Non-compact spectral triples

Raimar Wulkenhaar

Mathematisches Institut der Westfälischen Wilhelms-Universität
Münster, Germany

Spectral triples

see: A. Connes, “On the spectral characterization of manifolds,” 2008

Definition (commutative spectral triple $(\mathcal{A},\mathcal{H},\mathcal{D})$ of dimension $p \in \mathbb{N}$)

...given by a Hilbert space $\mathcal{H}$, a commutative involutive unital algebra $\mathcal{A}$ represented in $\mathcal{H}$, and a selfadjoint operator $\mathcal{D}$ in $\mathcal{H}$ with compact resolvent, with

1. Dimension: k^{th} characteristic value of resolvent of $\mathcal{D}$ is $\mathcal{O}(k^{-\frac{1}{p}})$
2. Order one: $[[\mathcal{D}, f], g] = 0 \quad \forall f, g \in \mathcal{A}$
3. Regularity: f and $[\mathcal{D}, f]$ belong to the domain of δ^k, for any $f \in \mathcal{A}$ and $k \in \mathbb{Z}$, where $\delta T := [|\mathcal{D}|, T]$
4. Orientability: $\exists$ Hochschild p-cycle $\boldsymbol{c} \in Z_p(\mathcal{A},\mathcal{A})$ s.t.$\pi_{\mathcal{D}}(\boldsymbol{c}) = 1$ for p odd, $\pi_{\mathcal{D}}(\boldsymbol{c}) = \gamma$ for p even with $\gamma = \gamma^*$, $\gamma^2 = 1$, $\gamma\mathcal{D} = -\mathcal{D}\gamma$
5. Finiteness and absolute continuity: $\mathcal{H}_\infty := \cap_k \mathrm{dom}(\mathcal{D}^k) \subset \mathcal{H}$ is finitely generated projective $\mathcal{A}$-module, $\mathcal{H}_\infty = e\mathcal{A}^n$, with $e = e^* = e^2 \in M_m(\mathcal{A})$. Hermitian structure $(\xi|a\eta) = \sum_{i=1}^n a\xi_i^*\eta_i \in \mathcal{A}$ satisfies $\langle \xi, \eta \rangle = ⨍(\xi|\eta)|\mathcal{D}|^{-p}$

Theorem (Connes, 2008)

Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be a commutative spectral triple and assume that

– all endomorphisms $T \in \mathrm{End}_{\mathcal{A}}(\mathcal{H}_\infty)$ are regular,

*– the Hochschild cycle **c** is antisymmetric.*

Then there exists a compact oriented smooth manifold X such that $\mathcal{A} = C^\infty(X)$ is the algebra of smooth functions on X, and every compact oriented smooth manifold appears in this manner.

It is known since [Milnor, 1964] that there exist isospectral manifolds which are not isometric

Connes' proof identifies the missing piece which in addition to the spectrum of $\mathcal{D}$ characterises the geometry:

It is the analogue of the Cabibbo-Kobayashi-Maskawa-Matrix in QFT which encodes the relative position of two bases in the same Hilbert space.

Spectral triples are interesting for physics!

- equivalence classes of spectral triples describe Yang-Mills theory (inner automorphisms; exist always in nc case) and possibly gravity (outer automorphisms)
- inner fluctuations: $\mathcal{D} \mapsto \mathcal{D}_A = \mathcal{D} + A\,, \quad A = \sum f[\mathcal{D}, g]$
  for almost-commutative manifolds: A=Yang-Mills+Higgs

Spectral action principle [Chamseddine-Connes, 1996]

As an automorphism-invariant object, the (bosonic) action functional of physics has to be a function of the spectrum of $\mathcal{D}_A$, i.e. $S(\mathcal{D}_A) = \mathrm{Tr}(\chi(\mathcal{D}_A))$.

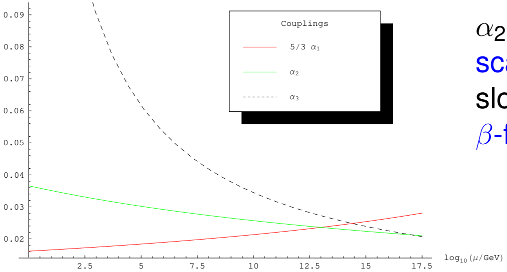
for almost-commutative 4-dim compact manifolds:

