

Noncommutative spacetime

Raimar Wolkenhaar

Mathematisches Institut der Westfälischen Wilhelms-Universität
Münster, Germany



Introduction

spectral triples / spectral actions need no further introduction

- the spectral action holds at a particular **scale** connected to experiment by **renormalisation group flow**
- relations at spectral scale translated to prediction of Weinberg angle; the result is wrong by a few percent
- this may indicate **new physics in the big desert**
 candidates:
 - new particles: susy, dark matter, . . .
 - spacetime noncommutativity
- modification of β -functions was demonstrated in the toy model of **ϕ^4 -theory on 4D-Moyal space**

The renormalisation of Moyal ϕ^4 -theory touches another issue with the spectral action: **Spacetime is required to be compact.**

The field theories I consider are not distinguished by their noncommutativity, but by the **different rôle of compactness.**

Supersymmetric quantum mechanics

Let X be a d -dimensional smooth manifold, T^*X trivial

- $a_\mu = e^{-\omega h} \partial_\mu e^{\omega h} = \partial_\mu + W_\mu$, $a_\mu^\dagger = -e^{\omega h} \partial_\mu e^{-\omega h} = -\partial_\mu + W_\mu$
 $h \in C^\infty(X)$ Morse function, $W_\mu = \omega \partial_\mu h$

- commutation relations:

$$[a_\mu, a_\nu] = [a_\mu^\dagger, a_\nu^\dagger] = 0, \quad [a_\mu, a_\nu^\dagger] = 2\omega \partial_\mu \partial_\nu h$$

- d fermionic ladder operators:

$$\{b_\mu, b_\nu\} = 0, \quad \{b_\mu^\dagger, b_\nu^\dagger\} = 0, \quad \{b_\mu, b_\nu^\dagger\} = \delta_{\mu\nu}$$

- supercharges:

$$\mathfrak{Q} := \sum_{\mu=1}^d a_\mu \otimes b_\mu^\dagger, \quad \mathfrak{Q}^\dagger := \sum_{\mu=1}^d a_\mu^\dagger \otimes b_\mu$$

- supersymmetry algebra:

$$\{\mathfrak{Q}, \mathfrak{Q}^\dagger\} = \mathfrak{H} = (-\partial_\mu \partial^\mu + \omega^2 (\partial_\mu h)(\partial^\mu h)) \otimes 1 + \omega (\partial^\mu \partial^\nu h) \otimes [b_\mu^\dagger, b_\nu]$$

$$\{\mathfrak{Q}, \mathfrak{Q}\} = \{\mathfrak{Q}^\dagger, \mathfrak{Q}^\dagger\} = 0, \quad [\mathfrak{Q}, \mathfrak{H}] = [\mathfrak{Q}^\dagger, \mathfrak{H}] = 0$$

cohomology of \mathfrak{Q} related to Morse theory for h [Witten, 1982]

Harmonic oscillator spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D}_i)$

Morse function $h = \frac{1}{2} \|x\|^2$

implies constant $[a_\mu, a_\nu^\dagger] = 2\omega\delta_{\mu\nu}$

Hilbert space $\mathcal{H} = \ell^2(\mathbb{N}^d) \otimes \wedge(\mathbb{C}^d)$: declare ONB

$\{ (a_1^\dagger)^{n_1} \dots (a_d^\dagger)^{n_d} \otimes (b_1^\dagger)^{s_1} \dots (b_d^\dagger)^{s_d} |0\rangle : n_\mu \in \mathbb{N}, s_\mu \in \{0, 1\} \}$

TWO Dirac operators $\mathcal{D}_1 = \Omega + \Omega^\dagger, \quad \mathcal{D}_2 = i\Omega - i\Omega^\dagger$

$$\begin{aligned} \mathcal{D}_1^2 = \mathcal{D}_2^2 = \mathfrak{H} &= (a_\mu^\dagger a^\mu \otimes 1 + 2\omega \otimes b_\mu^\dagger b^\mu) \\ &= 2\omega(N_b + N_f) = H \otimes 1 + \omega \otimes \Sigma \end{aligned}$$

where

$$H = -\frac{\partial^2}{\partial x_\mu \partial x^\mu} + \omega^2 x_\mu x^\mu \quad - \text{harmonic oscillator hamiltonian}$$

$$\Sigma = [b_\mu^\dagger, b^\mu] \quad - \text{spin matrix}$$

algebra $\mathcal{A} = \mathcal{S}(\mathbb{R}^d)$ uniquely determined by smoothness

