

The standard model in NCG

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Real structure

Definition

A spectral triple $(\mathcal{A}, \mathcal{H}, D)$, with \mathcal{A} not necessarily commutative, satisfies the **noncommutative order-one condition** if there exists an anti-unitary operator J on \mathcal{H} such that $[a, JbJ^{-1}] = 0$ and $[[D, a], JbJ^{-1}] = 0$ for all $a, b \in \mathcal{A}$.

J defines a **real structure of KO-dimension** $k \in \mathbb{Z}/8$ if

$$J^2 = \epsilon, \quad JD = \epsilon' DJ, \quad J\gamma = \epsilon'' \gamma J \quad \text{for } p \text{ even}$$

where the signs $\epsilon, \epsilon', \epsilon'' = \pm 1$ are functions of $k \pmod 8$:

k	0	1	2	3	4	5	6	7
ϵ	1	1	-1	-1	-1	-1	1	1
ϵ'	1	-1	1	1	1	-1	1	1
ϵ''	1		-1		1		-1	

Matrix-valued spectral triples in KO-dimension 6

We look for matrix solutions of the axioms for $(\mathcal{A}, \mathcal{H}, D, J, \gamma)$ with

$$J^2 = 1, \quad JD = DJ, \quad J\gamma = -\gamma J, \quad [a, JbJ] = 0, \quad [[D, a], JbJ] = 0, \\ \gamma = \gamma^*, \quad \gamma^2 = 1, \quad D\gamma = -\gamma D,$$

The conditions are implemented step by step and restrict an initial matrix algebra \mathcal{A} eventually to \mathcal{A}_F .

Requirements

- 1 \mathcal{H} has a **separating vector**, i.e. there is a $\xi \in \mathcal{H}$ such that $\mathcal{A} \ni a \mapsto a\xi \in \mathcal{H}$ is injective.
- 2 $(\mathcal{A}, \mathcal{H}, J)$ is **irreducible**, i.e. there is no non-trivial ($e \neq 0, 1$) projector $e \in \mathcal{B}(\mathcal{H})$ which commutes with \mathcal{A} and J .

Lemma

- 1 Let $e \neq 1$ be a projector in the *centre* $Z(\mathcal{A})$. Then $eJeJ^{-1} = 0$.
- 2 Let $e_1, e_2 \in Z(\mathcal{A})$ be projectors with $e_1e_2 = 0$. Then $e_1Je_2J^{-1} + e_2Je_1J^{-1} \in \{0, 1\}$

Proposition

Let $\mathcal{A}_{\mathbb{C}}$ be the *complexification* of \mathcal{A} . Then one of the following cases is realised:

- 1 $Z(\mathcal{A}_{\mathbb{C}}) = \mathbb{C}1$.
- 2 $Z(\mathcal{A}_{\mathbb{C}}) = \mathbb{C}e_1 \oplus \mathbb{C}e_2$ and $e_1 = Je_2J^{-1}$ for the minimal projectors e_1, e_2 .

Proposition

Let $Z(\mathcal{A}_{\mathbb{C}}) = \mathbb{C}1$ and γ a \mathbb{Z}_2 -grading with $\gamma\mathcal{A}\gamma^{-1} = \mathcal{A}$. Then $\gamma\mathcal{J} = \mathcal{J}\gamma$.

In particular, $Z(\mathcal{A}_{\mathbb{C}}) = \mathbb{C}e_1 \oplus \mathbb{C}e_2$ in KO-dimension 6.

First steps in the proof:

- $A_{\mathbb{C}} = M_k(\mathbb{C})$ for some $k \in \mathbb{N}^{\times}$
- Existence of separating vector implies that \mathcal{H} contains subspace isomorphic to $A_{\mathbb{C}} \otimes \mathcal{J}A_{\mathbb{C}}\mathcal{J}^{-1} = M_{k^2}(\mathbb{C})$
- Irreducibility implies $\mathcal{H} = M_{k^2}(\mathbb{C}) \ni x$ with $\mathcal{J}x = x^*$