- $S(\mathcal{D}_A) = \int_X d\,vol\,(\mathcal{L}_\Lambda + \mathcal{L}_{\text{EH+W}} + \mathcal{L}_{\text{YM}} + \mathcal{L}_{\text{Higgs-kin}} + \mathcal{L}_{\text{Higgs-pot}})$
  for any function of the spectrum (universality of RG)
- structure of the standard model more or less unique

The rôle of scales

to be precise, an effective action is produced which is spectral at a distiguished scale Λ_{GUT}

- structural relations at Λ_{GUT}: YM couplings $\alpha_3 = \alpha_2$, $\sin^2\theta_W = \frac{3}{8}$
  Higgs coupling $\lambda = \frac{16\pi}{3}\alpha_2$
- connection to experiment: renormalisation group flow:



$\alpha_2, \alpha_3, \theta_W, \lambda$ scale-dependent, slope given by β-functions

- Higgs mass $m_H = \sqrt{\frac{2\lambda}{\pi\alpha_2}}\, m_W$
  from λ, α_2 evaluated at m_Z one finds $m_H \sim 170\,\text{GeV}$

Standard model on noncommutative space-time

Toy model: ϕ^4-theory on 4D Moyal space

$$S[\phi] = \int d^4x \Big(\frac{1}{2}\phi \star (-\Delta + \Omega^2 \tilde{x}^2 + \mu^2)\phi + \frac{\lambda}{4}\phi \star \phi \star \phi \star \phi\Big)(x)$$

Moyal product $\star$ defined by Θ and $\tilde{x} := 2\Theta^{-1} \cdot x$

parameters: $\mu^2, \lambda \in \mathbb{R}_+$, $\Omega \in [0,1]$, redef'n $\phi \mapsto Z^{\frac{1}{2}}\phi$, $Z \in \mathbb{R}_+$

- renormalisable as formal power series in λ [Grosse-W., 2004]
  means: well-defined perturbative quantum field theory
- β-function vanishes to all orders in λ for $\Omega = 1$ [Disertori-Gurau-Magnen-Rivasseau, 2006]
  means: model is believed to exist non-perturbatively

Does this model arise from a spectral triple?

Supersymmetric quantum mechanics

Let X be a d-dimensional smooth manifold, T^*X trivial

- $a_\mu = e^{-\omega h}\partial_\mu e^{\omega h} = \partial_\mu + W_\mu$, $\quad a_\mu^\dagger = -e^{\omega h}\partial_\mu e^{-\omega h} = -\partial_\mu + W_\mu$
  $h \in C^\infty(X)$ Morse function, $W_\mu = \omega \partial_\mu h$
- commutation relations:
  $[a_\mu, a_\nu] = [a_\mu^\dagger, a_\nu^\dagger] = 0$, $\qquad [a_\mu, a_\nu^\dagger] = 2\omega\partial_\mu\partial_\nu h$
- d fermionic ladder operators:
  $\{b_\mu, b_\nu\} = 0$, $\qquad \{b_\mu^\dagger, b_\nu^\dagger\} = 0$, $\qquad \{b_\mu, b_\nu^\dagger\} = \delta_{\mu\nu}$
- supercharges:
  $\mathfrak{Q} := \sum_{\mu=1}^d a_\mu \otimes b_\mu^\dagger$, $\qquad \mathfrak{Q}^\dagger := \sum_{\mu=1}^d a_\mu^\dagger \otimes b_\mu$
- supersymmetry algebra:
  $\{\mathfrak{Q}, \mathfrak{Q}^\dagger\} = \mathfrak{H} = \big(-\partial_\mu\partial^\mu + \omega^2(\partial_\mu h)(\partial^\mu h)\big)\otimes 1 + \omega(\partial^\mu\partial^\nu h)\otimes[b_\mu^\dagger, b_\nu]$
  $\{\mathfrak{Q}, \mathfrak{Q}\} = \{\mathfrak{Q}^\dagger, \mathfrak{Q}^\dagger\} = 0$, $\quad [\mathfrak{Q}, \mathfrak{H}] = [\mathfrak{Q}^\dagger, \mathfrak{H}] = 0$

cohomology of $\mathfrak{Q}$ related to Morse theory for h [Witten, 1982]

Harmonic oscillator spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D}_i)$

Morse function $h = \frac{1}{2}\|x\|^2$

implies constant $[a_\mu, a_\nu^\dagger] = 2\omega\delta_{\mu\nu}$

Hilbert space $\mathcal{H} = \ell^2(\mathbb{N}^d) \otimes \bigwedge(\mathbb{C}^d)$: declare ONB