- $\mathcal{H}_\infty = \bigcap_{k \in \mathbb{N}} \text{dom}(\mathcal{D}^{2k}) = \mathcal{S}(\mathbb{R}^d) \otimes \wedge(\mathbb{C}^d) \simeq (\mathcal{S}(\mathbb{R}^d))^{2^d}$
(trivial projector of rank 2^d)

- Hermitian structure takes values in Schwartz class functions: We may choose

- 1 the commutative algebra $\mathcal{A} = \mathcal{S}(\mathbb{R}^d)$

$$[\mathcal{D}_1, f] = \partial^\mu f \otimes (b_\mu^\dagger - b_\mu), \quad [\mathcal{D}_2, f] = \partial^\mu f \otimes (ib_\mu^\dagger + ib_\mu)$$

bounded & order-one

- 2 the noncommutative Moyal algebra \mathcal{A}_Θ which provides an isospectral deformation (work with H. Grosse)

$$[\mathcal{D}_1, L_\star(f)] = L_\star(\partial_\mu f) \otimes (\delta^{\mu\nu} (b_\nu^\dagger - b_\nu) + \frac{1}{2}\omega\Theta^{\mu\nu} (ib_\nu^\dagger + ib_\nu))$$

$$[\mathcal{D}_2, L_\star(f)] = L_\star(\partial_\mu f) \otimes (\delta^{\mu\nu} (ib_\nu^\dagger + ib_\nu) + \frac{1}{2}\omega\Theta^{\mu\nu} (b_\nu^\dagger - b_\nu))$$

All axioms of spectral triples satisfied, with minor adaptation:

① The asymptotics of eigenvalues of $\langle \mathcal{D} \rangle^{-1} = (\mathcal{D}^2 + 1)^{-\frac{1}{2}}$ gives the **wrong dimension $2d$** .

– Metric dimension is maximum $d = \max\{r \in \mathbb{R} \cap \text{Sd}\}$ of **dimension spectrum Sd**

– For any ϕ belonging to algebra generated by $\delta^n f$ and $\delta^n[\mathcal{D}, f]$, with $\delta\phi = [\langle \mathcal{D} \rangle, \phi]$ and $f \in \mathcal{A}$,

$$\zeta_\phi(z) := \text{Tr}(\phi \langle \mathcal{D} \rangle^{-z})$$

extends holomorphically to $\mathbb{C} \setminus \text{Sd}$, for Sd discrete.

– Computation possible due to **Mehler kernel**

– **Sd = $d - \mathbb{N}$** , all poles simple, d is the **oscillator dimension**.

– All residues are **local**, i.e. finite linear combinations of

$$\int_{\mathbb{R}^d} dx x^{\alpha_0} \partial^{\alpha_1} f_1 \cdots \partial^{\alpha_k} f_k$$

2 **c** takes values in unitisation

$$\mathcal{B} = \{b \in \mathcal{A}'' : b, [\mathcal{D}, b] \in \bigcap_{k \in \mathbb{N}} \text{dom}(\delta^k)\} \ni u_\mu = e^{iX_\mu}$$

$$\mathbf{c} = \sum_{\sigma \in S_d} \epsilon(\sigma) \frac{i^{\frac{d(d-1)}{2}}}{d!} (u_1 \cdots u_d)^{-1} \otimes u_{\sigma(1)} \otimes \cdots \otimes u_{\sigma(d)} \in Z_d(\mathcal{B}, \mathcal{B})$$

Remark: if $1 \in \mathcal{A}$, then $\mathcal{B} = \mathcal{A}$ and $d = \text{asyp. dimension}$

3 There are **TWO images of c** in $\mathcal{B}(\mathcal{H})$:

$$\gamma_1 := \pi_{\mathcal{D}_1}(\mathbf{c}) = i^{\frac{d(d+1)}{2}} (b_1^\dagger - b_1) \cdots (b_d^\dagger - b_d),$$

$$\gamma_2 := \pi_{\mathcal{D}_2}(\mathbf{c}) = i^{\frac{d(d+3)}{2}} (b_1^\dagger + b_1) \cdots (b_d^\dagger + b_d)$$

