

Noncommutative Quantum Field Theory

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Spectral triples

see: A. Connes, "On the spectral characterization of manifolds," arXiv:0810.2088

Definition (commutative spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ of dimension $p \in \mathbb{N}$)

... given by a Hilbert space \mathcal{H} , a commutative involutive unital algebra \mathcal{A} represented in \mathcal{H} , and a selfadjoint operator \mathcal{D} in \mathcal{H} with compact resolvent, with

- 1 *Dimension:* k^{th} characteristic value of resolvent of \mathcal{D} is $\mathcal{O}(k^{-\frac{1}{p}})$
- 2 *Order one:* $[[\mathcal{D}, f], g] = 0 \quad \forall f, g \in \mathcal{A}$
- 3 *Regularity:* f and $[\mathcal{D}, f]$ belong to the domain of δ^k , for any $f \in \mathcal{A}$ and $k \in \mathbb{Z}$, where $\delta T := [[\mathcal{D}], T]$
- 4 *Orientability:* \exists Hochschild p -cycle $\mathbf{c} \in Z_p(\mathcal{A}, \mathcal{A})$ s.t. $\pi_{\mathcal{D}}(\mathbf{c}) = 1$ for p odd, $\pi_{\mathcal{D}}(\mathbf{c}) = \gamma$ for p even with $\gamma = \gamma^*$, $\gamma^2 = 1$, $\gamma\mathcal{D} = -\mathcal{D}\gamma$
- 5 *Finiteness and absolute continuity:* $\mathcal{H}_{\infty} := \bigcap_k \text{dom}(\mathcal{D}^k) \subset \mathcal{H}$ is finitely generated projective \mathcal{A} -module, $\mathcal{H}_{\infty} = e\mathcal{A}^n$, with $e = e^* = e^2 \in M_m(\mathcal{A})$. Hermitian structure $(\xi|\eta) = \sum_{i=1}^n a\xi_i^*\eta_i \in \mathcal{A}$ satisfies $\langle \xi, \eta \rangle = f(\xi|\eta)|\mathcal{D}|^{-p}$

Theorem (Connes; 2008)

Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be a *commutative spectral triple* and assume that

- all endomorphisms $T \in \text{End}_{\mathcal{A}}(\mathcal{H}_{\infty})$ are regular,
- the Hochschild cycle \mathbf{c} is antisymmetric.

Then *there exists a compact oriented smooth manifold X such that $\mathcal{A} = C^{\infty}(X)$ is the algebra of smooth functions on X , and every compact oriented smooth manifold appears in this manner.*

It is known since [Milnor; 1964] that there exist **isospectral manifolds which are not isometric**

Connes' proof identifies the missing piece which in addition to the spectrum of \mathcal{D} characterises the geometry:

It is the **analogue of the Cabibbo-Kobayashi-Maskawa-Matrix** in QFT which encodes the relative position of two bases in the same Hilbert space.

Spectral triples are interesting for physics!

- equivalence classes of spectral triples describe **Yang-Mills theory** (inner automorphisms; exist always in nc case) and possibly **gravity** (outer automorphisms)
- **inner fluctuations**: $\mathcal{D} \mapsto \mathcal{D}_A = \mathcal{D} + A$, $A = \sum f[\mathcal{D}, g]$
for almost-commutative manifolds: **A=Yang-Mills+Higgs**

Spectral action principle [Chamseddine+Connes; 1996]

As an automorphism-invariant object, the **(bosonic) action functional of physics** is a function of the **spectrum of \mathcal{D}_A** , i.e.
 $S(\mathcal{D}_A) = \text{Tr}(\chi(\mathcal{D}_A))$.