$\Big\{(a_1^\dagger)^{n_1} \dots (a_d^\dagger)^{n_d} \otimes (b_1^\dagger)^{s_1} \dots (b_d^\dagger)^{s_d}|0\rangle \ : \ n_\mu \in \mathbb{N}\,, s_\mu \in \{0,1\}\Big\}$

TWO Dirac operators $\mathcal{D}_1 = \mathfrak{Q} + \mathfrak{Q}^\dagger\,, \qquad \mathcal{D}_2 = \mathrm{i}\mathfrak{Q} - \mathrm{i}\mathfrak{Q}^\dagger$

$$\mathcal{D}_1^2 = \mathcal{D}_2^2 = \mathfrak{H} = \sum_{\mu=1}^d \big(a_\mu^\dagger a_\mu \otimes 1 + 2\omega \otimes b_\mu^\dagger b_\mu\big)$$
$$= 2\omega(N_b + N_f) = H \otimes 1 + \omega \otimes \Sigma$$

where

$H = -\frac{\partial^2}{\partial x_\mu \partial x^\mu} + \omega^2 x_\mu x^\mu$ – harmonic oscillator hamiltonian

$\Sigma = \sum_{\mu=1}^d [b_\mu^\dagger, b_\mu]$ – spin matrix

algebra $\mathcal{A} = \mathcal{S}(\mathbb{R}^d)$ uniquely determined by smoothness

- $\mathcal{H}_\infty = \bigcap_{k\in\mathbb{N}} \mathrm{dom}(\mathcal{D}^{2k}) = \mathcal{S}(\mathbb{R}^d) \otimes \bigwedge(\mathbb{C}^d) \simeq \big(\mathcal{S}(\mathbb{R}^d)\big)^{2^d}$
  (trivial projector of rank 2^d)
- Hermitian structure takes values in Schwartz class functions: We may choose
  1. the commutative algebra $\mathcal{A} = \mathcal{S}(\mathbb{R}^d)$
     $[\mathcal{D}_1, f] = \partial^\mu f \otimes (b^\dagger_\mu - b_\mu)$, $\quad [\mathcal{D}_2, f] = \partial^\mu f \otimes (\mathrm{i}b^\dagger_\mu + \mathrm{i}b_\mu)$
     bounded & order-one
  2. the noncommutative Moyal algebra $\mathcal{A}_\Theta$ which provides an isospectral deformation (work with H. Grosse)
     $$[\mathcal{D}_1, L_\star(f)] = L_\star(\partial_\mu f) \otimes \big(\delta^{\mu\nu}(b^\dagger_\nu - b_\nu) + \tfrac{1}{2}\omega\Theta^{\mu\nu}(\mathrm{i}b^\dagger_\nu + \mathrm{i}b_\nu)\big)$$
     $$[\mathcal{D}_2, L_\star(f)] = L_\star(\partial_\mu f) \otimes \big(\delta^{\mu\nu}(\mathrm{i}b^\dagger_\nu + \mathrm{i}b_\nu) + \tfrac{1}{2}\omega\Theta^{\mu\nu}(b^\dagger_\nu - b_\nu)\big)$$

All axioms of spectral triples satisfied, with minor adaptation:

1. The asymptotics of eigenvalues of $\langle\mathcal{D}\rangle^{-1} = (\mathcal{D}^2+1)^{-\frac{1}{2}}$ gives the wrong dimension $2d$.

   We require discrete dimension spectrum: For any ϕ belonging to algebra generated by $\delta^n f$ and $\delta^n[\mathcal{D}, f]$, the function $\zeta_\phi(z) := \mathrm{Tr}(\phi\langle\mathcal{D}\rangle^{-z})$ extends holomorphically to $\mathbb{C}\setminus\mathrm{Sd}$, for Sd discrete.
   The metric dimension is $d = \max\{r \in \mathbb{R}\cap\mathrm{Sd}\}$

   Computation possible due to Mehler kernel

   $$e^{-tH}(x,y) = \Big(\frac{\omega}{2\pi\sinh(2\omega t)}\Big)^{\frac{d}{2}} e^{-\frac{\omega}{4}\coth(\omega t)\|x-y\|^2-\frac{\omega}{4}\tanh(\omega t)\|x+y\|^2}$$

   $\mathrm{Sd} = d - \mathbb{N}$, all poles simple, d is the oscillator dimension.
   All residues are local, i.e. finite linear combinations of
   $\int_{\mathbb{R}^d} dx\, x^{\alpha_0}\partial^{\alpha_1} f_1 \cdots \partial^{\alpha_k} f_k$