$$Z(\mathcal{A}_\mathbb{C}) = \mathbb{C}e_1 \oplus \mathbb{C}e_2$$

- $\mathcal{A}_\mathbb{C} = M_{k_1}(\mathbb{C}) \oplus M_{k_2}(\mathbb{C})$ for $k_1, k_2 \in \mathbb{N}^\times$
- $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ with $\mathcal{H}_i = e_i \mathcal{H}$
- $e_2 = J e_1 J^{-1} \Rightarrow J \mathcal{H}_1 = \mathcal{H}_2$ and $J \mathcal{H}_2 = \mathcal{H}_1$
- separating vector and irreducibility:
 $\mathcal{H}_1 \simeq M(k_1 \times k_2, \mathbb{C}) \ni x$ and $\mathcal{H}_2 \simeq M(k_2 \times k_1, \mathbb{C}) \ni y$
 with $J(x, y) = (y^*, x^*)$
- $\dim(\mathcal{A}_\mathbb{C}) = k_1^2 + k_2^2$ and $\dim(\mathcal{H}) = 2k_1 k_2$
 separating vector $\Rightarrow k_1 = k_2 = k$.

Conclusion: $\mathcal{A}_\mathbb{C} = M_k(\mathbb{C}) \oplus M_k(\mathbb{C})$ and $\mathcal{H} = M_{k^2}(\mathbb{C}) \oplus M_{k^2}(\mathbb{C})$

Requirements from particle physics

- 1 One block of the **real algebra** \mathcal{A} is quaternionic

$$\mathcal{A} = M_k(\mathbb{H}) \oplus M_{2k}(\mathbb{C})$$

- 2 The quaternionic block has a non-trivial \mathbb{Z}_2 -grading γ_1

The **minimal solution** of these requirements is

$$\mathcal{A} = M_2(\mathbb{H}) \oplus M_4(\mathbb{C}).$$

Up to isomorphisms, we have

$$J \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y^* \\ x^* \end{pmatrix},$$

$$\gamma = (\gamma_1, 0) - J(\gamma_1, 0)J^{-1} \in \mathcal{A}\mathcal{A}^{op}, \quad \gamma \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \gamma_1 x \\ -y\gamma_1 \end{pmatrix}$$

with $\gamma_1 = \text{diag}(1_2, -1_2) \in M_2(\mathbb{H})$.

The **even part** \mathcal{A}^{ev} of \mathcal{A} is

$$\mathcal{A}^{ev} = (\mathbb{H} \oplus \mathbb{H}) \oplus M_4(\mathbb{C})$$

Order-one condition

We look for a D which

- satisfies the order-one condition
- connects the two blocks of the algebra, $e_1 D e_2 \neq 0$

We show that this is only possible if \mathcal{A}^{ev} is further restricted to \mathcal{A}_F .

- Let $\pi_i(a) = e_i a$. The representations π_i are **disjoint** if there are no common subrepresentations.
- Let $T \in \text{End}(\mathcal{H})$ with $[T, a] = 0$. Then $e_1 T e_2 \pi_2(a) = \pi_1(a) e_1 T e_2$.
- If π_i are disjoint, then $e_1 T e_2 = 0 = e_2 T e_1$.
- Same arguments for $[T, J b J^{-1}] = 0$. Disjoint π_i imply $e_1 T e_2 = 0 = e_2 T e_1$.

Conclusion: The π_i are not disjoint.

Proposition

Up to isomorphisms of \mathcal{A}^{ev} there is a unique involutive subalgebra $\mathcal{A}_F \subset \mathcal{A}^{ev}$ of maximal dimension which under the order-one condition permits a connecting D . This solution is the matrix algebra of the standard model

$$\mathcal{A}_F = \mathbb{H} \oplus \mathbb{C} \oplus M_3(\mathbb{C}) \subset M_2(\mathbb{H}) \oplus M_4(\mathbb{C})$$

$$\ni \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & \bar{\lambda} & 0 & 0 \\ 0 & 0 & \alpha & \beta \\ 0 & 0 & -\bar{\beta} & \bar{\alpha} \end{pmatrix} \oplus \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & m_{11} & m_{12} & m_{13} \\ 0 & m_{21} & m_{22} & m_{23} \\ 0 & m_{31} & m_{32} & m_{33} \end{pmatrix}$$

The Hilbert space is $M_4(\mathbb{C}) \oplus M_4(\mathbb{C}) \in \binom{x}{y}$. We have

$$J \binom{x}{y} = \begin{pmatrix} y^* \\ x^* \end{pmatrix}, \quad \gamma \binom{x}{y} = \begin{pmatrix} \gamma_1 x \\ -y \gamma_1 \end{pmatrix}$$