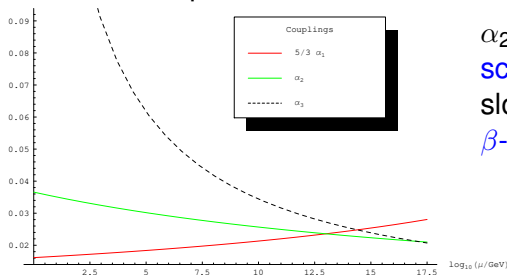
for almost-commutative 4-dim compact manifolds:

- $S(\mathcal{D}_A) = \int_X d \text{vol} (\mathcal{L}_\Lambda + \mathcal{L}_{\text{EH+W}} + \mathcal{L}_{\text{YM}} + \mathcal{L}_{\text{Higgs-kin}} + \mathcal{L}_{\text{Higgs-pot}})$
for **any** function of the spectrum (universality of RG)
- structure of the **standard model more or less unique**

The rôle of scales

to be precise, an effective action is produced which is **spectral**
at a distinguished scale Λ_{GUT}

- **structural relations** at Λ_{GUT} : YM couplings $\alpha_3 = \alpha_2$, $\sin^2 \theta_W = \frac{3}{8}$
Higgs coupling $\lambda = \frac{16\pi}{3} \alpha_2$
- connection to experiment: **renormalisation group flow**:



$\alpha_2, \alpha_3, \theta_W, \lambda$
scale-dependent,
slope given by
 β -functions

- Higgs mass $m_H = \sqrt{\frac{2\lambda}{\pi\alpha_2}} m_W$
from λ, α_2 evaluated at m_Z one finds $m_H \sim 170 \text{ GeV}$

Message: Geometry of Nature is scale dependent

Rough picture

- 1 at $E < \Lambda_{electroweak}$: Riemannian geometry
- 2 at $E = \Lambda_{electroweak} \dots \Lambda_{GUT}$: almost-commutative geometry
- 3 at $E > \Lambda_{GUT}$ truly noncommutative geometry??

Needs correction:

Yang-Mills couplings don't intersect in single point

- supersymmetry?
- dark matter?
- space-time noncommutativity at intermediate scale?
... asymptotic freedom of “non-Abelian” $U(1)$?

We focus on space-time noncommutativity.

Up to now, all proposed models are far from realistic!

- They lead to **long-range contradictions** to observation.
- They are **rigid**: Usual gravity seems impossible to realise.

These are **features** of noncommutative geometries, **not bugs**!

Space-time noncommutativity as an episode of gravity¹

- appears at some scale Λ_0 through **phase transition from exotic form of geometry** (strings?, quantum gravity?)
- disappears at scale $\Lambda_1 < \Lambda_0$ through **phase transition into almost-commutative geometry**

Space-time noncommutativity is **not a universal theory of everything**. It is **not present at macroscopic distances**.

¹remember: Yang-Mills-Higgs is a version of gravity, too!

Quantum field theory on noncommutative spaces

- 1 We are interested in noncommutative geometries described by **spectral triples**.
- 2 These define **Yang-Mills-Higgs models** via **spectral action principle** at **initial (Planck?) scale Λ_0** .
- 3 **Renormalisation group flow** evolves initial Yang-Mills-Higgs models into **effective action** at **transition (GUT?) scale $\Lambda_1 \ll \Lambda_0$** where almost-commutative geometry sets in.
- 4 Fit to **standard model** requires **domination of Yang-Mills-Higgs action** at Λ_1

Yang-Mills-Higgs is required to be renormalisable quantum field theory on noncommutative space-time, too!

Noncommutative torus and Moyal space

Simplest examples:

Noncommutative torus $u^k u^l = e^{-i\pi k\Theta l} u^{k+l} \quad k, l \in \mathbb{Z}^d$

- mathematically well-studied
- **discreteness of momenta** challenging for QFT
- some estimates need **Diophantine deformation matrix** Θ

Moyal space $(f \star g)(x) = \int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{dy dk}{(2\pi)^d} f(x + \frac{1}{2}\Theta k) g(x+y) e^{iky}$

- formulation of (naïve) physical models very easy: add everywhere **oscillating phase factors** [Filk; 1996]
- mathematically challenging: **non-compact spectral triple** [Gayral, Gracia-Bondía, Ioachim, Schücker, Várilly; 2003]
- main problem: compactness+dim. of $L_\Theta(f)(\Delta + m^2)^{-\frac{1}{2}}$

