

Non-compact spectral triples with finite volume

Towards noncommutative gauge theory

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Spectral triples

see: A. Connes, "On the spectral characterization of manifolds," 2008

Definition (commutative spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ of dimension $p \in \mathbb{N}$)

... given by a Hilbert space \mathcal{H} , a commutative involutive unital algebra \mathcal{A} represented in \mathcal{H} , and a selfadjoint operator \mathcal{D} in \mathcal{H} with compact resolvent, with

- 1 *Dimension:* k^{th} characteristic value of resolvent of \mathcal{D} is $\mathcal{O}(k^{-\frac{1}{p}})$
- 2 *Order one:* $[[\mathcal{D}, f], g] = 0 \quad \forall f, g \in \mathcal{A}$
- 3 *Regularity:* f and $[\mathcal{D}, f]$ belong to the domain of δ^k , for any $f \in \mathcal{A}$ and $k \in \mathbb{Z}$, where $\delta T := [|\mathcal{D}|, T]$
- 4 *Orientability:* \exists Hochschild p -cycle $\mathbf{c} \in Z_p(\mathcal{A}, \mathcal{A})$ s.t. $\pi_{\mathcal{D}}(\mathbf{c}) = 1$ for p odd, $\pi_{\mathcal{D}}(\mathbf{c}) = \gamma$ for p even with $\gamma = \gamma^*$, $\gamma^2 = 1$, $\gamma\mathcal{D} = -\mathcal{D}\gamma$
- 5 *Finiteness and absolute continuity:* $\mathcal{H}_{\infty} := \bigcap_k \text{dom}(\mathcal{D}^k) \subset \mathcal{H}$ is finitely generated projective \mathcal{A} -module, $\mathcal{H}_{\infty} = e\mathcal{A}^n$, with $e = e^* = e^2 \in M_m(\mathcal{A})$. Hermitian structure $(\xi|\eta) = \sum_{i=1}^n a\xi_i^* \eta_i \in \mathcal{A}$ satisfies $\langle \xi, \eta \rangle = f(\xi|\eta)|\mathcal{D}|^{-p}$

Theorem (Connes, 2008)

Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be a *commutative spectral triple* and assume that

- all endomorphisms $T \in \text{End}_{\mathcal{A}}(\mathcal{H}_{\infty})$ are regular,
- the Hochschild cycle \mathbf{c} is antisymmetric.

Then *there exists a compact oriented smooth manifold X such that $\mathcal{A} = C^{\infty}(X)$ is the algebra of smooth functions on X , and every compact oriented smooth manifold appears in this manner.*

It is known since [Milnor, 1964] that there exist **isospectral manifolds which are not isometric**

Connes' proof identifies the missing piece which in addition to the spectrum of \mathcal{D} characterises the geometry:

It is the **analogue of the Cabibbo-Kobayashi-Maskawa-Matrix** in QFT which encodes the relative position of two bases in the same Hilbert space.

Spectral triples are interesting for physics!

- equivalence classes of spectral triples describe **Yang-Mills theory** (inner automorphisms; exist always in nc case) and possibly **gravity** (outer automorphisms)
- **inner fluctuations**: $\mathcal{D} \mapsto \mathcal{D}_A = \mathcal{D} + A$, $A = \sum f[\mathcal{D}, g]$
for almost-commutative manifolds: **A=Yang-Mills+Higgs**

Spectral action principle [Chamseddine+Connes, 1996]

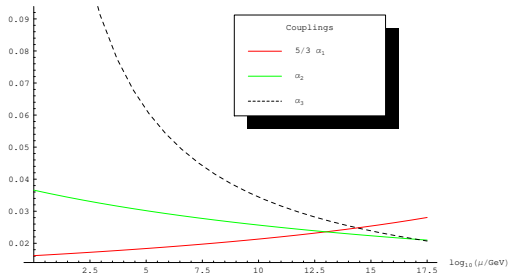
As an automorphism-invariant object, the **(bosonic) action functional of physics** has to be a function of the **spectrum of \mathcal{D}_A** , i.e. **$S(\mathcal{D}_A) = \text{Tr}(\chi(\mathcal{D}_A))$** .