The Hilbert space is $M_4(\mathbb{C}) \oplus M_4(\mathbb{C})$. Its elements are parametrised by elementary fermions:

$$\mathcal{H} \ni \left(\begin{array}{c|ccc} \nu_R & u_{Rr} & u_{Rb} & u_{Rg} \\ e_R & d_{Rr} & d_{Rb} & d_{Rg} \\ \hline \nu_L & u_{Lr} & u_{Lb} & u_{Lg} \\ e_L & d_{Lr} & d_{Lb} & d_{Lg} \\ \hline \nu_R^c & e_R^c & \nu_L^c & e_L^c \\ \hline u_{Rr}^c & d_{Rr}^c & u_{Lr}^c & d_{Lr}^c \\ u_{Rb}^c & d_{Rb}^c & u_{Lb}^c & d_{Lb}^c \\ u_{Rg}^c & d_{Rg}^c & u_{Lg}^c & d_{Lg}^c \end{array} \right)$$

We now look for the complete D (not only $e_1 D e_2$) which is even and satisfies order-one.

Lepton/quark decomposition

We want to represent D as a matrix acting on \mathcal{H}_F :

$$\mathcal{H}_F = \mathcal{H}_1 \oplus \mathcal{H}_3, \quad \mathcal{H}_1 \simeq \mathbb{C}^8 \ni \begin{pmatrix} l_R \\ l_L \\ l_R^c \\ l_L^c \end{pmatrix}, \quad \mathcal{H}_3 \simeq \mathbb{C}^{24} \ni \begin{pmatrix} \mathbf{q}_R \\ \mathbf{q}_L \\ \mathbf{q}_R^c \\ \mathbf{q}_L^c \end{pmatrix}$$

with leptons $\nu, \mathbf{e} \in \mathbb{C}$

$$l_R = \begin{pmatrix} \nu_R \\ \mathbf{e}_R \end{pmatrix}, \quad l_L = \begin{pmatrix} \nu_L \\ \mathbf{e}_L \end{pmatrix}, \quad l_R^c = \begin{pmatrix} \nu_R^c \\ \mathbf{e}_R^c \end{pmatrix}, \quad l_L^c = \begin{pmatrix} \nu_L^c \\ \mathbf{e}_L^c \end{pmatrix}$$

and quarks $\mathbf{u}, \mathbf{d} \in \mathbb{C}^3$

$$\mathbf{q}_R = \begin{pmatrix} \mathbf{u}_R \\ \mathbf{d}_R \end{pmatrix}, \quad \mathbf{q}_L = \begin{pmatrix} \mathbf{u}_L \\ \mathbf{d}_L \end{pmatrix}, \quad \mathbf{q}_R^c = \begin{pmatrix} \mathbf{u}_R^c \\ \mathbf{d}_R^c \end{pmatrix}, \quad \mathbf{q}_L^c = \begin{pmatrix} \mathbf{u}_L^c \\ \mathbf{d}_L^c \end{pmatrix}.$$

Write operators on \mathcal{H}_F as $T_F = \begin{pmatrix} T_{11} & T_{13} \\ T_{31} & T_{33} \end{pmatrix}$ with

$T_{11} \in M_8(\mathbb{C})$ and $T_{33} \in M_{24}(\mathbb{C})$.

- $\mathcal{A}_F, J_F, \gamma_F$ diagonal (no 13,31-blocks)
- $D_{13} = 0$ and $D_{31} = 0$ from order-one and $D_F \gamma_F = -\gamma_F D_F$

diagonal parts of \mathcal{A}_F :

$$(q, \lambda, m)_{11} = \begin{pmatrix} q_\lambda & 0 & 0 & 0 \\ 0 & q & 0 & 0 \\ 0 & 0 & \lambda 1_2 & 0 \\ 0 & 0 & 0 & \lambda 1_2 \end{pmatrix}, \quad q_\lambda = \begin{pmatrix} \lambda & 0 \\ 0 & \bar{\lambda} \end{pmatrix}$$

$$(q, \lambda, m)_{33} = \begin{pmatrix} q_\lambda \otimes 1_3 & 0 & 0 & 0 \\ 0 & q \otimes 1_3 & 0 & 0 \\ 0 & 0 & 1_2 \otimes m & 0 \\ 0 & 0 & 0 & 1_2 \otimes m \end{pmatrix},$$

with $q \in \mathbb{H}$, $\lambda \in \mathbb{C}$, $m \in M_3(\mathbb{C})$

diagonal part of J_F :