UV/IR-mixing

- Moyal-type interaction is **non-local**: $\delta_z \star_{\Theta} \delta_z = \frac{\text{const}}{\det \Theta} 1$
- **highly localised subgraphs** lead to **non-local divergences which increase with the distance**
consequence: **non-local interactions dominate** at $\Lambda_1 \ll \Lambda_0$ over the local Yang-Mills-Higgs action

- **UV/IR-mixing** discovered for ϕ^4 on Moyal space by [Minwalla, van Raamsdonk, Seiberg; 1999]
confirmed for Yang-Mills on Moyal space by [Matusis, Susskind, Toumbas; 2000]
- only solution is **restriction of space-time noncommutativity to finite range of scales**

prototype: Gaußian $e(x) = 2^{\frac{d}{2}} \exp\left(-\frac{x_1^2+x_2^2}{\theta_1} - \dots - \frac{x_{d-1}^2+x_d^2}{\theta_{d/2}}\right)$

is projector $e = \bar{e} = e \star e$

Euclidean vs. Minkowskian signature

I used the term “**space-time noncommutativity**”, but “**time**” in noncommutative geometry is tricky

- noncommutative **von Neumann algebras** carry their own time evolution through the **modular automorphism group**
- together with **KMS condition** for thermal state of **cosmic background radiation**, this could be the source of the physical time flow [Connes, Rovelli; 1994]
- disconnected so far from spectral geometry; reconciliation in **locally covariant QFT**? [Paschke, Verch; 2004]

Modest approach: IF restricted to finite range of scales, **NCQFT experiences temperature** with non-zero imaginary part of time
UV/IR-mixing for **Minkowskian noncommutative QFT** is different, but long-range contradictions remain [Bahns; 2002. . .]

UV+IR-regularisation for QFT on Moyal space

Rigorous treatment of QFT divergences requires

- ① IR+UV-regularisation
- ② renormalisation allowing for thermodynamic+continuum limits

Problem: Moyal product does not preserve supports

- **Gauß projector** $e = \bar{e} = e \star e$ is ground state for harmonic oscillator of $([x, x] = i\Theta)$ -quantum mechanics
- left and right creation operators (in 2D):

$$f_{mn}(x_1, x_2) = \frac{(x_1 + ix_2)^{\star m}}{\sqrt{m!(2\theta)^m}} \star \left(2e^{-\frac{1}{\theta}(x_1^2 + x_2^2)} \right) \star \frac{(x_1 - ix_2)^{\star n}}{\sqrt{n!(2\theta)^n}}$$

- $(f_{mn} \star f_{kl})(x) = \delta_{nk} f_{ml}(x)$ $\int dx f_{mn}(x) = \sqrt{\det(2\pi\Theta)} \delta_{mn}$

- expansion $\phi(x) = \sum_{k, l \in \mathbb{Z}^d} \phi_{kl} f_{kl}(x)$ in matrix basis:

\star -product is **infinite matrix product**, **UV-cutoff** $|k|, |l| \leq N$

Laplacian in matrix basis

in 2D:
$$\Delta_{mn;kl} = \frac{2}{\theta}(m+n+1)\delta_{nk}\delta_{ml} - \frac{2}{\theta}(\sqrt{kl}\delta_{n+1,k}\delta_{m+1,l} + \sqrt{mn}\delta_{n-1,k}\delta_{m-1,l})$$

resolvent is non-compact, related to UV/IR-problem

IR-cutoff means compactness of resolvent [Grosse, W.; 2004]

$$\Delta_{mn;kl}^{\Omega} = \frac{2}{\theta}(1 + \Omega^2)(m+n+1)\delta_{nk}\delta_{ml} - \frac{2}{\theta}(1 - \Omega^2)(\sqrt{kl}\delta_{n+1,k}\delta_{m+1,l} + \sqrt{mn}\delta_{n-1,k}\delta_{m-1,l})$$

- in function basis: **harmonic oscillator with frequency Ω** :
 $(\Delta^{\Omega}f)(x) = (\Delta f)(x) + 4\Omega^2\|\Theta^{-1}x\|^2f(x)$
- **resonance $\Omega = 1$** with $([x, x] = i\Theta)$ -harmonic oscillator has special properties (**Langmann-Szabo duality**)