2. c takes values in unitisation
$\mathcal{B} = \{b \in \mathcal{A}'' \ : \ b, [\mathcal{D}, b] \in \bigcap_{k\in\mathbb{N}} \mathrm{dom}(\mathcal{D}^k)\} \ni u_\mu = e^{\mathrm{i}x_\mu}$

$$c = \sum_{\sigma\in S_d} \epsilon(\sigma) \frac{\mathrm{i}^{\frac{d(d-1)}{2}}}{d!} (u_1 \cdots u_d)^{-1} \otimes u_{\sigma(1)} \otimes \cdots \otimes u_{\sigma(d)} \in Z_d(\mathcal{B}, \mathcal{B})$$

Remark: if $1 \in \mathcal{A}$, Lemma 2.1. of [Connes, 2008] yields $\mathcal{B} = \mathcal{A}$, and d is Connes' metric dimension

3. There are TWO images of c in $\mathcal{B}(\mathcal{H})$:

$$\gamma_1 := \pi_{\mathcal{D}_1}(c) = \mathrm{i}^{\frac{d(d+1)}{2}} (b_1^\dagger - b_1) \cdots (b_d^\dagger - b_d) ,$$
$$\gamma_2 := \pi_{\mathcal{D}_2}(c) = \mathrm{i}^{\frac{d(d+3)}{2}} (b_1^\dagger + b_1) \cdots (b_d^\dagger + b_d)$$

They fulfill $\gamma_i^2 = 1$, $\gamma_i^* = \gamma_i$, but not $\mathcal{D}\gamma_i + (-1)^d \gamma_i \mathcal{D} = 0$!

product $(-\mathrm{i})^d \gamma_1\gamma_2 = \mathrm{i}^d \gamma_2\gamma_1 = (-1)^{N_f}$ is $\mathbb{Z}_2$-grading!

Remark: reconstruction theorem uses γ being $\mathbb{Z}_2$-grading to obtain state-independence of Dixmier trace

Index formula

$$\mathcal{H} = \mathcal{H}_{ev} \oplus \mathcal{H}_{odd}, \qquad \mathcal{D}_i^+ = \mathcal{D}_i\big|_{\mathcal{H}_{ev}} : \mathcal{H}_{ev} \to \mathcal{H}_{odd}$$

- to $\mathcal{P}_{\mathfrak{a}} : \mathcal{S}(\mathbb{R}^n; \mathbb{C}^k) \to \mathcal{S}(\mathbb{R}^n; \mathbb{C}^k)$ associate symbol $\mathfrak{a} \in M_k(C^\infty(T^*\mathbb{R}^n))$ by

$$(\mathcal{P}_{\mathfrak{a}}\eta)(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n \times \mathbb{R}^n} d\xi\, dy\, e^{\mathrm{i}\langle x-y,\xi\rangle}\, \mathfrak{a}_i(x,\xi)\, \eta(y)$$

- $\mathfrak{a}$ is elliptic of order m if $\exists C, R > 0$ such that

$$\mathfrak{a}(x,\xi)^*\mathfrak{a}(x,\xi) \geq C(\|x\|^2 + \|\xi\|^2)^m 1_k \text{ for } \|x\|^2 + \|\xi\|^2 \geq R$$

- graph projector

$$e_{\mathfrak{a}} = \begin{pmatrix} (1+\mathfrak{a}^*\mathfrak{a})^{-1} & (1+\mathfrak{a}^*\mathfrak{a})^{-1}\mathfrak{a} \\ \mathfrak{a}^*(1+\mathfrak{a}^*\mathfrak{a})^{-1} & \mathfrak{a}^*(1+\mathfrak{a}^*\mathfrak{a})^{-1}\mathfrak{a} \end{pmatrix} \qquad \hat{e}_{\mathfrak{a}} = e_{\mathfrak{a}} - \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

Theorem (Elliott-Natsume-Nest, 1996)

If $\mathcal{P}_\mathfrak{a}$ is an elliptic ΨDO of positive order, then

$$\mathrm{index}\,(\mathcal{P}_\mathfrak{a}) = \frac{1}{(2\pi \mathrm{i})^n n!} \int_{T^*\mathbb{R}^n} \mathrm{tr}\big(\hat{e}_\mathfrak{a}(d\hat{e}_\mathfrak{a})^{2n}\big)\;,$$

for $\mathcal{D}_1^+ : \mathcal{S}(\mathbb{R}^d; \mathbb{C}^{2^{d-1}}) \to \mathcal{S}(\mathbb{R}^d; \mathbb{C}^{2^{d-1}})$ in $n = d = 2$ dimensions:

- in bases $\binom{|0,0\rangle_f}{|1,1\rangle_f}$ of $\big(\bigwedge(\mathbb{C}^2)\big)_{ev}$ and $\binom{|1,0\rangle_f}{|0,1\rangle_f}$ of $\big(\bigwedge(\mathbb{C}^2)\big)_{odd}$:

$$\mathfrak{a}_1(x_1, x_2, \xi_1, \xi_2) = \begin{pmatrix} \mathrm{i}\xi_1 + \omega x_1 & -(-\mathrm{i}\xi_2 + \omega x_2) \\ \mathrm{i}\xi_2 + \omega x_2 & -\mathrm{i}\xi_1 + \omega x_1 \end{pmatrix}$$

- symbol elliptic of order 1, with

$$\mathrm{tr}\big(\hat{e}_{\mathfrak{a}_1}\, d\hat{e}_{\mathfrak{a}_1} \wedge d\hat{e}_{\mathfrak{a}_1} \wedge d\hat{e}_{\mathfrak{a}_1} \wedge d\hat{e}_{\mathfrak{a}_1}\big) = -\frac{96\omega^2\, dx_1 \wedge d\xi_1 \wedge dx_2 \wedge d\xi_2}{(1+\omega^2 x_1^2+\omega^2 x_2^2+\xi_1^2+\xi_2^2)^5}$$

$\mathrm{index}\,(\mathcal{D}_i^+) = \mathrm{dim}\ \ker \mathcal{D}_i^+ - \mathrm{dim}\ \ker \mathcal{D}_i^- = 1$ for any d and $\omega^2 > 0$

1-dim. kernel spanned by $e^{-\frac{\omega}{2}\|x\|^2}|0,\dots,0\rangle_f$, cokernel trivial

U(1)-Higgs model for commutative algebra

tensor $(\mathcal{A},\mathcal{H},\mathcal{D}_1)$ with $(\mathbb{C}\oplus\mathbb{C},\mathbb{C}^2,M\sigma_1,\sigma_3)$ [Connes+Lott]

- $\mathcal{D}=\mathcal{D}_1\otimes\sigma_3+1\otimes\sigma_1 M=\begin{pmatrix}\mathcal{D}_1 & M\\ M & -\mathcal{D}_1\end{pmatrix}$ $\quad\begin{pmatrix} f & 0\\ 0 & g\end{pmatrix}\in\mathcal{A}_{tot}$
- selfadjoint fluctuated Dirac operators $\mathcal{D}_A{:=}\mathcal{D}+\sum_i a_i[\mathcal{D},b_i]$, $a_i,b_i\in\mathcal{A}_{tot}=\mathcal{A}\oplus\mathcal{A}$, are of the form
  $$\mathcal{D}_A=\begin{pmatrix}\mathcal{D}_1+\mathrm{i}A^\mu\otimes(b_\mu^\dagger-b_\mu) & \phi\otimes 1\\ \overline{\phi}\otimes 1 & -(\mathcal{D}_1+\mathrm{i}B^\mu\otimes(b_\mu^\dagger-b_\mu))\end{pmatrix}$$
  for $A_\mu=\overline{A_\mu},\ B_\mu=\overline{B_\mu},\ \phi\in\mathcal{A}$
- $$\mathcal{D}_A^2=\begin{pmatrix}(H+|\phi|^2)\otimes 1+\omega\otimes\Sigma+F_A & D^\mu\phi\otimes(b_\mu^\dagger-b_\mu)\\ -\overline{D^\mu\phi}\otimes(b_\mu^\dagger-b_\mu) & (H+|\phi|^2)\otimes 1+\omega\otimes\Sigma+F_B\end{pmatrix}$$
- $D_\mu\phi=\partial_\mu\phi+i(A_\mu-B_\mu)\phi$
  $F_A=(-\{\partial^\mu,A_\mu\}-\mathrm{i}A^\mu A_\mu)\otimes 1+\frac{1}{4}F_A^{\mu\nu}\otimes[b_\mu^\dagger-b_\mu,b_\nu^\dagger-b_\nu]$

Spectral action principle

most general form of bosonic action is $S(\mathcal{D}_A) = Tr\big(\chi(\mathcal{D}_A^2)\big)$