for almost-commutative 4-dim compact manifolds:

- $S(\mathcal{D}_A) = \int_X d \text{vol} (\mathcal{L}_\Lambda + \mathcal{L}_{\text{EH+W}} + \mathcal{L}_{\text{YM}} + \mathcal{L}_{\text{Higgs-kin}} + \mathcal{L}_{\text{Higgs-pot}})$
for **any** function of the spectrum (universality of RG)
- structure of the **standard model more or less unique**

to be precise, an effective action is produced which is **spectral** at a distinguished scale Λ_{GUT}

- **structural relations** at Λ_{GUT} : YM couplings $\alpha_3 = \alpha_2$, $\sin^2 \theta_W = \frac{3}{8}$
Higgs coupling $\lambda = \frac{16\pi}{3}\alpha_2$
- connection to experiment: **renormalisation group flow**:



$\alpha_2, \alpha_3, \theta_W, \lambda$ **scale-dependent**, slope given by β -functions

- Higgs mass $m_H = \sqrt{\frac{2\lambda}{\pi\alpha_2}} m_W$
from λ, α_2 evaluated at m_Z one finds $m_H \sim 170 \text{ GeV}$

Message: Geometry of Nature is scale dependent

- 1 at $E < \Lambda_{\text{electroweak}}$: Riemannian geometry
- 2 at $E = \Lambda_{\text{electroweak}} \dots \Lambda_{\text{GUT}}$: almost-commutative geometry
- 3 at $E > \Lambda_{\text{GUT}}$ truly noncommutative geometry??

We study spectral triples over truly noncommutative algebras and their corresponding spectral actions.

- Simplest example: Moyal space (appears in String Theory)

$$(f \star g)(x) = \int d^4 y \frac{d^4 k}{(2\pi)^4} f(x + \frac{1}{2}\Theta k) g(x + y) e^{iky}$$

- Naïve replacement of local product by \star -product is ill-behaved: not renormalisable (UV/IR-mixing) and incompatible with spectral action principle
- Both problems disappear if we require discrete spectra!

NC ϕ_4^4 -theory develops additional marginal coupling

action functional for real-valued field ϕ on \mathbb{R}^4 :

$$S[\phi] = \int d^4x \left(\frac{1}{2} \phi \star (-\Delta + \Omega^2 \tilde{x}^2 + \mu^2) \phi + \frac{\lambda}{4} \phi \star \phi \star \phi \star \phi \right)(x)$$

Moyal product \star defined by Θ and $\tilde{x} := 2\Theta^{-1} \cdot x$

parameters: $\mu^2, \lambda \in \mathbb{R}_+$, $\Omega \in [0, 1]$, redef'n $\phi \mapsto \mathcal{Z}\phi$, $\mathcal{Z} \in \mathbb{R}_+$

- **renormalisable as formal power series** in λ
[H.Grosse+R.W.; 2004]
means: well-defined **perturbative** quantum field theory
- **β -function vanishes to all orders** in λ for $\Omega = 1$
[M.Disertori, R.Gurau, J.Magnen + V.Rivasseau; 2006]
means: model is believed to exist **non-perturbatively**

Does this model arise from a spectral triple?

Supersymmetric quantum mechanics

X a d -dimensional smooth manifold, T^*X trivial

- $a_\mu = e^{-\omega h} \partial_\mu e^{\omega h} = \partial_\mu + W_\mu$, $a_\mu^\dagger = -e^{\omega h} \partial_\mu e^{-\omega h} = -\partial_\mu + W_\mu$
 $h \in C^\infty(X)$ Morse function, $W_\mu = \omega \partial_\mu h$

- commutation relations:

$$[a_\mu, a_\nu] = [a_\mu^\dagger, a_\nu^\dagger] = 0, \quad [a_\mu, a_\nu^\dagger] = 2\omega \partial_\mu \partial_\nu h$$

- d fermionic ladder operators:

$$\{b_\mu, b_\nu\} = 0, \quad \{b_\mu^\dagger, b_\nu^\dagger\} = 0, \quad \{b_\mu, b_\nu^\dagger\} = \delta_{\mu\nu}$$