$$J_{11} = \begin{pmatrix} 0 & 0 & 1_2 & 0 \\ 0 & 0 & 0 & 1_2 \\ 1_2 & 0 & 0 & 0 \\ 0 & 1_2 & 0 & 0 \end{pmatrix} \circ CC$$

$$J_{33} = \begin{pmatrix} 0 & 0 & 1_2 \otimes 1_3 & 0 \\ 0 & 0 & 0 & 1_2 \otimes 1_3 \\ 1_2 \otimes 1_3 & 0 & 0 & 0 \\ 0 & 1_2 \otimes 1_3 & 0 & 0 \end{pmatrix} \circ CC$$

where CC means complex conjugation

diagonal part of γ_F :

$$\gamma_{11} = \begin{pmatrix} 1_2 & 0 & 0 & 0 \\ 0 & -1_2 & 0 & 0 \\ 0 & 0 & -1_2 & 0 \\ 0 & 0 & 0 & 1_2 \end{pmatrix},$$

$$\gamma_{33} = \begin{pmatrix} 1_2 \otimes 1_3 & 0 & 0 & 0 \\ 0 & -1_2 \otimes 1_3 & 0 & 0 \\ 0 & 0 & -1_2 \otimes 1_3 & 0 \\ 0 & 0 & 0 & 1_2 \otimes 1_3 \end{pmatrix}.$$

$\gamma_F^2 = 1$, $\gamma_F = \gamma_F^*$, $J_F^2 = 1$ and $\gamma_F J_F = -J_F \gamma_F$ satisfied

Diagonal part of D_F ; from $\{D_F, \gamma_F\} = 0$, $[D_F, J_F] = 0$ and connecting part:

$$D_{11} = \left(\begin{array}{cc|cc} 0 & Y_1 & T & 0 \\ Y_1^* & 0 & 0 & 0 \\ \hline T^* & 0 & 0 & Y_1 \\ 0 & 0 & Y_1^t & 0 \end{array} \right), \quad D_{33} = \left(\begin{array}{cc|cc} 0 & \mathbf{Y}_3 & 0 & 0 \\ \mathbf{Y}_3^* & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & \mathbf{Y}_3 \\ 0 & 0 & \mathbf{Y}_3^t & 0 \end{array} \right),$$

with $Y_1 \in M_2(\mathbb{C})$, $\mathbf{Y}_3 \in M_2(\mathbb{C}) \otimes M_3(\mathbb{C})$ and

$$T = T^t = \begin{pmatrix} Y_R & 0 \\ 0 & 0 \end{pmatrix} \text{ with } Y_R \in \mathbb{C}$$

- order-one: $\mathbf{Y}_3 = Y_3 \otimes 1_3$ with $Y_3 \in M_2(\mathbb{C})$
- **Physical condition:** D_F commutes with representation of \mathbb{C} in \mathcal{A}_F , i.e. with $(q_\lambda, \lambda, 0)$

- leads to $Y_1 = \begin{pmatrix} Y_\nu & 0 \\ 0 & Y_e \end{pmatrix}, \quad Y_3 = \begin{pmatrix} Y_u & 0 \\ 0 & Y_d \end{pmatrix}$

Cabibbo-Kobayashi-Maskawa matrices

- Standard model needs **3 copies** of \mathcal{H}_F .
- leptons $\nu, e \in \mathbb{C}^3$, quarks $\mathbf{u}, \mathbf{d} \in \mathbb{C}^3 \otimes \mathbb{C}^3$
- $Y_{\nu, e, u, d, R} \in M_3(\mathbb{C})$ in D

We say that D_F, D'_F are equivalent if $D'_F = UD_FU^*$ for a unitary matrix U which commutes with $\mathcal{A}_F, J_F, \gamma_F$.

$$U_{11} = \begin{pmatrix} \text{diag}(V_1, V_2) & 0 & 0 & 0 \\ 0 & \text{diag}(V_3, V_3) & 0 & 0 \\ 0 & 0 & \text{diag}(\overline{V_1}, \overline{V_2}) & 0 \\ 0 & 0 & 0 & \text{diag}(\overline{V_3}, \overline{V_3}) \end{pmatrix}$$