- Laplace transf. + asympt. exp'n $\mathrm{Tr}(e^{-t\mathcal{D}_A^2}) \sim \sum_{n=-dim/2}^{\infty} a_n(\mathcal{D}_A^2)t^n$

  leads to $S(\mathcal{D}_A) = \sum_{n=-dim/2}^{\infty} \chi_n \mathrm{Tr}(a_n(\mathcal{D}_A^2))$

  with $\chi_z = \frac{1}{\Gamma(-z)}\int_0^\infty ds\, s^{-z-1}\chi(s)$ for $z \notin \mathbb{N}$

  $\chi_k = (-1)^k \chi^{(k)}(0)$ for $k \in \mathbb{N}$
- a_n – Seeley coefficients, must be computed from scratch

Duhamel expansion: $\mathcal{D}_A^2 = \mathrm{H}_0 - V$

$$e^{-t(\mathrm{H}_0-V)} = e^{-t\mathrm{H}_0} - \int_0^t dt_1 \frac{d}{dt_1}\Big(e^{-(t-t_1)(\mathrm{H}_0-V)}e^{-t_1\mathrm{H}_0}\Big)$$

$$= e^{-t\mathrm{H}_0} + \int_0^t dt_1 \Big(e^{-(t-t_1)(\mathrm{H}_0-V)}Ve^{-t_1\mathrm{H}_0}\Big) \qquad \ldots\text{iteration}$$

Vacuum trace

Mehler kernel (in 4D)

$$e^{-t(H+\omega\Sigma)}(x,y) =$$

$$\frac{\omega^2(1-\tanh^2(\omega t))^2}{16\pi^2\tanh^2(\omega t)} e^{-t\omega\Sigma 1_2 - \frac{\omega}{4}\frac{\|x-y\|^2}{\tanh(\omega t)} - \frac{\omega}{4}\tanh(\omega t)\|x+y\|^2}$$

$$\mathrm{tr}(e^{-t\omega\Sigma}) = (2\cosh(\omega t))^d$$

$$\begin{aligned}\mathrm{Tr}(e^{-t(H+\omega\Sigma)\otimes 1_2}) &= 2\mathrm{tr}\Big(\int d^4x\,(e^{-t(\mathrm{H}+\omega\Sigma)})(x,x)\Big) \\ &= \frac{2}{\tanh^4(\omega t)} = 2(\omega t)^{-4} + \frac{8}{3}(\omega t)^{-2} + \frac{52}{45} + \mathcal{O}(t^2)\end{aligned}$$

- Spectral action is finite, in contrast to standard $\mathbb{R}^4$!
- expansion starts with t^{-4} $\Rightarrow$ corresponds to 8-dim. space

Vertices

only one-vertex contribution from
$V = \mathrm{diag}(-(A_\mu A^\mu + |\phi|^2) \otimes 1, -(B_\mu B^\mu + |\phi|^2) \otimes 1)$
(no tadpole, in contrast to Moyal where $\{\partial_\mu + \omega x_\mu, \pi(A^\mu)\}$ contributes)

$$\int_0^t dt_1 \,\mathrm{Tr}\big(e^{-(t-t_1)(H+\omega\Sigma)1_2} V e^{-t_1(H+\omega\Sigma)1_2}\big)$$
$$= \frac{\omega^2 t}{\pi^2 \tanh^2(\omega t)} \int d^4x \big(-2|\phi|^2 - A_\mu A^\mu - B_\mu B^\mu \big) e^{-\omega \tanh(\omega t)\|x\|^2}$$

- yields $-\mu^2\phi^2$ for Higgs potential and oscillator potential $+x^2\phi^2$
- A^2, B^2 eliminated by two vertices with $\{\partial^\mu, A_\mu\}$, $\{\partial^\mu, B_\mu\}$; these also give Yang-Mills with negative sign, exceeded by two curvature-vertices

The spectral action

$$S(\mathcal{D}_A) = \frac{2\chi_{-4}}{\omega^4} + \frac{8\chi_{-2}}{3\omega^2} + \frac{52\chi_0}{45} + \frac{\chi_0}{\pi^2}\int d^4x \,\Big\{ D^\mu\phi\overline{D_\mu\phi} + \frac{5}{12}(F^A_{\mu\nu}F_A^{\mu\nu} + F^B_{\mu\nu}F_B^{\mu\nu}) + \Big((|\phi|^2)^2 - \frac{2\chi_{-1}}{\chi_0}|\phi|^2 + 2\omega^2\|x\|^2|\phi|^2\Big)\Big\} + \mathcal{O}(\chi_1)$$