- satisfy $\gamma_i^2 = 1$, $\gamma_i^* = \gamma_i$, but not $\mathcal{D}_i \gamma_i + (-1)^d \gamma_i \mathcal{D}_i = 0!$
- product $(-i)^d \gamma_1 \gamma_2 = i^d \gamma_2 \gamma_1 = (-1)^{N_f}$ is \mathbb{Z}_2 -grading!
- \mathcal{D}_i^\dagger elliptic ΨDO in the sense of Shubin, of order 1
- index formula of Elliott-Natsume-Nest integrates $\text{tr}(\hat{e}_a (d\hat{e}_a)^{2d})$ over $T^*\mathbb{R}^d$, where \hat{e}_a graph projector

U(1)-Higgs model

tensor $(\mathcal{A}, \mathcal{H}, \mathcal{D}_1)$ with $(\mathbb{C} \oplus \mathbb{C}, \mathbb{C}^2, M\sigma_1, \sigma_3)$ [Connes+Lott]

- $\mathcal{D} = \mathcal{D}_1 \otimes \sigma_3 + 1 \otimes \sigma_1 M = \begin{pmatrix} \mathcal{D}_1 & M \\ M & -\mathcal{D}_1 \end{pmatrix} \quad \begin{pmatrix} f & 0 \\ 0 & g \end{pmatrix} \in \mathcal{A}_{tot}$

- selfadjoint **fluctuated Dirac operators** $\mathcal{D}_A := \mathcal{D} + \sum_i a_i [\mathcal{D}, b_i]$, $a_i, b_i \in \mathcal{A}_{tot} = \mathcal{A} \oplus \mathcal{A}$, are of the form

$$\mathcal{D}_A = \begin{pmatrix} \mathcal{D}_4 + iA^\mu \otimes (b_\mu^\dagger - b_\mu) & \phi \otimes 1 \\ \bar{\phi} \otimes 1 & -(\mathcal{D}_4 + iB^\mu \otimes (b_\mu^\dagger - b_\mu)) \end{pmatrix}$$

for $A_\mu = \overline{A_\mu}$, $B_\mu = \overline{B_\mu}$, $\phi \in \mathcal{A}$

- $\mathcal{D}_A^2 = \begin{pmatrix} (H + |\phi|^2) \otimes 1 + \omega \otimes \Sigma + F_A & D^\mu \phi \otimes (b_\mu^\dagger - b_\mu) \\ -\overline{D^\mu \phi} \otimes (b_\mu^\dagger - b_\mu) & (H + |\phi|^2) \otimes 1 + \omega \otimes \Sigma + F_B \end{pmatrix}$

- $D_\mu \phi = \partial_\mu \phi + i(A_\mu - B_\mu)\phi$
 $F_A = (-\{\partial^\mu, A_\mu\} - iA^\mu A_\mu) \otimes 1 + \frac{1}{4} F_A^{\mu\nu} \otimes [b_\mu^\dagger - b_\mu, b_\nu^\dagger - b_\nu]$

Spectral action principle

most general form of bosonic action is $S(\mathcal{D}_A) = \text{Tr}(\chi(\mathcal{D}_A^2))$

write $\mathcal{D}_A^2 = H + \omega\Sigma - V$

Mehler kernel (in 4D)

$$e^{-t(H+\omega\Sigma)}(x, y) = \frac{\omega^2(1 - \tanh^2(\omega t))^2}{16\pi^2 \tanh^2(\omega t)} e^{-t\omega\Sigma} 1_2 - \frac{\omega}{4} \frac{\|x-y\|^2}{\tanh(\omega t)} - \frac{\omega}{4} \tanh(\omega t) \|x+y\|^2$$

Vacuum trace:

$$\begin{aligned} \text{Tr}(e^{-t(H+\omega\Sigma)} \otimes 1_2) &= 2\text{tr} \left(\int d^4x (e^{-t(H+\omega\Sigma)})(x, x) \right) \\ &= \frac{2}{\tanh^4(\omega t)} = 2(\omega t)^{-4} + \frac{8}{3}(\omega t)^{-2} + \frac{52}{45} + \mathcal{O}(t^2) \end{aligned}$$

- Spectral action is finite, in contrast to standard \mathbb{R}^4 !
- expansion starts with $t^{-4} \Rightarrow$ corresponds to 8-dim. space

One-vertex contribution

$$\int_0^t dt_1 \operatorname{Tr}(e^{-(t-t_1)(H+\omega\Sigma)} 1_2 V e^{-t_1(H+\omega\Sigma)} 1_2)$$

$$= \frac{\omega^2 t}{\pi^2 \tanh^2(\omega t)} \int d^4 x (-2|\phi|^2 - A_\mu A^\mu - B_\mu B^\mu) e^{-\omega \tanh(\omega t) \|x\|^2}$$