$$U_{33} = \begin{pmatrix} \text{diag}(W_1, W_2) \otimes 1_3 & 0 & 0 & 0 \\ 0 & \text{diag}(W_3, W_3) \otimes 1_3 & 0 & 0 \\ 0 & 0 & \text{diag}(\overline{W_1}, \overline{W_2}) \otimes 1_3 & 0 \\ 0 & 0 & 0 & \text{diag}(\overline{W_3}, \overline{W_3}) \otimes 1_3 \end{pmatrix}$$

with $V_j, W_j \in U(3)$

Product of spectral triples

$$\begin{aligned} \mathcal{A} &= \mathcal{A}_M \otimes \mathcal{A}_F, & \mathcal{H} &= \mathcal{H}_M \otimes \mathcal{H}_F, & D &= D_M \otimes 1_F + \gamma_M \otimes D_F, \\ \mathcal{J} &= \mathcal{J}_M \otimes \mathcal{J}_F, & \gamma &= \gamma_M \otimes \gamma_F \end{aligned}$$

with

- $\mathcal{A}_M = C^\infty(M)$, $\mathcal{H}_M = L^2(M, \mathcal{S})$, $\gamma_M = \gamma^5$, $\mathcal{J}_M = \gamma^2 \circ CC$ and

$$D_M = ie_\mu^a \gamma^a \nabla_\mu^{\mathcal{S}}, \quad \nabla_\mu^{\mathcal{S}} = \partial_\mu + \frac{1}{8} \omega_\mu^{ab} [\gamma_a, \gamma_b]$$

- e_μ^a vierbein, ω_μ^{ab} spin connection form:

$$\delta_{ab} e_\mu^a e_\nu^b = g_{\mu\nu}, \quad \partial_\mu e_a^\nu - \Gamma_{\mu\rho}^\nu e_a^\rho + \omega_\mu^{ab} e_b^\nu = 0$$

- Lichnerowicz formula

$$D_M^2 = \Delta^{LC} + \frac{1}{4} R, \quad \Delta^{LC} = -g^{\mu\nu} (\partial_\mu \partial_\nu - \Gamma_{\mu\nu}^\rho \partial_\rho), \quad R = g^{\mu\nu} R_{\mu\nu}$$

Fluctuations

- replace D by **fluctuated** operator $D_A = D + A + JAJ^{-1}$, where $A = A^* = \sum_{\alpha} a_{\alpha}[D, b_{\alpha}]$
- D_A satisfies the same axioms as D
- $D_M \otimes 1$ generates 1-forms $G = e_a^{\mu} \gamma^a G_{\mu}$ with $G_{\mu} = \sum_{\alpha} a_{\alpha} \partial_{\mu}(b_{\alpha}) \in \mathcal{A}$
- $\gamma_M \otimes D_F$ generates 1-forms $\gamma^5 \phi = \gamma^5 \sum_{\alpha} a_{\alpha}[D_F, b_{\alpha}]$

$$\Phi_{11} = \left(\begin{array}{cc|cc} 0 & \Phi_{\ell} & 0 & 0 \\ \Phi_{\ell}^* & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right),$$

$$\Phi_{\ell} = \begin{pmatrix} M_{\nu} \overline{C_{\ell} \phi_1} & M_{\nu} \overline{C_{\ell} \phi_2} \\ -M_e \phi_2 & M_e \phi_1 \end{pmatrix}$$

$$\Phi_{33} = \left(\begin{array}{cc|cc} 0 & \Phi_q \otimes 1_3 & 0 & 0 \\ \Phi_q \otimes 1_3 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right),$$

$$\Phi_q = \begin{pmatrix} M_U \overline{C_q \phi_1} & M_U \overline{C_q \phi_2} \\ -M_d \phi_2 & M_d \phi_1 \end{pmatrix}$$

lepton part of D_A is:

$$D_{A,11} = \left(\begin{array}{cc|cc} i\gamma^a e_a^\mu \nabla_\mu^{S,B} & \gamma_5 \Phi_\ell & T & 0 \\ \gamma_5 \Phi_\ell^* & i\gamma^a e_a^\mu \nabla_\mu^{S,B,W} & 0 & 0 \\ \hline T^* & 0 & i\gamma^a e_a^\mu \nabla_\mu^{S,B} & \gamma_5 \overline{\Phi_\ell} \\ 0 & 0 & \gamma_5 \Phi_\ell^t & i\gamma^a e_a^\mu \nabla_\mu^{S,B,W} \end{array} \right)$$

with

$$\nabla_\mu^{S,B} = \begin{pmatrix} \nabla_\mu^S & 0 \\ 0 & \nabla_\mu^S - 2iB_\mu \end{pmatrix} 1_\gamma,$$

$$\nabla_\mu^{S,B,W} = \begin{pmatrix} \nabla_\mu^S + i(-B_\mu - W_\mu^3) & -i(W_\mu^1 - iW_\mu^2) \\ -i(W_\mu^1 + iW_\mu^2) & \nabla_\mu^S + i(-B_\mu + W_\mu^3) \end{pmatrix} 1_3$$