- supercharges:

$$\mathfrak{Q} := \sum_{\mu=1}^d a_\mu \otimes b_\mu^\dagger, \quad \mathfrak{Q}^\dagger := \sum_{\mu=1}^d a_\mu^\dagger \otimes b_\mu$$

- supersymmetry algebra:

$$\{\mathfrak{Q}, \mathfrak{Q}^\dagger\} = \mathfrak{H} = (-\partial_\mu \partial^\mu + \omega^2 (\partial_\mu h)(\partial^\mu h)) \otimes 1 + \omega (\partial^\mu \partial^\nu h) \otimes [b_\mu^\dagger, b_\nu]$$

$$\{\mathfrak{Q}, \mathfrak{Q}\} = \{\mathfrak{Q}^\dagger, \mathfrak{Q}^\dagger\} = 0, \quad [\mathfrak{Q}, \mathfrak{H}] = [\mathfrak{Q}^\dagger, \mathfrak{H}] = 0$$

cohomology of \mathfrak{Q} related to Morse theory for h [Witten, 1982]

Harmonic oscillator spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D}_i)$

Morse function $h = \frac{1}{2} \|x\|^2$

implies constant $[a_\mu, a_\nu^\dagger] = 2\omega\delta_{\mu\nu}$

Hilbert space $\mathcal{H} = \ell^2(\mathbb{N}^d) \otimes \wedge(\mathbb{C}^d)$: declare ONB

$\{(a_1^\dagger)^{n_1} \dots (a_d^\dagger)^{n_d} \otimes (b_1^\dagger)^{s_1} \dots (b_d^\dagger)^{s_d} | 0\rangle : n_\mu \in \mathbb{N}, s_\mu \in \{0, 1\}\}$

TWO Dirac operators $\mathcal{D}_1 = \mathcal{Q} + \mathcal{Q}^\dagger, \quad \mathcal{D}_2 = i\mathcal{Q} - i\mathcal{Q}^\dagger$

$$\begin{aligned} \mathcal{D}_1^2 = \mathcal{D}_2^2 = \mathfrak{H} &= \sum_{\mu=1}^d (a_\mu^\dagger a_\mu \otimes 1 + 2\omega \otimes b_\mu^\dagger b_\mu) \\ &= 2\omega(N_b + N_f) = H \otimes 1 + \omega \otimes \Sigma \end{aligned}$$

where

$$H = -\frac{\partial^2}{\partial x_\mu \partial x^\mu} + \omega^2 x_\mu x^\mu \quad - \text{harmonic oscillator hamiltonian}$$

$$\Sigma = \sum_{\mu=1}^d [b_\mu^\dagger, b_\mu] \quad - \text{spin matrix}$$

algebra $\mathcal{A} = \mathcal{S}(\mathbb{R}^d)$ uniquely determined by smoothness

- $\mathcal{H}_\infty = \bigcap_{k \in \mathbb{N}} \text{dom}(\mathcal{D}^{2k}) = \mathcal{S}(\mathbb{R}^d) \otimes \wedge(\mathbb{C}^d) \simeq (\mathcal{S}(\mathbb{R}^d))^{2^d}$
(trivial projector of rank 2^d)

- Hermitian structure takes values in Schwartz class functions: We may choose

- 1 the commutative algebra $\mathcal{A} = \mathcal{S}(\mathbb{R}^d)$

$$[\mathcal{D}_1, f] = \partial^\mu f \otimes (b_\mu^\dagger - b_\mu), \quad [\mathcal{D}_2, f] = \partial^\mu f \otimes (ib_\mu^\dagger + ib_\mu)$$

bounded & order-one

- 2 the noncommutative Moyal algebra \mathcal{A}_Θ which provides an isospectral deformation (work with H. Grosse)

$$[\mathcal{D}_1, L_\star(f)] = L_\star(\partial_\mu f) \otimes (\delta^{\mu\nu} (b_\nu^\dagger - b_\nu) + \frac{1}{2}\omega \Theta^{\mu\nu} (ib_\nu^\dagger + ib_\nu))$$

$$[\mathcal{D}_2, L_\star(f)] = L_\star(\partial_\mu f) \otimes (\delta^{\mu\nu} (ib_\nu^\dagger + ib_\nu) + \frac{1}{2}\omega \Theta^{\mu\nu} (b_\nu^\dagger - b_\nu))$$

All axioms of spectral triples satisfied, with minor adaptation:

- 1 The metric dimension is different from the asymptotics of eigenvalues of \mathcal{D}^{-1} .