NC ϕ_4^4 -theory develops additional marginal couplingaction functional for real-valued field ϕ on \mathbb{R}^4 :

$$S[\phi] = \int d^4x \left(\frac{1}{2} \phi \star (\Delta + \Omega^2 \tilde{x}^2 + \mu^2) \phi + \frac{\lambda}{4} \phi \star \phi \star \phi \star \phi \right) (x)$$

Moyal product \star defined by Θ and $\tilde{x} := 2\Theta^{-1} \cdot x$ parameters: $\mu^2, \lambda \in \mathbb{R}_+$, $\Omega \in [0, 1]$, redef'n $\phi \mapsto \mathcal{Z}\phi$, $\mathcal{Z} \in \mathbb{R}_+$

- **renormalisable as formal power series** in λ
[Grosse, W.; 2004]
means: well-defined **perturbative** quantum field theory
- **β -function vanishes to all orders** in λ for $\Omega = 1$
[Disertori, Gurau, Magnen, Rivasseau; 2006]
means: model is believed to exist **non-perturbatively**

Does this model arise from a spectral triple?

Supersymmetric quantum mechanics

X a d -dimensional smooth manifold, T^*X trivial

- $a_\mu = e^{-\omega h} \partial_\mu e^{\omega h} = \partial_\mu + W_\mu$, $a_\mu^\dagger = -e^{\omega h} \partial_\mu e^{-\omega h} = -\partial_\mu + W_\mu$
 $h \in C^\infty(X)$ Morse function, $W_\mu = \omega \partial_\mu h$

- commutation relations:

$$[a_\mu, a_\nu] = [a_\mu^\dagger, a_\nu^\dagger] = 0, \quad [a_\mu, a_\nu^\dagger] = 2\omega \partial_\mu \partial_\nu h$$

- d fermionic ladder operators:

$$\{b_\mu, b_\nu\} = 0, \quad \{b_\mu^\dagger, b_\nu^\dagger\} = 0, \quad \{b_\mu, b_\nu^\dagger\} = \delta_{\mu\nu}$$

- supercharges:

$$\mathfrak{Q} := \sum_{\mu=1}^d a_\mu \otimes b_\mu^\dagger, \quad \mathfrak{Q}^\dagger := \sum_{\mu=1}^d a_\mu^\dagger \otimes b_\mu$$

- supersymmetry algebra:

$$\{\mathfrak{Q}, \mathfrak{Q}^\dagger\} = \mathfrak{H} = (-\partial_\mu \partial^\mu + \omega^2 (\partial_\mu h)(\partial^\mu h)) \otimes 1 + \omega (\partial^\mu \partial^\nu h) \otimes [b_\mu^\dagger, b_\nu]$$

$$\{\mathfrak{Q}, \mathfrak{Q}\} = \{\mathfrak{Q}^\dagger, \mathfrak{Q}^\dagger\} = 0, \quad [\mathfrak{Q}, \mathfrak{H}] = [\mathfrak{Q}^\dagger, \mathfrak{H}] = 0$$

cohomology of \mathfrak{Q} related to Morse theory for h [Witten; 1982]

Harmonic oscillator spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D}_i)$

Morse function $h = \frac{1}{2} \|x\|^2$

implies constant $[a_\mu, a_\nu^\dagger] = 2\omega \delta_{\mu\nu}$

Hilbert space $\mathcal{H} = \ell^2(\mathbb{N}^d) \otimes \wedge(\mathbb{C}^d)$: declare ONB

$\{(a_1^\dagger)^{n_1} \dots (a_d^\dagger)^{n_d} \otimes (b_1^\dagger)^{s_1} \dots (b_d^\dagger)^{s_d} | 0\rangle : n_\mu \in \mathbb{N}, s_\mu \in \{0, 1\}\}$