- yields $-\mu^2 \phi^2$ for Higgs potential and oscillator potential $+x^2 \phi^2$
- A^2, B^2 eliminated by two vertices with $\{\partial^\mu, A_\mu\}, \{\partial^\mu, B_\mu\}$; these also give Yang-Mills with negative sign, exceeded by two curvature-vertices
- no tadpole, in contrast to Moyal where $\{\partial_\mu + \omega x_\mu, \pi(A^\mu)\}$ contributes

The spectral action

$$\begin{aligned}
 S(\mathcal{D}_A) &= \frac{2\chi_{-4}}{\omega^4} + \frac{8\chi_{-2}}{3\omega^2} + \frac{52\chi_0}{45} \\
 &+ \frac{\chi_0}{\pi^2} \int d^4x \left\{ D^\mu \phi \overline{D_\mu \phi} + \frac{5}{12} (F_{\mu\nu}^A F_A^{\mu\nu} + F_{\mu\nu}^B F_B^{\mu\nu}) \right. \\
 &\quad \left. + \left((|\phi|^2)^2 - \frac{2\chi_{-1}}{\chi_0} |\phi|^2 + 2\omega^2 \|x\|^2 |\phi|^2 \right) \right\} + \mathcal{O}(\chi_1)
 \end{aligned}$$

- spectral action is finite
- only difference in field equations to infinite volume is **additional harmonic oscillator potential for the Higgs**
- Yang-Mills is unchanged (in contrast to Moyal)
- vacuum is at $A_\mu = B_\mu = 0$ and (after gauge transformation) $\phi \in \mathbb{R}$, **rotationally invariant**

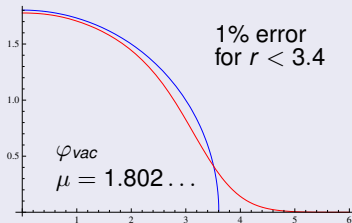
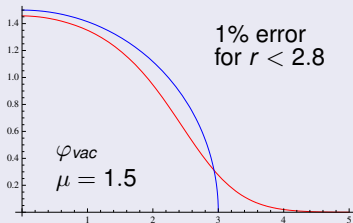
$$\text{rescale } r = 2^{\frac{1}{4}} \sqrt{\omega} \|x\|, \quad \phi = \frac{\pi}{\sqrt{2}\chi_0} \varphi, \quad \mu^2 = \frac{\chi_{-1}}{\sqrt{8}\omega\chi_0}, \quad \lambda = \frac{\pi^2}{\sqrt{2}\omega\chi_0}$$

Field equation

$$-\varphi''(r) - \frac{3}{r}\varphi'(r) + (r^2 - 4\mu^2)\varphi(r) = -\lambda\varphi^3(r)$$

expand $\phi = \sum_{n=0}^{\infty} c_n \varphi_n$ in eigenfunctions φ_n of 4D-harm. oscill.
yields system of cubic equations cut-off at $n \leq N$

$\varphi(r)$ in units of $\frac{2}{\sqrt{\lambda}}$, cutoff at $N = 10$

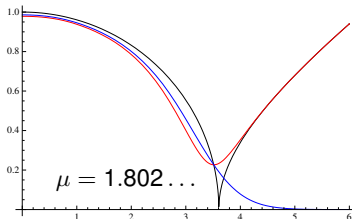
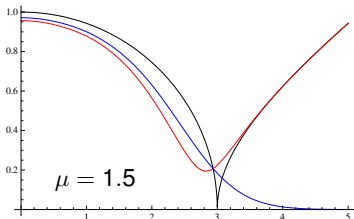


- vacuum solution $\varphi_{vac}(r) < \sqrt{\mu^2 - \frac{1}{4}r^2}$ due to its **negative curvature** and approaches it for $\mu \rightarrow \infty$

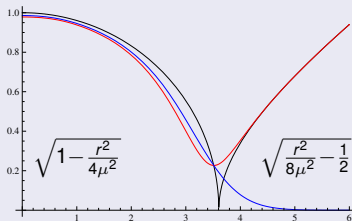
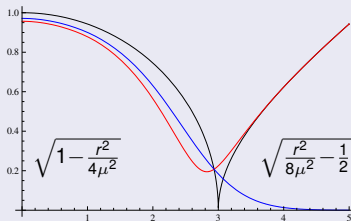
Masses