This is taken into account by the **dimension spectrum** computed from the **Mehler kernel**

$$e^{-tH}(x, y) = \left(\frac{\omega}{2\pi \sinh(2\omega t)} \right)^{\frac{d}{2}} e^{-\frac{\omega}{4} \coth(\omega t) \|x-y\|^2 - \frac{\omega}{4} \tanh(\omega t) \|x+y\|^2}$$

- 2 \mathbf{c} takes values in unitisation

$$\mathcal{B} = \{b \in \mathcal{A}'' : b, [\mathcal{D}, b] \in \bigcap_{k \in \mathbb{N}} \text{dom}(\mathcal{D}^k)\} \ni u_\mu = e^{iX_\mu}$$

$$\mathbf{c} = \sum_{\sigma \in S_d} \epsilon(\sigma) \frac{i^{\frac{d(d-1)}{2}}}{d!} (u_1 \cdots u_d)^{-1} \otimes u_{\sigma(1)} \otimes \cdots \otimes u_{\sigma(d)} \in Z_d(\mathcal{B}, \mathcal{B})$$

- 3 $\gamma_1 := \pi_{\mathcal{D}_1}(\mathbf{c}) = i^{\frac{d(d+1)}{2}} (b_1^\dagger - b_1) \cdots (b_d^\dagger - b_d)$,

$$\gamma_2 := \pi_{\mathcal{D}_2}(\mathbf{c}) = i^{\frac{d(d+3)}{2}} (b_1^\dagger + b_1) \cdots (b_d^\dagger + b_d)$$

$$\gamma_i^2 = 1, \gamma_i^* = \gamma_i, \quad (-i)^d \gamma_1 \gamma_2 = i^d \gamma_2 \gamma_1 = (-1)^{N_f} \mathbb{Z}_2\text{-grading}$$

U(1)-Higgs model for commutative algebra

tensor $(\mathcal{A}, \mathcal{H}, \mathcal{D}_1)$ with $(\mathbb{C} \oplus \mathbb{C}, \mathbb{C}^2, M\sigma_1, \sigma_3)$ [Connes+Lott]

- $\mathcal{D} = \mathcal{D}_1 \otimes \sigma_3 + 1 \otimes \sigma_1 M = \begin{pmatrix} \mathcal{D}_1 & M \\ M & -\mathcal{D}_1 \end{pmatrix} \quad \begin{pmatrix} f & 0 \\ 0 & g \end{pmatrix} \in \mathcal{A}_{tot}$

- selfadjoint **fluctuated Dirac operators** $\mathcal{D}_A := \mathcal{D} + \sum_i a_i [\mathcal{D}, b_i]$, $a_i, b_i \in \mathcal{A}_{tot} = \mathcal{A} \oplus \mathcal{A}$, are of the form

$$\mathcal{D}_A = \begin{pmatrix} \mathcal{D}_4 + iA^\mu \otimes (b_\mu^\dagger - b_\mu) & \phi \otimes 1 \\ \bar{\phi} \otimes 1 & -(\mathcal{D}_4 + iB^\mu \otimes (b_\mu^\dagger - b_\mu)) \end{pmatrix}$$

for $A_\mu = \overline{A_\mu}$, $B_\mu = \overline{B_\mu}$, $\phi \in \mathcal{A}$

- $\mathcal{D}_A^2 = \begin{pmatrix} (H + |\phi|^2) \otimes 1 + \omega \otimes \Sigma + F_A & D^\mu \phi \otimes (b_\mu^\dagger - b_\mu) \\ -\overline{D^\mu \phi} \otimes (b_\mu^\dagger - b_\mu) & (H + |\phi|^2) \otimes 1 + \omega \otimes \Sigma + F_B \end{pmatrix}$

