^\xi d\rho\,\frac{\alpha\, G_{\alpha\rho}}{(1-\alpha\rho)} \qquad \mathcal{L}^\xi_\alpha := \int_0^\xi d\rho\,\frac{G_{\alpha\rho} - G_{0\rho}}{(1-\rho)}$$

$$\mathcal{Y}^\xi := \lim_{\alpha\to 0}\frac{\mathcal{M}^\xi_\alpha - \mathcal{L}^\xi_\alpha}{\alpha} \qquad \mathcal{N}^\xi_{\alpha\beta} := \int_0^\xi d\rho\,\frac{G_{\rho\beta} - G_{\alpha\beta}}{(\rho-\alpha)}$$

- $\mathcal{K}^\xi_\alpha$, $\mathcal{X}^\xi$ and $\ln(1-\xi)$ are singular for $\xi \to 1$, but these singularities cancel!

Lemma

$$G_{\alpha 0}\Big(\frac{(1-\alpha)\mathcal{K}^\xi_\alpha - \alpha\mathcal{X}^\xi + \mathcal{M}^\xi_\alpha - \mathcal{L}^\xi_\alpha}{1+\lambda(\mathcal{X}^\xi - \mathcal{Y}^\xi)} - \alpha\ln(1-\xi)\Big) = \mathcal{M}^\xi_\alpha - \mathcal{L}^\xi_\alpha + \alpha\mathcal{N}^\xi_{\alpha 0}$$

Proof.

$$G_{0\beta} = 1 + \lambda\Big(((1-\beta)G_{0\beta} - 1)\mathcal{Y}^\xi + \mathcal{M}^\xi_\beta - \mathcal{L}^\xi_\beta\Big)$$

$$G_{\alpha 0} = 1 + \lambda\Big(G_{\alpha 0}\Big\{\frac{(1-\alpha)\mathcal{K}^\xi_\alpha - \alpha\mathcal{X}^\xi + \mathcal{M}^\xi_\alpha - \mathcal{L}^\xi_\alpha}{1+\lambda(\mathcal{X}^\xi - \mathcal{Y}^\xi)} - \alpha\ln(1-\xi)\Big\} + ((1-\alpha)G_{\alpha 0} - 1)\mathcal{Y}^\xi - \alpha\mathcal{N}^\xi_{\alpha 0}\Big)$$

Two point function is symmetric. □

Because the model is renormalisable, we can now take the limit $\xi \to 1$. All integrals will exist, at least perturbatively.

Theorem

The renormalised planar connected two-point function $G_{\alpha\beta}$ of self-dual noncommutative ϕ^4_4-theory satisfies

$$G_{\alpha\beta} = 1 + \lambda\Big(\frac{1-\alpha}{1-\alpha\beta}(\mathcal{M}_\beta - \mathcal{L}_\beta - \beta\mathcal{Y}) + \frac{1-\beta}{1-\alpha\beta}(\mathcal{M}_\alpha - \mathcal{L}_\alpha - \alpha\mathcal{Y})$$
$$+ \frac{(1-\alpha)(1-\beta)}{1-\alpha\beta}(G_{\alpha\beta}-1)\mathcal{Y} - \frac{\alpha(1-\beta)}{1-\alpha\beta}(\mathcal{L}_\beta + \mathcal{N}_{\alpha\beta} - \mathcal{N}_{\alpha 0})$$
$$+ \frac{1-\beta}{1-\alpha\beta}\Big(\frac{G_{\alpha\beta}}{G_{0\alpha}} - 1\Big)(\mathcal{M}_\alpha - \mathcal{L}_\alpha + \alpha\mathcal{N}_{\alpha 0})\Big)$$

with $\alpha, \beta \in [0,1)$ and

$$\mathcal{L}_\alpha := \int_0^1 d\rho \, \frac{G_{\alpha\rho} - G_{0\rho}}{1-\rho} \qquad \mathcal{M}_\alpha := \int_0^1 d\rho \, \frac{\alpha \, G_{\alpha\rho}}{1-\alpha\rho}$$

$$\mathcal{N}_{\alpha\beta} := \int_0^1 d\rho \, \frac{G_{\rho\beta} - G_{\alpha\beta}}{\rho - \alpha} \qquad \mathcal{Y} = \lim_{\alpha\to 0} \frac{\mathcal{M}_\alpha - \mathcal{L}_\alpha}{\alpha}$$