- scale of bare fermion and gauge field masses given by vacuum expectation value $\sqrt{\frac{4\mu^2}{\lambda} \frac{\varphi_{vac}}{\mu}} = \sqrt{\frac{2\chi-1}{\pi^2} \frac{\varphi_{vac}}{\mu}}$
- bare Higgs mass given by difference function

$$\sqrt{\sqrt{2\omega((r^2 - 4\mu^2) + 12\mu^2 \frac{\varphi_{vac}^2}{\mu^2})}} = \sqrt{\frac{4\chi-1}{\chi_0} \frac{\sqrt{\frac{3}{2}\varphi_{vac}^2 - \frac{1}{2}\mu^2 + \frac{1}{8}r^2}}{\mu}}$$



Smooth transition between two phases



- 1 *Spontaneously broken phase* $\omega^2 \|x\|^2 < \frac{\chi-1}{\chi_0}$
fermions, gauge fields and Higgs are massive, with **Higgs mass slightly smaller than NCG-prediction**
- 2 *Unbroken phase* $\omega^2 \|x\|^2 \geq \frac{\chi-1}{\chi_0}$
fermions + gauge fields massless, Higgs remains massive

vacuum energy is finite; masses of gauge fields and fermions dissipate into **variable cosmological constant**

The spectral action: noncommutative case

$$\begin{aligned}
 S(\mathcal{D}_A) = & \frac{\theta^4 \chi_{-4}}{8\Omega^4} + \frac{2\theta^2 \chi_{-2}}{3\Omega^2} + \frac{52\chi_0}{45} + \frac{\chi_0}{2\pi^2(1+\Omega^2)^2} \int d^4x \left\{ 2D_\mu \phi \star \overline{D_\mu \phi} \right. \\
 & + \left(\frac{(1-\Omega^2)^2}{2} - \frac{(1-\Omega^2)^4}{3(1+\Omega^2)^2} \right) (F_{\mu\nu}^A \star F_A^{\mu\nu} + F_{\mu\nu}^B \star F_B^{\mu\nu}) \\
 & + \left(\phi \star \bar{\phi} + \frac{4\Omega^2}{1+\Omega^2} X_A^\mu \star X_{A\mu} - \frac{\chi_{-1}}{\chi_0} \right)^2 \\
 & + \left(\bar{\phi} \star \phi + \frac{4\Omega^2}{1+\Omega^2} X_B^\mu \star X_{B\mu} - \frac{\chi_{-1}}{\chi_0} \right)^2 \\
 & \left. - 2 \left(\frac{4\Omega^2}{1+\Omega^2} X_0^\mu \star X_{0\mu} - \frac{\chi_{-1}}{\chi_0} \right)^2 \right\} (x) + \mathcal{O}(\chi_1)
 \end{aligned}$$

$$\Theta = \begin{pmatrix} 0 & \theta & 0 & 0 \\ -\theta & 0 & 0 & 0 \\ 0 & 0 & 0 & \theta \\ 0 & 0 & -\theta & 0 \end{pmatrix} \quad \omega = \frac{2\Omega}{\theta}$$

deeper entanglement of gauge and Higgs fields

covariant coordinates $X_{A\mu}(x) = (\Theta^{-1})_{\mu\nu} x^\nu + A_\mu(x)$ appear with Higgs field ϕ in **unified potential**; vacuum is non-trivial!

potential cannot be restricted to Higgs part if distinction into discrete and continuous geometries no longer possible

The vacuum

vacuum field equations

$$(\phi^{vac} = \overline{\phi^{vac}}, \quad A_{\mu}^{vac} = B_{\mu}^{vac})$$

$$\begin{aligned} \frac{1}{g^2} [X_{A\nu}, [X_A^{\mu}, X_A^{\nu}]_{\star}]_{\star} + 2[\phi, [X_A^{\mu}, \phi]_{\star}]_{\star} \\ = \frac{4\Omega^2}{1+\Omega^2} \left\{ \phi \star \phi + \frac{4\Omega^2}{1+\Omega^2} X_A^{\nu} \star X_{A\nu} - \mu^2, X_A^{\mu} \right\}_{\star} \end{aligned}$$

$$2[X_{A\nu}, [\phi, X_A^{\nu}]_{\star}]_{\star} = \left\{ \phi \star \phi + \frac{4\Omega^2}{1+\Omega^2} X_A^{\nu} \star X_{A\nu} - \mu^2, \phi \right\}_{\star}$$

$$\left(\text{with } \frac{1}{4g^2} = \frac{(1-\Omega^2)^2}{2} - \frac{(1-\Omega^2)^4}{3(1+\Omega^2)^2} \quad \mu^2 = \frac{\chi-1}{\chi_0} \right)$$

spirit of **emerging geometry** through phase transitions

- $\Omega = 0 \Rightarrow$ solution: $\phi = \mu 1, \quad [X_{\mu}, X_{\nu}] = \begin{cases} \Theta_{\mu\nu} \\ 0 \end{cases}$