TWO Dirac operators $\mathcal{D}_1 = \mathcal{Q} + \mathcal{Q}^\dagger, \quad \mathcal{D}_2 = i\mathcal{Q} - i\mathcal{Q}^\dagger$

$$\begin{aligned} \mathcal{D}_1^2 = \mathcal{D}_2^2 = \mathfrak{H} &= \sum_{\mu=1}^d (a_\mu^\dagger a_\mu \otimes 1 + 2\omega \otimes b_\mu^\dagger b_\mu) \\ &= 2\omega(N_b + N_f) = H \otimes 1 + \omega \otimes \Sigma \end{aligned}$$

where

$$H = -\frac{\partial^2}{\partial x_\mu \partial x^\mu} + \omega^2 x_\mu x^\mu \quad - \text{harmonic oscillator Hamiltonian}$$

$$\Sigma = \sum_{\mu=1}^d [b_\mu^\dagger, b_\mu] \quad - \text{spin matrix}$$

algebra $\mathcal{A} = \mathcal{S}(\mathbb{R}^d)$ uniquely determined by smoothness

- $\mathcal{H}_\infty = \bigcap_{k \in \mathbb{N}} \text{dom}(\mathcal{D}^{2k}) = \mathcal{S}(\mathbb{R}^d) \otimes \wedge(\mathbb{C}^d) \simeq (\mathcal{S}(\mathbb{R}^d))^{2^d}$
(trivial projector of rank 2^d)

- Hermitian structure takes values in Schwartz class functions: We may choose

- 1 the commutative algebra $\mathcal{A} = \mathcal{S}(\mathbb{R}^d)$

$$[\mathcal{D}_1, f] = \partial^\mu f \otimes (b_\mu^\dagger - b_\mu), \quad [\mathcal{D}_2, f] = \partial^\mu f \otimes (ib_\mu^\dagger + ib_\mu)$$

bounded & order-one

- 2 the noncommutative Moyal algebra \mathcal{A}_Θ which provides an isospectral deformation (work with H. Grosse)

$$[\mathcal{D}_1, L_\star(f)] = L_\star(\partial_\mu f) \otimes (\delta^{\mu\nu} (b_\nu^\dagger - b_\nu) + \frac{1}{2}\omega \Theta^{\mu\nu} (ib_\nu^\dagger + ib_\nu))$$

$$[\mathcal{D}_2, L_\star(f)] = L_\star(\partial_\mu f) \otimes (\delta^{\mu\nu} (ib_\nu^\dagger + ib_\nu) + \frac{1}{2}\omega \Theta^{\mu\nu} (b_\nu^\dagger - b_\nu))$$

All axioms of spectral triples satisfied, with minor adaptation:

- 1 The metric dimension is different from the asymptotics of eigenvalues of \mathcal{D}^{-1} .

This is taken into account by the **dimension spectrum** computed from the **Mehler kernel**

$$e^{-tH}(x, y) = \left(\frac{\omega}{2\pi \sinh(2\omega t)} \right)^{\frac{d}{2}} e^{-\frac{\omega}{4} \coth(\omega t) \|x-y\|^2 - \frac{\omega}{4} \tanh(\omega t) \|x+y\|^2}$$

- 2 \mathbf{c} takes values in unitisation

$$\mathcal{B} = \{b \in \mathcal{A}'' : b, [\mathcal{D}, b] \in \bigcap_{k \in \mathbb{N}} \text{dom}(\mathcal{D}^k)\} \ni u_\mu = e^{iX_\mu}$$

$$\mathbf{c} = \sum_{\sigma \in S_d} \epsilon(\sigma) \frac{i^{\frac{d(d-1)}{2}}}{d!} (u_1 \cdots u_d)^{-1} \otimes u_{\sigma(1)} \otimes \cdots \otimes u_{\sigma(d)} \in Z_d(\mathcal{B}, \mathcal{B})$$

- 3 $\gamma_1 := \pi_{\mathcal{D}_1}(\mathbf{c}) = i^{\frac{d(d+1)}{2}} (b_1^\dagger - b_1) \cdots (b_d^\dagger - b_d)$,

$$\gamma_2 := \pi_{\mathcal{D}_2}(\mathbf{c}) = i^{\frac{d(d+3)}{2}} (b_1^\dagger + b_1) \cdots (b_d^\dagger + b_d)$$

$$\gamma_i^2 = 1, \gamma_i^* = \gamma_i, \quad (-i)^d \gamma_1 \gamma_2 = i^d \gamma_2 \gamma_1 = (-1)^{N_f} \mathbb{Z}_2\text{-grading}$$