- $D_\mu \phi = \partial_\mu \phi + i(A_\mu - B_\mu)\phi$
 $F_A = (-\{\partial^\mu, A_\mu\} - iA^\mu A_\mu) \otimes 1 + \frac{1}{4} F_A^{\mu\nu} \otimes [b_\mu^\dagger - b_\mu, b_\nu^\dagger - b_\nu]$

Spectral action principle

most general form of bosonic action is $S(\mathcal{D}_A) = \text{Tr}(\chi(\mathcal{D}_A^2))$

- Laplace transf. + asympt. expansion $e^{-t\mathcal{D}_A^2} = \sum_{n=-\dim/2}^{\infty} a_n(\mathcal{D}_A^2)t^n$

leads to $S(\mathcal{D}_A) = \sum_{n=-\dim/2}^{\infty} \chi_n \text{Tr}(a_n(\mathcal{D}_A^2))$

with $\chi_z = \frac{1}{\Gamma(-z)} \int_0^{\infty} ds s^{-z-1} \chi(s)$ for $z \notin \mathbb{N}$

$\chi_k = (-1)^k \chi^{(k)}(0)$ for $k \in \mathbb{N}$

- a_n – Seeley coefficients, must be computed from scratch

Duhamel expansion: $\mathcal{D}_A^2 = H_0 - V$

$$e^{-t(H_0 - V)} = e^{-tH_0} - \int_0^t dt_1 \frac{d}{dt_1} (e^{-(t-t_1)(H_0 - V)} e^{-t_1 H_0})$$

$$= e^{-tH_0} + \int_0^t dt_1 (e^{-(t-t_1)(H_0 - V)} V e^{-t_1 H_0}) \quad \dots \text{iteration}$$

Mehler kernel (in 4D)

$$e^{-t(H+\omega\Sigma)}(x, y) = \frac{\omega^2(1-\tanh^2(\omega t))^2}{16\pi^2 \tanh^2(\omega t)} e^{-t\omega\Sigma} e^{-\frac{\omega}{4} \frac{\|x-y\|^2}{\tanh(\omega t)} - \frac{\omega}{4} \tanh(\omega t) \|x+y\|^2}$$

$$\mathrm{tr}(e^{-t\omega\Sigma}) = (2 \cosh(\omega t))^d$$

$$\begin{aligned} \mathrm{Tr}(e^{-t(H+\omega\Sigma)} \otimes 1_2) &= 2 \mathrm{tr} \left(\int d^4x (e^{-t(H+\omega\Sigma)})(x, x) \right) \\ &= \frac{2}{\tanh^4(\omega t)} = 2(\omega t)^{-4} + \frac{8}{3}(\omega t)^{-2} + \frac{52}{45} + \mathcal{O}(t^2) \end{aligned}$$

- Spectral action is finite, in contrast to standard \mathbb{R}^4 !
- expansion starts with $t^{-4} \Rightarrow$ corresponds to 8-dim. space

Vertices

only one-vertex contribution from

$$V = \text{diag}(-(\mathbf{A}_\mu \mathbf{A}^\mu + |\phi|^2) \otimes 1, -(\mathbf{B}_\mu \mathbf{B}^\mu + |\phi|^2) \otimes 1)$$

(no tadpole, in contrast to Moyal where $\{\partial_\mu + \omega x_\mu, \pi(\mathbf{A}^\mu)\}$ contributes)

$$\begin{aligned} & \int_0^t dt_1 \text{Tr}(e^{-(t-t_1)(H+\omega\Sigma)} 1_2 V e^{-t_1(H+\omega\Sigma)} 1_2) \\ &= \frac{\omega^2 t}{\pi^2 \tanh^2(\omega t)} \int d^4 x (-2|\phi|^2 - \mathbf{A}_\mu \mathbf{A}^\mu - \mathbf{B}_\mu \mathbf{B}^\mu) e^{-\omega \tanh(\omega t) \|x\|^2} \end{aligned}$$