$B_\mu, W_\mu^a \in C^\infty(X)$ real-valued

quark part of D_A is

$$D_{A,33} = \left(\begin{array}{cc|cc} i\gamma^a e_a^\mu \nabla_\mu^{S,B,G} & \gamma_5 \Phi_q & 0 & 0 \\ \gamma_5 \Phi_q^* & i\gamma^a e_a^\mu \nabla_\mu^{S,W,G} & 0 & 0 \\ \hline 0 & 0 & i\gamma^a e_a^\mu \nabla_\mu^{S,B,G} & \gamma_5 \overline{\Phi}_q \\ 0 & 0 & \gamma_5 \Phi_q^t & i\gamma^a e_a^\mu \nabla_\mu^{S,W,G} \end{array} \right)$$

with

$$\nabla_\mu^{S,B,G} = \begin{pmatrix} \nabla_\mu^S 1_3 + i((-G_\mu^0 + B_\mu)1_3 - G_\mu) & 0 \\ 0 & \nabla_\mu^S 1_3 + i(-(G_\mu^0 + B_\mu)1_3 - G_\mu) \end{pmatrix} 1_3$$

$$\nabla_\mu^{S,W,G} = \begin{pmatrix} \nabla_\mu^S 1_3 + i((-G_\mu^0 - W_\mu^3)1_3 + G_\mu) & -i(W_\mu^1 - iW_\mu^2)1_3 \\ -i(W_\mu^1 + iW_\mu^2)1_3 & \nabla_\mu^S 1_3 + i((-G_\mu^0 + W_\mu^3) + G_\mu) \end{pmatrix} 1_3$$

$G_\mu^0 \in C^\infty(X)$ real $G_\mu \in M_3(C^\infty(X))$ hermitian and traceless

Unimodularity condition $\text{tr}(A) = 0 \Rightarrow G_\mu^0 = -\frac{1}{3}B_\mu$

The action functional

Fermionic action functional

$$S_F = \langle J\psi, D_A\psi \rangle$$

where $\psi = \gamma\psi \in \mathcal{H}^+$ are **Grassmann-valued**

Spectral action principle [Chanseddine-Connes]

The bosonic action is a functional only of the **spectrum of D_A^2** .

functional calculus and Laplace transformation

$$S_A = \text{Tr}(\chi(D_A^2)) = \int_0^\infty dt \hat{\chi}(t) \text{Tr}(e^{-tD_A^2}),$$

with $\chi(s) = \int_0^\infty dt e^{-st} \hat{\chi}(t)$

Proposition (heat kernel expansion)

Let F be a vector bundle over (M, g) . A second-order differential operator $P = -(g^{\mu\nu} \partial_\mu \partial_\nu + A^\rho \partial_\rho + B)$ (locally), where $A^\mu, B \in \text{End}(F)$, has an asymptotic expansion

$$\text{Tr}(e^{-tP}) \sim \sum_{k=0}^{\infty} t^{\frac{k-p}{2}} \int_M dx a_k(x, P),$$

where $a_k(x, P)$ are the Seeley-de Witt coefficients.

inversion of Laplace transformation

$$S_A \sim \sum_{k=0}^{\infty} \chi_{\frac{k-p}{2}} \int_M dx a_k(x, D_A^2),$$

$$\chi_z = \int_0^\infty dt t^z \hat{\chi}(t) = \begin{cases} \frac{1}{\Gamma(-z)} \int_0^\infty ds s^{-z-1} \chi(s) & \text{für } z \notin \mathbb{N} \\ (-1)^k \chi^{(k)}(0) & \text{für } z = k \in \mathbb{N} \end{cases}$$