$U(1)$ -Higgs model for commutative algebra

tensor $(\mathcal{A}, \mathcal{H}, \mathcal{D}_1)$ with $(\mathbb{C} \oplus \mathbb{C}, \mathbb{C}^2, M\sigma_1, \sigma_3)$ [Connes+Lott]

- $\mathcal{D} = \mathcal{D}_1 \otimes \sigma_3 + 1 \otimes \sigma_1 M = \begin{pmatrix} \mathcal{D}_1 & M \\ M & -\mathcal{D}_1 \end{pmatrix} \quad \begin{pmatrix} f & 0 \\ 0 & g \end{pmatrix} \in \mathcal{A}_{tot}$

- selfadjoint **fluctuated Dirac operators** $\mathcal{D}_A := \mathcal{D} + \sum_i a_i [\mathcal{D}, b_i]$, $a_i, b_i \in \mathcal{A}_{tot} = \mathcal{A} \oplus \mathcal{A}$, are of the form

$$\mathcal{D}_A = \begin{pmatrix} \mathcal{D}_4 + iA^\mu \otimes (b_\mu^\dagger - b_\mu) & \phi \otimes 1 \\ \bar{\phi} \otimes 1 & -(\mathcal{D}_4 + iB^\mu \otimes (b_\mu^\dagger - b_\mu)) \end{pmatrix}$$

for $A_\mu = \overline{A_\mu}$, $B_\mu = \overline{B_\mu}$, $\phi \in \mathcal{A}$

- $\mathcal{D}_A^2 = \begin{pmatrix} (H + |\phi|^2) \otimes 1 + \omega \otimes \Sigma + F_A & D^\mu \phi \otimes (b_\mu^\dagger - b_\mu) \\ -\overline{D^\mu \phi} \otimes (b_\mu^\dagger - b_\mu) & (H + |\phi|^2) \otimes 1 + \omega \otimes \Sigma + F_B \end{pmatrix}$

- $D_\mu \phi = \partial_\mu \phi + i(A_\mu - B_\mu)\phi$

$$F_A = (-\{\partial^\mu, A_\mu\} - iA^\mu A_\mu) \otimes 1 + \frac{1}{4} F_A^{\mu\nu} \otimes [b_\mu^\dagger - b_\mu, b_\nu^\dagger - b_\nu]$$

The spectral action

$$\begin{aligned}
 S(\mathcal{D}_A) &= \frac{2\chi_{-4}}{\omega^4} + \frac{8\chi_{-2}}{3\omega^2} + \frac{52\chi_0}{45} \\
 &+ \frac{\chi_0}{\pi^2} \int d^4x \left\{ D^\mu \phi \overline{D_\mu \phi} + \frac{5}{12} (F_{\mu\nu}^A F_A^{\mu\nu} + F_{\mu\nu}^B F_B^{\mu\nu}) \right. \\
 &\quad \left. + \left((|\phi|^2)^2 - \frac{2\chi_{-1}}{\chi_0} |\phi|^2 + 2\omega^2 \|x\|^2 |\phi|^2 \right) \right\} + \mathcal{O}(\chi_1)
 \end{aligned}$$

- spectral action is finite (in contrast to standard \mathbb{R}^4)
- only difference in field equations to infinite volume is **additional harmonic oscillator potential for the Higgs field**
- Yang-Mills is unchanged (in contrast to Moyal)
- vacuum is at $A_\mu = B_\mu = 0$ and (after gauge transformation) ϕ real-valued, rotationally invariant

Field equations

rescaling $r = 2^{\frac{1}{4}} \sqrt{\omega} \|\mathbf{x}\|$, $\phi = \frac{\pi}{\sqrt{2}\chi_0} \varphi$, $\mu^2 = \frac{\chi-1}{\sqrt{8}\omega\chi_0}$, $\lambda = \frac{\pi^2}{\sqrt{2}\omega\chi_0}$

$$-\varphi''(r) - \frac{3}{r}\varphi'(r) + (r^2 - 4\mu^2)\varphi(r) = -\lambda\varphi^3(r)$$