- spectral action is finite
- only difference in field equations to infinite volume is additional harmonic oscillator potential for the Higgs
- Yang-Mills is unchanged (in contrast to Moyal)
- vacuum is at $A_\mu = B_\mu = 0$ and (after gauge transformation) $\phi \in \mathbb{R}$, rotationally invariant

Field equations

rescaling $r = 2^{\frac{1}{4}}\sqrt{\omega}\|x\|$, $\phi = \frac{\pi}{\sqrt{2}\chi_0}\varphi$, $\mu^2 = \frac{\chi_{-1}}{\sqrt{8}\,\omega\chi_0}$, $\lambda = \frac{\pi^2}{\sqrt{2}\,\omega\chi_0}$

$$-\varphi''(r) - \frac{3}{r}\varphi'(r) + (r^2 - 4\mu^2)\varphi(r) = -\lambda\varphi^3(r)$$

- inhomogeneous confluent hypergeometric diff. eq.
- expand $\varphi = \frac{2}{\sqrt{\lambda}}\sum_{n=0}^{\infty} c_n\varphi_n$ in terms of eigenfunctions of 4-dim. harmonic oscillator:

$$\varphi_n := e^{-\frac{r^2}{2}} L_n^1(r^2) \qquad \Big(-\frac{d^2}{dr^2} - \frac{3}{r}\frac{d}{dr} + r^2\Big)\varphi_n = 4(n+1)\varphi_n$$

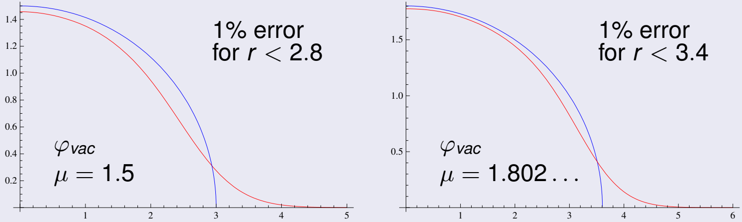
- yields

$$c_n(\mu^2-n-1) = \sum_{k,l,m=0}^{\infty} \frac{c_k c_l c_m}{k!l!m!}\Big(\frac{d^k}{dw^k}\frac{d^l}{dy^l}\frac{d^m}{dz^m}\frac{(1-wy-wz-yz+2wyz)^n}{(2-w-y-z+wyz)^{n+2}}\Big)_{w=y=z=0}$$

- numerical solution: cut-off at N

$-\varphi''(r) - \frac{3}{r}\varphi'(r) + (r^2 - 4\mu^2)\varphi(r) = -\lambda\varphi^3(r)$

$\varphi(r)$ in units of $\frac{2}{\sqrt{\lambda}}$, cutoff at $N = 10$

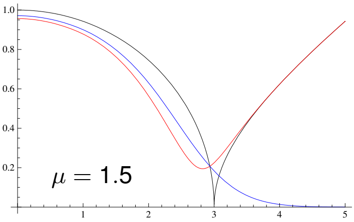


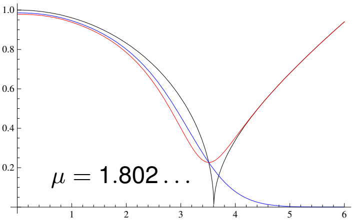
- vacuum solution $\varphi_{vac}(r)$ is smaller than asymptotic curve $\sqrt{\mu^2 - \frac{1}{4}r^2}$ due to its negative curvature and approaches it for $\mu \to \infty$
- $\varphi_{vac} = 0$ for $r \geq 2\mu$, transition is smoothly
- vacuum solution is integrable

Masses

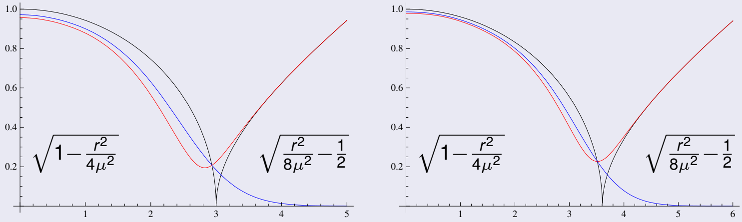
- scale of bare fermion and gauge field masses given by vacuum expectation value $\sqrt{\frac{4\mu^2}{\lambda}}\frac{\varphi_{vac}}{\mu} = \sqrt{\frac{2\chi_{-1}}{\pi^2}}\frac{\varphi_{vac}}{\mu}$
- bare Higgs mass given by difference function

$$\sqrt{\sqrt{2}\omega((r^2-4\mu^2)+12\mu^2\frac{\varphi_{vac}^2}{\mu^2})} = \sqrt{\frac{4\chi_{-1}}{\chi_0}}\frac{\sqrt{\frac{3}{2}\varphi_{vac}^2-\frac{1}{2}\mu^2+\frac{1}{8}r^2}}{\mu}$$