- yields $-\mu^2 \phi^2$ for Higgs potential and oscillator potential $+x^2 \phi^2$
- A^2, B^2 eliminated by two vertices with $\{\partial^\mu, \mathbf{A}_\mu\}, \{\partial^\mu, \mathbf{B}_\mu\}$; these also give Yang-Mills with negative sign, exceeded by two curvature-vertices

The spectral action

$$S(\mathcal{D}_A) = \frac{2\chi_{-4}}{\omega^4} + \frac{8\chi_{-2}}{3\omega^2} + \frac{52\chi_0}{45} \\ + \frac{\chi_0}{\pi^2} \int d^4x \left\{ D^\mu \phi \overline{D_\mu \phi} + \frac{5}{12} (F_{\mu\nu}^A F_A^{\mu\nu} + F_{\mu\nu}^B F_B^{\mu\nu}) \right. \\ \left. + \left((|\phi|^2)^2 - \frac{2\chi_{-1}}{\chi_0} |\phi|^2 + 2\omega^2 \|x\|^2 |\phi|^2 \right) \right\} + \mathcal{O}(\chi_1)$$

- spectral action is finite
- only difference in field equations to infinite volume is **additional harmonic oscillator potential for the Higgs**
- Yang-Mills is unchanged (in contrast to Moyal)
- vacuum is at $A_\mu = B_\mu = 0$ and (after gauge transformation) $\phi \in \mathbb{R}$, **rotationally invariant**

Field equations

rescaling $r = 2^{\frac{1}{4}}\sqrt{\omega}\|\mathbf{x}\|$, $\phi = \frac{\pi}{\sqrt{2}\chi_0}\varphi$, $\mu^2 = \frac{\chi-1}{\sqrt{8}\omega\chi_0}$, $\lambda = \frac{\pi^2}{\sqrt{2}\omega\chi_0}$

$$-\varphi''(r) - \frac{3}{r}\varphi'(r) + (r^2 - 4\mu^2)\varphi(r) = -\lambda\varphi^3(r)$$

- inhomogeneous confluent hypergeometric diff. eq.
- expand $\varphi = \frac{2}{\sqrt{\lambda}} \sum_{n=0}^{\infty} c_n \varphi_n$ in terms of eigenfunctions of 4-dim. harmonic oscillator:

$$\varphi_n := e^{-\frac{r^2}{2}} L_n^1(r^2) \quad \left(-\frac{d^2}{dr^2} - \frac{3}{r} \frac{d}{dr} + r^2 \right) \varphi_n = 4(n+1)\varphi_n$$

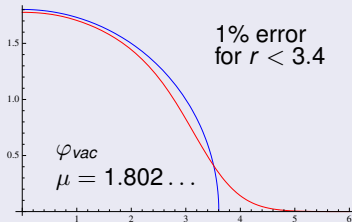
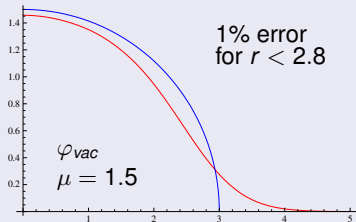
- yields

$$c_n(\mu^2 - n - 1) = \sum_{k,l,m=0}^{\infty} \frac{c_k c_l c_m}{k!l!m!} \left(\frac{d^k}{dw^k} \frac{d^l}{dy^l} \frac{d^m}{dz^m} \frac{(1-wy-wz-yz+2wyz)^n}{(2-w-y-z+wyz)^{n+2}} \right)_{w=y=z=0}$$

- numerical solution: cut-off at N

$$-\varphi''(r) - \frac{3}{r}\varphi'(r) + (r^2 - 4\mu^2)\varphi(r) = -\lambda\varphi^3(r)$$

$\varphi(r)$ in units of $\frac{2}{\sqrt{\lambda}}$, cutoff at $N = 10$

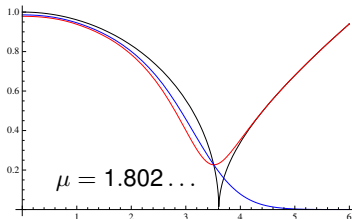
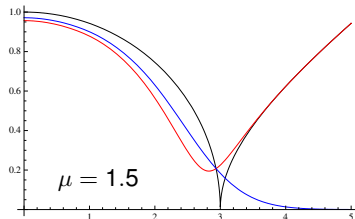