- The previous integral equation for Γ_{ab} is non-perturbatively defined. Unfortunately, it resisted an exact treatment.
- We look for an iterative solution $G_{\alpha\beta} = \sum_{n=0}^{\infty} \lambda^n G^{(n)}_{\alpha\beta}$.
- This involves iterated integrals labelled by rooted trees.

Up to $\mathcal{O}(\lambda^3)$ we need

$$I_{\alpha} := \int_0^1 dx\, \frac{\alpha}{1-\alpha x} = -\ln(1-\alpha)\,,$$

$$I_{\overset{\alpha}{\bullet}} := \int_0^1 dx\, \frac{\alpha\, I_x}{1-\alpha x} = \mathrm{Li}_2(\alpha) + \frac{1}{2}\big(\ln(1-\alpha)\big)^2$$

$$I_{\overset{\alpha}{\bullet\ \bullet}} = \int_0^1 dx\, \frac{\alpha\, I_x \cdot I_x}{1-\alpha x} = -2\,\mathrm{Li}_3\Big(-\frac{\alpha}{1-\alpha}\Big)$$

$$I_{\overset{\alpha}{\overset{\bullet}{\bullet}}} = \int_0^1 dx\, \frac{\alpha\, I_{\overset{x}{\bullet}}}{1-\alpha x} = -2\mathrm{Li}_3\Big(-\frac{\alpha}{1-\alpha}\Big) - 2\mathrm{Li}_3(\alpha) - \ln(1-\alpha)\zeta(2)$$
$$+ \ln(1-\alpha)\mathrm{Li}_2(\alpha) + \frac{1}{6}\big(\ln(1-\alpha)\big)^3$$

In terms of I_t and $A := \frac{1-\alpha}{1-\alpha\beta}$, $B := \frac{1-\beta}{1-\alpha\beta}$:

$$
\begin{aligned}
G_{\alpha\beta} = 1 &+ \lambda\Big\{A(I_\beta - \beta) + B(I_\alpha - \alpha)\Big\} \\
&+ \lambda^2\Big\{A\big(\beta I_{\underset{\bullet}{\beta}} - \beta I_\beta\big) - \alpha AB\big((I_\beta)^2 - 2\beta I_\beta + I_\beta\big) \\
&\quad + B\big(\alpha I_{\underset{\bullet}{\alpha}} - \alpha I_\alpha\big) - \beta BA\big((I_\alpha)^2 - 2\alpha I_\alpha + I_\alpha\big) \\
&\quad + AB\big((I_{\underset{\bullet}{\alpha}} - \alpha) + (I_{\underset{\bullet}{\beta}} - \beta) + (I_\alpha - \alpha)(I_\beta - \beta) + \alpha\beta(\zeta(2)+1)\big)\Big\} \\
&+ \lambda^3\Big\{A\mathcal{W}_\beta + \alpha AB\big(-\mathcal{U}_\beta + I_\alpha I_\beta + I_{\underset{\bullet}{\alpha}} I_\beta\big) + \alpha A^2 B\mathcal{V}_\beta \\
&\quad + B\mathcal{W}_\alpha + \beta BA\big(-\mathcal{U}_\alpha + I_\beta I_\alpha + I_{\underset{\bullet}{\beta}} I_\alpha\big) + \beta B^2 A\mathcal{V}_\alpha \\
&\quad + AB\big(\mathcal{T}_\beta + \mathcal{T}_\alpha - I_\beta(I_\alpha)^2 - I_\alpha(I_\beta)^2 - 6I_\alpha I_\beta\big) \\
&\quad + AB^2\big((1-\alpha)(I_{\underset{\bullet}{\alpha}} - \alpha) + 3I_\alpha I_\beta + I_{\underset{\bullet}{\beta}} I_\alpha + I_\beta(I_\alpha)^2\big) \\
&\quad + BA^2\big((1-\beta)(I_{\underset{\bullet}{\beta}} - \beta) + 3I_\alpha I_\beta + I_{\underset{\bullet}{\alpha}} I_\beta + I_\alpha(I_\beta)^2\big)\Big\} + \mathcal{O}(\lambda^4)
\end{aligned}
$$