$\Omega \neq 0$ gives some **dynamical geometry**

- analytical solution seems impossible

\Rightarrow **need numerical simulations**

ϕ_4^4 -theory on Moyal space with oscillator potential

We consider the previous model for vanishing Yang-Mills fields
 $A = B = 0$ and **reversed sign in Higgs mass**

action functional

$$S[\phi] = \int d^4x \left(\frac{1}{2} \phi \star (-\Delta + \Omega^2 \tilde{x}^2 + \mu^2) \phi + \frac{\lambda}{4} \phi \star \phi \star \phi \star \phi \right) (x)$$

Moyal product \star defined by Θ and $\tilde{x} := 2\Theta^{-1} \cdot x$

parameters: $\mu^2, \lambda \in \mathbb{R}_+$, $\Omega \in [0, 1]$, redef'n $\phi \mapsto Z^{\frac{1}{2}} \phi$, $Z \in \mathbb{R}_+$

- **renormalisable as formal power series** in λ [Grosse-W.]
 means: well-defined **perturbative** quantum field theory
- **β -function vanishes to all orders** in λ for $\Omega = 1$
 [Disertori-Gurau-Magnen-Rivasseau]
 means: model is believed to exist **non-perturbatively**

The action functional for $\Omega = 1$ in matrix basis

- ensemble of selfadjoint large matrices $(\phi_{mn})_{m,n \in \mathbb{N}_\Lambda^2} \in M_\Lambda$ with **cut-off** Λ in the matrix size
- correlation functions generated by **partition function**

$$\mathcal{Z}[J] = N \int \left(\prod_{m,n \in \mathbb{N}_\Lambda^2} d\phi_{mn} \right) \exp \left(-S[\phi] + \text{tr}(\phi J) \right)$$

We are interested in $\mathbb{N}_\Lambda^2 \rightarrow \mathbb{N}^2$. Correlation functions ill-defined unless $S[\phi]$ is a suitably divergent function of Λ :

$$S[\phi] = \sum_{m,n \in \mathbb{N}_\Lambda^2} \frac{1}{2} \phi_{mn} H_{mn} \phi_{nm} + V(\phi)$$

$$H_{mn} = Z(\mu_{bare}^2 + |m| + |n|) \quad V(\phi) = \frac{Z^2 \lambda}{4} \sum_{m,n,k,l \in \mathbb{N}_\Lambda^2} \phi_{mn} \phi_{nk} \phi_{kl} \phi_{lm}$$

with $|m| = m_1 + m_2$ and divergent $\mu_{bare}[\Lambda, \lambda], Z[\Lambda, \lambda]$

There is no separate Λ -dependence in λ !

Schwinger-Dyson equations and Ward identities

- 1 Schwinger-Dyson equations express n -point function in terms of $(m > n)$ -point functions:

$$\Gamma_{ab} = \text{circle with } a, b \text{ lines} = \text{circle with } a \text{ loop} + \text{circle with } b \text{ loop} + \text{circle with } a \text{ tadpole}$$

- 2 The model admits a Ward identity which expresses n -point functions in terms of $(n - 2)$ -point functions:

$$Z(|a| - |b|) \text{circle with } a, b \text{ lines and dots} = \text{circle with } b \text{ lines and dots} - \text{circle with } a \text{ lines and dots}$$

Result: closed formula for 2-point function alone

$$\Gamma_{ab} = Z^2 \lambda \sum_p \left(\frac{1}{H_{bp} - \Gamma_{bp}} + \frac{1}{H_{ap} - \Gamma_{ap}} - \frac{1}{H_{bp} - \Gamma_{bp}} \frac{(\Gamma_{bp} - \Gamma_{ab})}{Z(|p| - |a|)} \right)$$

Direct renormalisation of SD-equation for Γ_{ab}

- Taylor: $\Gamma_{ab} = Z\mu_{bare}^2 - \mu^2 + (Z-1)(|a|+|b|) + \Gamma_{ab}^{ren}$
with $\Gamma_{00}^{ren} = 0$ and $(\partial\Gamma^{ren})_{00} = 0$

$$\Rightarrow \quad G_{ab}^{-1} = H_{ab} - \Gamma_{ab} = |a| + |b| + \mu^2 - \Gamma_{ab}^{ren}$$

- We replace discrete indices $a, b, p \in \mathbb{N}^2$ by **continuous indices** $a, b, p \in (\mathbb{R}_+)^2$, and sums by integrals.