- vacuum solution $\varphi_{vac}(r)$ is smaller than asymptotic curve $\sqrt{\mu^2 - \frac{1}{4}r^2}$ due to its **negative curvature** and approaches it for $\mu \rightarrow \infty$
- $\varphi_{vac} = 0$ for $r \geq 2\mu$, transition is smoothly
- **vacuum solution is integrable**

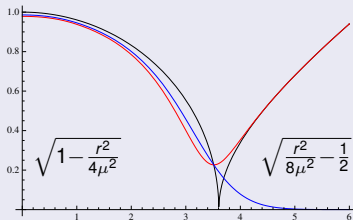
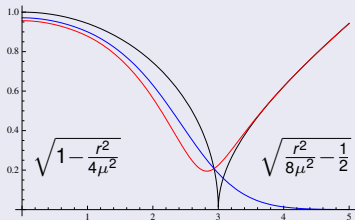
Masses

- scale of bare fermion and gauge field masses given by vacuum expectation value $\sqrt{\frac{4\mu^2}{\lambda} \frac{\varphi_{vac}}{\mu}} = \sqrt{\frac{2\chi_{-1}}{\pi^2} \frac{\varphi_{vac}}{\mu}}$
- bare Higgs mass given by difference function

$$\sqrt{\sqrt{2\omega((r^2 - 4\mu^2) + 12\mu^2 \frac{\varphi_{vac}^2}{\mu^2})}} = \sqrt{\frac{4\chi_{-1}}{\chi_0} \frac{\sqrt{\frac{3}{2}\varphi_{vac}^2 - \frac{1}{2}\mu^2 + \frac{1}{8}r^2}}{\mu}}$$



Smooth transition between two phases



- 1 *Spontaneously broken phase* $\omega^2 \|x\|^2 < \frac{\chi-1}{\chi_0}$
fermions, gauge fields and Higgs are massive, with **Higgs mass slightly smaller than NCG-prediction**
- 2 *Unbroken phase* $\omega^2 \|x\|^2 \geq \frac{\chi-1}{\chi_0}$
fermions + gauge fields massless, Higgs remains massive

Mass of gauge fields and fermions dissipates into cosmological constant!

The spectral action: noncommutative case

$$\begin{aligned}
 S(\mathcal{D}_A) = & \frac{\theta^4 \chi_{-4}}{8\Omega^4} + \frac{2\theta^2 \chi_{-2}}{3\Omega^2} + \frac{52\chi_0}{45} + \frac{\chi_0}{2\pi^2(1+\Omega^2)^2} \int d^4x \left\{ 2D_\mu \phi \star \overline{D_\mu \phi} \right. \\
 & + \left(\frac{(1-\Omega^2)^2}{2} - \frac{(1-\Omega^2)^4}{3(1+\Omega^2)^2} \right) (F_{\mu\nu}^A \star F_A^{\mu\nu} + F_{\mu\nu}^B \star F_B^{\mu\nu}) \\
 & + \left(\phi \star \bar{\phi} + \frac{4\Omega^2}{1+\Omega^2} X_A^\mu \star X_{A\mu} - \frac{\chi_{-1}}{\chi_0} \right)^2 \\
 & + \left(\bar{\phi} \star \phi + \frac{4\Omega^2}{1+\Omega^2} X_B^\mu \star X_{B\mu} - \frac{\chi_{-1}}{\chi_0} \right)^2 \\
 & \left. - 2 \left(\frac{4\Omega^2}{1+\Omega^2} X_0^\mu \star X_{0\mu} - \frac{\chi_{-1}}{\chi_0} \right)^2 \right\} (x) + \mathcal{O}(\chi_1)
 \end{aligned}$$

$$\Theta = \begin{pmatrix} 0 & \theta & 0 & 0 \\ -\theta & 0 & 0 & 0 \\ 0 & 0 & 0 & \theta \\ 0 & 0 & -\theta & 0 \end{pmatrix} \quad \omega = \frac{2\Omega}{\theta}$$

deeper entanglement of gauge and Higgs fields

covariant coordinates $X_{A\mu}(x) = (\Theta^{-1})_{\mu\nu} x^\nu + A_\mu(x)$ appear with Higgs field ϕ in **unified potential**; vacuum is non-trivial!