where

$$\mathcal{T}_\beta := \beta \underset{\bullet}{I}_{\beta} - \beta I_\beta + (I_\beta - \beta) \;,$$

$$\begin{aligned}\mathcal{U}_\beta := & -\beta \underset{\bullet\;\bullet}{I}_{\beta} - (I_\beta)^3 + \beta \underset{\bullet}{I}_{\beta} I_\beta + 2\underset{\bullet}{I}_{\beta} I_\beta + \beta\zeta(2) I_\beta - 2\beta\zeta(3) \\ & - 2(I_\beta)^2 + \beta(I_\beta)^2 + \underset{\bullet}{I}_{\beta} + \beta\underset{\bullet}{I}_{\beta} + 2I_\beta - \beta^2 \;,\end{aligned}$$

$$\begin{aligned}\mathcal{V}_\beta := & \; \beta \underset{\substack{\bullet\\ \bullet}}{I}_{\beta} - \beta^2 \underset{\bullet\;\bullet}{I}_{\beta} - 2\beta^2\zeta(3) + 2\beta \underset{\bullet}{I}_{\beta} I_\beta - I_\beta^3 + 2\beta I_\beta \zeta(2) - 3\beta^2\zeta(2) \\ & + (1-\beta)\big(2\beta \underset{\bullet}{I}_{\beta} - 3I_\beta^2 + 3\beta I_\beta - 3I_\beta + \beta\big) \;,\end{aligned}$$

$$\mathcal{W}_\beta := (\underset{\substack{\bullet\\ \bullet}}{I}_{\beta} - \beta\zeta(2)) - \frac{1}{2} I_\beta \frac{I_\beta - \beta}{\beta} + \frac{1}{2}(I_\beta)^2 - (\underset{\bullet}{I}_{\beta} - \beta) - \frac{1}{2}(I_\beta - \beta) - \frac{1}{2}\beta^2$$

Remark: $\frac{I_\beta - \beta}{\beta} = \displaystyle\int_0^1 dx \; \frac{\beta x}{1 - \beta x}$

(optimal family of iterated integrals not yet determined)

Observations

Polylogarithms and multiple zeta values appear in singular part of individual graphs of e.g. ϕ^4-theory [Broadhurst-Kreimer]
We encounter them for regular part of all graphs together

Conjecture

- $G_{\alpha\beta}$ takes values in a polynom ring with generators $A, B, \alpha, \beta, \{I_t\}$, where t is a rooted tree with root label α or β
- at order n the degree of A, B is $\leq n$,
  the degree of α, β is $\leq n$,
  the number of vertices in the forest is $\leq n$.

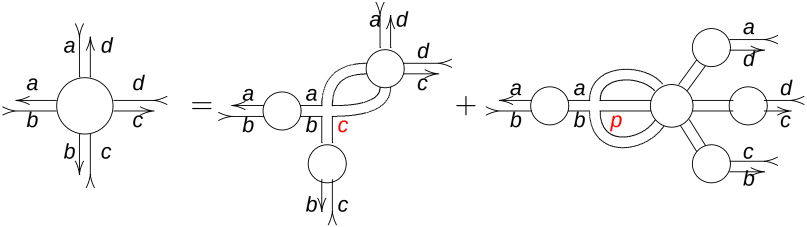
If true:

- There are $n!$ forests of rooted trees with n vertices at order n
- plausible estimation: $|G^{(n)}_{\alpha\beta}| \leq n!(C_{\alpha\beta})^n$

This is a main step to prove Borel summability.