- This **captures the behaviour at $\Lambda \rightarrow \infty$** of the discrete version, or defines another interesting field theory.

- perturbative calculation suggests change of variables

$$|a| = \frac{\alpha\mu^2}{1-\alpha}, \quad \Gamma_{ab}^{ren} = \frac{\Gamma_{\alpha\beta}\mu^2}{(1-\alpha)(1-\beta)}, \quad \Lambda = \frac{\xi\mu^2}{1-\xi}$$

Elimination of Z shows that limit $\xi \rightarrow 1$ exists

Theorem [Grosse-W., 2009]

The renormalised planar connected two-point function $G_{\alpha\beta}$ of Moyal-harmonic-oscillator ϕ_4^4 -theory satisfies

$$\begin{aligned}
 G_{\alpha\beta} = & 1 + \lambda \left(\frac{1-\alpha}{1-\alpha\beta} (\mathcal{M}_\beta - \mathcal{L}_\beta - \beta\mathcal{Y}) + \frac{1-\beta}{1-\alpha\beta} (\mathcal{M}_\alpha - \mathcal{L}_\alpha - \alpha\mathcal{Y}) \right. \\
 & + \frac{(1-\alpha)(1-\beta)}{1-\alpha\beta} (G_{\alpha\beta} - 1)\mathcal{Y} - \frac{\alpha(1-\beta)}{1-\alpha\beta} (\mathcal{L}_\beta + \mathcal{N}_{\alpha\beta} - \mathcal{N}_{\alpha 0}) \\
 & \left. + \frac{1-\beta}{1-\alpha\beta} \left(\frac{G_{\alpha\beta}}{G_{0\alpha}} - 1 \right) (\mathcal{M}_\alpha - \mathcal{L}_\alpha + \alpha\mathcal{N}_{\alpha 0}) \right)
 \end{aligned}$$

with $\alpha, \beta \in [0, 1)$ and

$$\begin{aligned}
 \mathcal{L}_\alpha &:= \int_0^1 d\rho \frac{G_{\alpha\rho} - G_{0\rho}}{1-\rho} & \mathcal{M}_\alpha &:= \int_0^1 d\rho \frac{\alpha G_{\alpha\rho}}{1-\alpha\rho} \\
 \mathcal{N}_{\alpha\beta} &:= \int_0^1 d\rho \frac{G_{\rho\beta} - G_{\alpha\beta}}{\rho-\alpha} & \mathcal{Y} &= \lim_{\alpha \rightarrow 0} \frac{\mathcal{M}_\alpha - \mathcal{L}_\alpha}{\alpha}
 \end{aligned}$$

- The previous integral equation for $G_{\alpha\beta}$ is **non-perturbatively** defined. Unfortunately, it resisted an exact treatment.
- We look for an iterative solution $G_{\alpha\beta} = \sum_{n=0}^{\infty} \lambda^n G_{\alpha\beta}^{(n)}$.
- This involves **iterated integrals labelled by rooted trees**.

Up to $\mathcal{O}(\lambda^3)$ we need

$$I_{\alpha} := \int_0^1 dx \frac{\alpha}{1 - \alpha x} = -\ln(1 - \alpha),$$

$$I_{\bullet} := \int_0^1 dx \frac{\alpha I_x}{1 - \alpha x} = \text{Li}_2(\alpha) + \frac{1}{2} (\ln(1 - \alpha))^2$$

$$I_{\bullet\bullet} := \int_0^1 dx \frac{\alpha I_x \cdot I_x}{1 - \alpha x} = -2 \text{Li}_3\left(-\frac{\alpha}{1 - \alpha}\right)$$

$$I_{\bullet\bullet\bullet} := \int_0^1 dx \frac{\alpha I_x \cdot \bullet}{1 - \alpha x} = -2 \text{Li}_3\left(-\frac{\alpha}{1 - \alpha}\right) - 2 \text{Li}_3(\alpha) - \ln(1 - \alpha) \zeta(2) \\ + \ln(1 - \alpha) \text{Li}_2(\alpha) + \frac{1}{6} (\ln(1 - \alpha))^3$$