- inhomogeneous confluent hypergeometric diff. eq.
- expand $\varphi = \frac{2}{\sqrt{\lambda}} \sum_{n=0}^{\infty} c_n \varphi_n$ in terms of eigenfunctions of 4-dim. harmonic oscillator:

$$\varphi_n := e^{-\frac{r^2}{2}} L_n^1(r^2) \quad \left(-\frac{d^2}{dr^2} - \frac{3}{r} \frac{d}{dr} + r^2 \right) \varphi_n = 4(n+1)\varphi_n$$

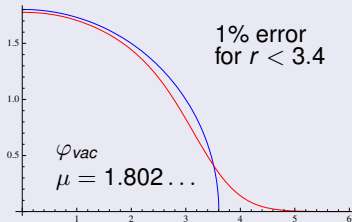
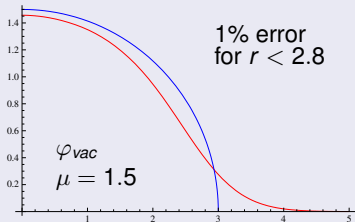
- yields

$$c_n(\mu^2 - n - 1) = \sum_{k,l,m=0}^{\infty} \frac{c_k c_l c_m}{k!l!m!} \left(\frac{d^k}{dw^k} \frac{d^l}{dy^l} \frac{d^m}{dz^m} \frac{(1-wy-wz-yz+2wyz)^n}{(2-w-y-z+wyz)^{n+2}} \right)_{w=y=z=0}$$

- numerical solution: cut-off at N

$$-\varphi''(r) - \frac{3}{r}\varphi'(r) + (r^2 - 4\mu^2)\varphi(r) = -\lambda\varphi^3(r)$$

$\varphi(r)$ in units of $\frac{2}{\sqrt{\lambda}}$, cutoff at $N = 10$

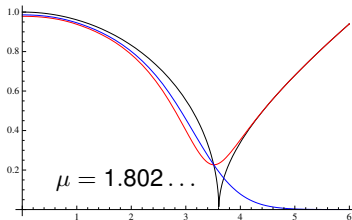
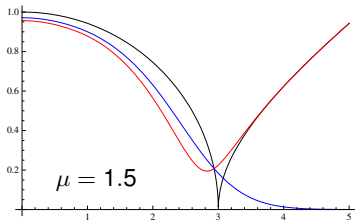


- vacuum solution $\varphi_{vac}(r)$ is smaller than asymptotic curve $\sqrt{\mu^2 - \frac{1}{4}r^2}$ due to its **negative curvature** and approaches it for $\mu \rightarrow \infty$
- $\varphi_{vac} = 0$ for $r \geq 2\mu$, transition is smoothly
- **vacuum solution is integrable**

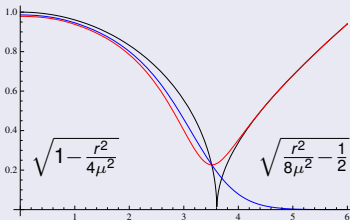
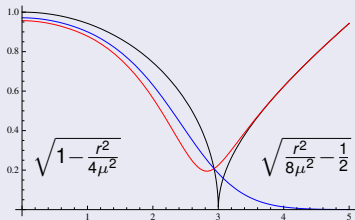
Masses

- scale of bare fermion and gauge field masses given by vacuum expectation value $\sqrt{\frac{4\mu^2}{\lambda} \frac{\varphi_{vac}}{\mu}} = \sqrt{\frac{2\chi_{-1}}{\pi^2} \frac{\varphi_{vac}}{\mu}}$
- bare Higgs mass given by difference function

$$\sqrt{\sqrt{2\omega}((r^2 - 4\mu^2) + 12\mu^2 \frac{\varphi_{vac}^2}{\mu^2})} = \sqrt{\frac{4\chi_{-1}}{\chi_0} \frac{\sqrt{\frac{3}{2}\varphi_{vac}^2 - \frac{1}{2}\mu^2 + \frac{1}{8}r^2}}{\mu}}$$



Smooth transition between two phases



- 1 *Spontaneously broken phase* $\omega^2 \|x\|^2 < \frac{\chi-1}{\chi_0}$
fermions, gauge fields and Higgs are massive, with **Higgs mass slightly smaller than NCG-prediction**
- 2 *Unbroken phase* $\omega^2 \|x\|^2 \geq \frac{\chi-1}{\chi_0}$
fermions + gauge fields massless, Higgs remains massive

Mass of gauge fields and fermions dissipates into cosmological constant!