potential cannot be restricted to Higgs part if distinction into discrete and continuous geometries no longer possible

The vacuum

vacuum field equations

$$(\phi^{vac} = \overline{\phi^{vac}}, \quad A_{\mu}^{vac} = B_{\mu}^{vac})$$

$$\begin{aligned} \frac{1}{g^2} [X_{A\nu}, [X_A^{\mu}, X_A^{\nu}]_{\star}]_{\star} + 2[\phi, [X_A^{\mu}, \phi]_{\star}]_{\star} \\ = \frac{4\Omega^2}{1+\Omega^2} \left\{ \phi \star \phi + \frac{4\Omega^2}{1+\Omega^2} X_A^{\nu} \star X_{A\nu} - \mu^2, X_A^{\mu} \right\}_{\star} \end{aligned}$$

$$2[X_{A\nu}, [\phi, X_A^{\nu}]_{\star}]_{\star} = \left\{ \phi \star \phi + \frac{4\Omega^2}{1+\Omega^2} X_A^{\nu} \star X_{A\nu} - \mu^2, \phi \right\}_{\star}$$

$$\left(\text{with } \frac{1}{4g^2} = \frac{(1-\Omega^2)^2}{2} - \frac{(1-\Omega^2)^4}{3(1+\Omega^2)^2} \quad \mu^2 = \frac{\chi-1}{\chi_0} \right)$$

spirit of **emerging geometry** through phase transitions

- $\Omega = 0 \Rightarrow$ solution: $\phi = \mu 1, \quad [X_{\mu}, X_{\nu}] = \begin{cases} \Theta_{\mu\nu} \\ 0 \end{cases}$

$\Omega \neq 0$ gives some **dynamical geometry**

- analytical solution seems impossible

\Rightarrow **need numerical simulations**

Matrix representation

- matrix base in two dimensions (radial coordinates):

$$f_{mn}(\rho, \varphi) = 2(-1)^m \sqrt{\frac{m!}{n!}} e^{i\varphi(n-m)} \left(\sqrt{\frac{2}{\theta}} \rho \right)^{n-m} e^{-\frac{\rho^2}{\theta}} L_m^{n-m} \left(\frac{2}{\theta} \rho^2 \right)$$

satisfies:

$$(f_{mn} \star f_{kl})(x) = \delta_{nk} f_{ml}(x)$$
$$\int d^2x f_{mn}(x) = \sqrt{\det(2\pi\Theta)} \delta_{mn}$$

- expansion $X^\mu(x) = \sum_{m_i, n_i \in \mathbb{N}} X_{m_1 n_1}^{\mu} f_{m_1 n_1}(x^1, x^2) f_{m_2 n_2}(x^3, x^4)$

$$\phi(x) = \sum_{m_i, n_i \in \mathbb{N}} \phi_{m_1 n_1} f_{m_1 n_1}(x^1, x^2) f_{m_2 n_2}(x^3, x^4)$$

- We can forget the Moyal product:
Action and field equations correspond to **5-matrix model**

Can we use knowledge from lattice gauge theory?

- Action is different, but adaptation seems possible
- Main difficulty is the **asymptotics** (and its realisation by finite matrices):

$\phi, A = X - \Theta^{-1} \cdot x$ vanish rapidly at ∞

$$(X_1 + iX_2)_{\substack{m_1 & n_1 \\ m_2 & n_2}} = \sqrt{\theta n_1} \delta_{m_1, n_1 - 1} \delta_{m_2, n_2}$$

$$(X_1 - iX_2)_{\substack{m_1 & n_1 \\ m_2 & n_2}} = \sqrt{\theta m_1} \delta_{m_1 - 1, n_1} \delta_{m_2, n_2}$$

- Choose X_μ or A_μ as variables?
- **Minima of action** vs. **solutions of field equations**?
- Dependence on μ, g, λ, Ω ? Phase transitions?