In terms of I_t and $A := \frac{1-\alpha}{1-\alpha\beta}$, $B := \frac{1-\beta}{1-\alpha\beta}$:

$$\begin{aligned}
 G_{\alpha\beta} = & 1 + \lambda \left\{ A(I_\beta - \beta) + B(I_\alpha - \alpha) \right\} \\
 & + \lambda^2 \left\{ A(\beta I_\beta - \beta I_\beta) - \alpha AB((I_\beta)^2 - 2\beta I_\beta + I_\beta) \right. \\
 & \quad + B(\alpha I_\alpha - \alpha I_\alpha) - \beta BA((I_\alpha)^2 - 2\alpha I_\alpha + I_\alpha) \\
 & \quad \left. + AB((I_\alpha - \alpha) + (I_\beta - \beta) + (I_\alpha - \alpha)(I_\beta - \beta) + \alpha\beta(\zeta(2) + 1)) \right\} \\
 & + \lambda^3 \left\{ AW_\beta + \alpha AB(-\mathcal{U}_\beta + I_\alpha I_\beta + I_\alpha I_\beta) + \alpha A^2 B \mathcal{V}_\beta \right. \\
 & \quad + BW_\alpha + \beta BA(-\mathcal{U}_\alpha + I_\beta I_\alpha + I_\beta I_\alpha) + \beta B^2 A \mathcal{V}_\alpha \\
 & \quad + AB(\mathcal{T}_\beta + \mathcal{T}_\alpha - I_\beta(I_\alpha)^2 - I_\alpha(I_\beta)^2 - 6I_\alpha I_\beta) \\
 & \quad + AB^2((1-\alpha)(I_\alpha - \alpha) + 3I_\alpha I_\beta + I_\beta I_\alpha + I_\beta(I_\alpha)^2) \\
 & \quad \left. + BA^2((1-\beta)(I_\beta - \beta) + 3I_\alpha I_\beta + I_\alpha I_\beta + I_\alpha(I_\beta)^2) \right\} + \mathcal{O}(\lambda^4)
 \end{aligned}$$

where

$$\mathcal{T}_\beta := \beta \mathcal{I}_\beta - \beta I_\beta + (I_\beta - \beta),$$

$$\begin{aligned} \mathcal{U}_\beta := & -\beta \mathcal{I}_\beta - (I_\beta)^3 + \beta \mathcal{I}_\beta I_\beta + 2 \mathcal{I}_\beta I_\beta + \beta \zeta(2) I_\beta - 2\beta \zeta(3) \\ & - 2(I_\beta)^2 + \beta (I_\beta)^2 + \mathcal{I}_\beta + \beta I_\beta + 2I_\beta - \beta^2, \end{aligned}$$

$$\begin{aligned} \mathcal{V}_\beta := & \beta \mathcal{I}_\beta - \beta^2 \mathcal{I}_\beta - 2\beta^2 \zeta(3) + 2\beta \mathcal{I}_\beta I_\beta - I_\beta^3 + 2\beta I_\beta \zeta(2) - 3\beta^2 \zeta(2) \\ & + (1 - \beta)(2\beta \mathcal{I}_\beta - 3I_\beta^2 + 3\beta I_\beta - 3I_\beta + \beta), \end{aligned}$$

$$\mathcal{W}_\beta := (\mathcal{I}_\beta - \beta \zeta(2)) - \frac{1}{2} I_\beta \frac{I_\beta - \beta}{\beta} + \frac{1}{2} (I_\beta)^2 - (I_\beta - \beta) - \frac{1}{2} (I_\beta - \beta) - \frac{1}{2} \beta^2$$

Remark: $\frac{I_\beta - \beta}{\beta} = \int_0^1 dx \frac{\beta x}{1 - \beta x}$

(optimal family of iterated integrals not yet determined)

Observations

- Polylogarithms and multiple zeta values appear in **singular part** of **individual graphs** of e.g. ϕ^4 -theory [Broadhurst-Kreimer]
- We encounter them for **regular part** of **all graphs together**

We expect that the planar two-point function $\Gamma_{\alpha\beta}$ is analytic in λ

Strategies for the proof:

- 1 study the recursive generation of these iterated integrals
 - 2 establish bounds directly in the integral equation
- SD-equations for all **higher n -point functions G^n** (planar / non-planar) are expected to be **linear in G^n** with **inhomogeneity given by $G^{<n}$** (verified for planar $n = 4$)

From the known planar 2-point function we could compute or estimate the whole theory – which is an interacting QFT in 4D!