4-point Schwinger-Dyson equation

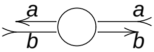
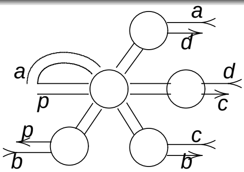
Following the *a*-face, there is a distinguished vertex at which the first *ab*-line starts:

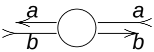
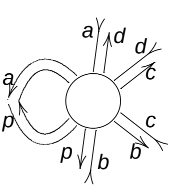
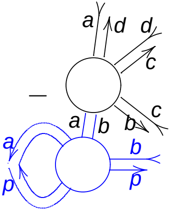


1. First graph has index c at to a opposite corner. It equals $Z^2 \lambda G_{ab} G_{bc} G^{ins}_{[ac]d}$
2. Second graph has summation index p at to a opposite corner. We open the p-face to get an insertion.

This is not into full connected four-point function, which would contain an *ab*-line not present in the graph.

second graph equals

$Z^2\lambda$  $\times \sum_p$ 

$= Z^2\lambda$  $\sum_p \Bigg($  $-$  $\Bigg)$

$$= Z^2\lambda\Big(\sum_p G_{ab}G^{ins}_{[ap]bcd} - G^{ins}_{[ap]b}G_{abcd}\Big)$$

1PI four-point function

$$\Gamma^{ren}_{abcd} = Z\lambda\Big\{\frac{G^{-1}_{ad} - G^{-}_{cd}}{|a|-|c|} + \sum_p \frac{G_{pb}}{|a|-|p|}\Big(\frac{G_{dp}}{G_{ad}}\Gamma^{ren}_{pbcd} - \Gamma^{ren}_{abcd}\Big)\Big\}$$

Theorem

The renormalised planar 1PI four-point function $\Gamma_{\alpha\beta\gamma\delta}$ of self-dual noncommutative ϕ^4_4-theory satisfies

$$\Gamma_{\alpha\beta\gamma\delta} = \lambda \cdot \frac{\begin{array}{c}\Big(1 - \dfrac{(1-\alpha)(1-\gamma\delta)(G_{\alpha\delta}-G_{\gamma\delta})}{G_{\gamma\delta}(1-\delta)(\alpha-\gamma)} \\ + \displaystyle\int_0^1 \rho\, d\rho \frac{(1-\beta)(1-\alpha\delta)G_{\beta\rho}G_{\delta\rho}}{(1-\beta\rho)(1-\delta\rho)} \frac{\Gamma_{\rho\beta\gamma\delta}-\Gamma_{\alpha\beta\gamma\delta}}{\rho-\alpha}\Big)\end{array}}{\begin{array}{c}G_{\alpha\delta}+\lambda\Big((\mathcal{M}_\beta-\mathcal{L}_\beta-\mathcal{Y})G_{\alpha\delta}+\displaystyle\int_0^1 d\rho \frac{G_{\alpha\delta}G_{\beta\rho}(1-\beta)}{(1-\delta\rho)(1-\beta\rho)} \\ + \displaystyle\int_0^1 \rho\, d\rho \frac{(1-\beta)(1-\alpha\delta)G_{\beta\rho}}{(1-\beta\rho)(1-\delta\rho)} \frac{(G_{\rho\delta}-G_{\alpha\delta})}{(\rho-\alpha)}\Big)\end{array}}$$

Corollary

$\Gamma_{\alpha\beta\gamma\delta} = 0$ is not a solution!

We have a non-trivial (interacting) QFT in four dimensions!

Perturbative solution

$$\Gamma_{\alpha\beta\gamma\delta} = \lambda - \lambda^2 \Big(\frac{(1-\gamma)(I_\alpha - \alpha) - (1-\alpha)(I_\gamma - \gamma)}{\alpha - \gamma} + \frac{(1-\delta)(I_\beta - \beta) - (1-\beta)(I_\delta - \delta)}{\beta - \delta} \Big) + \mathcal{O}(\lambda^3)$$

cyclic in the four indices