The Seeley-de Witt coefficients are given in the book of Gilkey, but expressed in terms of $P = \Delta^F - \mathcal{E}$, where Δ^F is the connection Laplacian for $\nabla f = dx^\mu \otimes (\partial_\mu f + \omega_\mu f)$. One finds

$$P = \Delta^F - \mathcal{E} \quad \Leftrightarrow \quad \omega_\mu = \frac{1}{2} g_{\mu\nu} (A^\nu + g^{\rho\sigma} \Gamma_{\rho\sigma}^\nu)$$

$$\mathcal{E} = B - g^{\mu\nu} (\omega_\mu \omega_\nu + \partial_\mu \omega_\nu - \Gamma_{\mu\nu}^\rho \omega_\rho)$$

The first coefficients are

$$a_0(x, P) = (4\pi)^{-\frac{p}{2}} \text{tr}(\text{id}) ,$$

$$a_2(x, P) = \frac{1}{6} (4\pi)^{-\frac{p}{2}} \text{tr}(-R \text{id} + 6\mathcal{E}) ,$$

$$a_4(x, P) = \frac{(4\pi)^{-\frac{p}{2}}}{360} \text{tr}((5R^2 - 2R_{\mu\nu}R^{\mu\nu} + 2R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 12\Delta^{LC}(R))\text{id} \\ + 60\Delta^F(\mathcal{E}) - 60R\mathcal{E} + 180\mathcal{E}^2 + 30\Omega_{\mu\nu}\Omega^{\mu\nu})$$

where $\Omega_{\mu\nu} = \partial_\mu\omega_\nu - \partial_\nu\omega_\mu + \omega_\mu\omega_\nu - \omega_\nu\omega_\mu$

Spectral action

$$\begin{aligned}
S_A = & \frac{1}{\pi^2} (48\chi_{-2} - c\chi_{-1} + d\chi_0) \int d^4x \sqrt{\det g} \\
& + \frac{1}{24\pi^2} (96\chi_{-1} - c\chi_0) \int d^4x \sqrt{\det g} R \\
& + \frac{\chi_0}{10\pi^2} \int d^4x \sqrt{\det g} \left(\frac{11}{6} R^* R^* - 3C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma} \right) \\
& + \frac{1}{\pi^2} (-2a\chi_{-1} + e\chi_0) \int d^4x \sqrt{\det g} |\phi|^2 \\
& + \frac{\chi_0}{2\pi^2} \int d^4x \sqrt{\det g} a \left(|D_\mu \phi|^2 - \frac{1}{6} R |\phi|^2 \right) \\
& + \frac{2\chi_0}{\pi^2} \int d^4x \sqrt{\det g} \left(\frac{1}{2} \text{tr}_3(G_{\mu\nu} G^{\mu\nu}) + \frac{1}{2} \text{tr}_2(W_{\mu\nu} W^{\mu\nu}) + \frac{5}{3} B_{\mu\nu} B^{\mu\nu} \right) \\
& + \frac{\chi_0}{2\pi^2} \int d^4x \sqrt{\det g} |\phi|^4
\end{aligned}$$

with

$$G_{\mu\nu} = \partial_\mu G_\nu - \partial_\nu G_\mu - i(G_\mu G_\nu - G_\nu G_\mu) \in M_3(C^\infty(M)) ,$$

$$W_{\mu\nu} = \partial_\mu W_\nu - \partial_\nu W_\mu - i(W_\mu W_\nu - W_\nu W_\mu) \in M_2(C^\infty(M)) ,$$

$$B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu \in C^\infty(M)$$

$$|\phi|^2 := |\phi_1|^2 + |\phi_2|^2 ,$$

$$\begin{pmatrix} D_\mu \phi_1 \\ D_\mu \phi_2 \end{pmatrix} = \begin{pmatrix} \partial_\mu \phi_1 \\ \partial_\mu \phi_2 \end{pmatrix} + i \begin{pmatrix} W_\mu^3 - B_\mu & W_\mu^1 - iW_\mu^2 \\ W_\mu^1 - iW_\mu^2 & -W_\mu^3 - B_\mu \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$$

and

$$a = \text{tr}(Y_\nu^* Y_\nu + Y_e^* Y_e + 3Y_u^* Y_u + Y_d^* Y_d) ,$$

$$b = \text{tr}((Y_\nu^* Y_\nu)^2 + (Y_e^* Y_e)^2 + 3(Y_u^* Y_u)^2 + (Y_d^* Y_d)^2) ,$$

$$c = \text{tr}(Y_R^* Y_R) , \quad d = 4\text{tr}((Y_R^* Y_R)^2) , \quad e = \text{tr}((Y_R^* Y_R)(Y_\nu^* Y_\nu)) .$$