Smooth transition between two phases



1. *Spontaneously broken phase* $\omega^2\|x\|^2 < \frac{\chi_{-1}}{\chi_0}$
   fermions, gauge fields and Higgs are massive, with Higgs mass slightly smaller than NCG-prediction
2. *Unbroken phase* $\omega^2\|x\|^2 \geq \frac{\chi_{-1}}{\chi_0}$
   fermions + gauge fields massless, Higgs remains massive

Mass of gauge fields and fermions dissipates into cosmological constant!

The spectral action: noncommutative case

$$
\begin{aligned}
S(\mathcal{D}_A) = {} & \frac{\theta^4\chi_{-4}}{8\Omega^4} + \frac{2\theta^2\chi_{-2}}{3\Omega^2} + \frac{52\chi_0}{45} + \frac{\chi_0}{2\pi^2(1+\Omega^2)^2}\int d^4x\,\Big\{2D_\mu\phi\star\overline{D_\mu\phi} \\
& + \Big(\frac{(1-\Omega^2)^2}{2} - \frac{(1-\Omega^2)^4}{3(1+\Omega^2)^2}\Big)\big(F^A_{\mu\nu}\star F^{\mu\nu}_A + F^B_{\mu\nu}\star F^{\mu\nu}_B\big) \\
& + \Big(\phi\star\bar\phi + \frac{4\Omega^2}{1+\Omega^2}X^\mu_A\star X_{A\mu} - \frac{\chi_{-1}}{\chi_0}\Big)^2 \\
& + \Big(\bar\phi\star\phi + \frac{4\Omega^2}{1+\Omega^2}X^\mu_B\star X_{B\mu} - \frac{\chi_{-1}}{\chi_0}\Big)^2 \\
& - 2\Big(\frac{4\Omega^2}{1+\Omega^2}X^\mu_0\star X_{0\mu} - \frac{\chi_{-1}}{\chi_0}\Big)^2\Big\}(x) + \mathcal{O}(\chi_1)
\end{aligned}
$$

$$
\Theta = \begin{pmatrix} 0 & \theta & 0 & 0 \\ -\theta & 0 & 0 & 0 \\ 0 & 0 & 0 & \theta \\ 0 & 0 & -\theta & 0 \end{pmatrix}
\qquad
\omega = \frac{2\Omega}{\theta}
$$

deeper entanglement of gauge and Higgs fields

covariant coordinates $X_{A\mu}(x) = (\Theta^{-1})_{\mu\nu}x^\nu + A_\mu(x)$ appear with Higgs field ϕ in unified potential; vacuum is non-trivial!

potential cannot be restricted to Higgs part if distinction into discrete and continuous geometries no longer possible

The vacuum

vacuum field equations $(\phi^{vac} = \overline{\phi^{vac}}, \quad A_\mu^{vac} = B_\mu^{vac})$

$$\frac{1}{g^2}[X_{A\nu},[X_A^\mu,X_A^\nu]_\star]_\star + 2[\phi,[X_A^\mu,\phi]_\star]_\star = \frac{4\Omega^2}{1+\Omega^2}\Big\{\phi\star\phi + \frac{4\Omega^2}{1+\Omega^2}X_A^\nu\star X_{A\nu} - \mu^2, X_A^\mu\Big\}_\star$$

$$2[X_{A\nu},[\phi,X_A^\nu]_\star]_\star = \Big\{\phi\star\phi + \frac{4\Omega^2}{1+\Omega^2}X_A^\nu\star X_{A\nu} - \mu^2, \phi\Big\}_\star$$

(with $\frac{1}{4g^2} = \frac{(1-\Omega^2)^2}{2} - \frac{(1-\Omega^2)^4}{3(1+\Omega^2)^2} \qquad \mu^2 = \frac{\chi_{-1}}{\chi_0}$)

spirit of emerging geometry through phase transitions

- $\Omega = 0 \quad \Rightarrow$ solution: $\phi = \mu 1$, $\quad [X_\mu, X_\nu] = \begin{cases} \Theta_{\mu\nu} \\ 0 \end{cases}$

  $\Omega \neq 0$ gives some dynamical geometry
- analytical solution seems impossible

  $\Rightarrow$ need numerical simulations