The spectral action: noncommutative case

$$\begin{aligned}
 S(\mathcal{D}_A) = & \frac{\theta^4 \chi_{-4}}{8\Omega^4} + \frac{2\theta^2 \chi_{-2}}{3\Omega^2} + \frac{52\chi_0}{45} + \frac{\chi_0}{2\pi^2(1+\Omega^2)^2} \int d^4x \left\{ 2D_\mu \phi \star \overline{D_\mu \phi} \right. \\
 & + \left(\frac{(1-\Omega^2)^2}{2} - \frac{(1-\Omega^2)^4}{3(1+\Omega^2)^2} \right) (F_{\mu\nu}^A \star F_A^{\mu\nu} + F_{\mu\nu}^B \star F_B^{\mu\nu}) \\
 & + \left(\phi \star \bar{\phi} + \frac{4\Omega^2}{1+\Omega^2} X_A^\mu \star X_{A\mu} - \frac{\chi_{-1}}{\chi_0} \right)^2 \\
 & + \left(\bar{\phi} \star \phi + \frac{4\Omega^2}{1+\Omega^2} X_B^\mu \star X_{B\mu} - \frac{\chi_{-1}}{\chi_0} \right)^2 \\
 & \left. - 2 \left(\frac{4\Omega^2}{1+\Omega^2} X_0^\mu \star X_{0\mu} - \frac{\chi_{-1}}{\chi_0} \right)^2 \right\} (x) + \mathcal{O}(\chi_1)
 \end{aligned}$$

$$\Theta = \begin{pmatrix} 0 & \theta & 0 & 0 \\ -\theta & 0 & 0 & 0 \\ 0 & 0 & 0 & \theta \\ 0 & 0 & -\theta & 0 \end{pmatrix} \quad \omega = \frac{2\Omega}{\theta}$$

deeper entanglement of gauge and Higgs fields

covariant coordinates $X_{A\mu}(x) = (\Theta^{-1})_{\mu\nu} x^\nu + A_\mu(x)$ appear with Higgs field ϕ in **unified potential**; vacuum is non-trivial!

potential cannot be restricted to Higgs part if distinction into discrete and continuous geometries no longer possible

The vacuum

vacuum field equations

$$(\phi^{vac} = \overline{\phi^{vac}}, \quad A_\mu^{vac} = B_\mu^{vac})$$

$$\frac{1}{g^2} [X_{A\nu}, [X_A^\mu, X_A^\nu]_\star]_\star + 2[\phi, [X_A^\mu, \phi]_\star]_\star$$

$$= \frac{4\Omega^2}{1+\Omega^2} \left\{ \phi \star \phi + \frac{4\Omega^2}{1+\Omega^2} X_A^\nu \star X_{A\nu} - \mu^2, X_A^\mu \right\}_\star$$

$$2[X_{A\nu}, [\phi, X_A^\nu]_\star]_\star = \left\{ \phi \star \phi + \frac{4\Omega^2}{1+\Omega^2} X_A^\nu \star X_{A\nu} - \mu^2, \phi \right\}_\star$$

$$\left(\text{with } \frac{1}{4g^2} = \frac{(1-\Omega^2)^2}{2} - \frac{(1-\Omega^2)^4}{3(1+\Omega^2)^2} \quad \mu^2 = \frac{\chi-1}{\chi_0} \right)$$

spirit of **emerging geometry** through phase transitions

- $\Omega = 0 \Rightarrow$ solution: $\phi = \mu 1, \quad [X_\mu, X_\nu] = \begin{cases} \Theta_{\mu\nu} \\ 0 \end{cases}$

$\Omega \neq 0$ gives some **dynamical geometry**

- analytical solution seems impossible

\Rightarrow **need numerical simulations**