Noncommutative Geometry

Raimar Wulkenhaar

Mathematisches Institut der Westfälischen Wilhelms-Universität

Münster, Germany

Topological spaces

topological space X: set of points together with collection T (topology) of open subsets

1. $Y \subset X$ closed if $X \setminus Y$ open
2. sufficies to define convergence of sequences
3. suffices to define continuity: $\phi : X \to Y$ continuous if for every open $Z \subset Y$ the pre-image $\phi^{-1}(Z)$ is open in X
4. $\phi : X \to Y$ homeomorphism if bijective and both ϕ, ϕ^{-1} continuous
5. X is compact if any open cover has a finite subcover; X is sequentially compact if any sequence has a convergent subsequence (equivalent for metric spaces)

X may carry different topologies which leads to different notions of convergence and continuity!

Hausdorff spaces

Hausdorff space X: topological space in which distinct points are separated by open neighbourhoods

1. limit of a convergent sequence is unique
2. compact subsets are closed
3. all metric spaces are Hausdorff

Hausdorff space X is locally compact if every point has a compact neighbourhood

a locally compact Hausdorff space can be embedded in a compact Hausdorff space which has only one extra point at infinity

C^*-algebras

- algebra A: vector space over $\mathbb{K}$ + ring + compatibility
  we assume $1 \in A$ and $\mathbb{K} = \mathbb{C}$
- normed algebra: $\| \ \| : A \to \mathbb{R}$ satisfying norm axioms of vector spaces, $\|ab\| \leq \|a\| \, \|b\|$ and $\|1\| = 1$
- Banach algebra: completeness, i.e. Cauchy sequences in A have a limit in A
- involution: $(a+\lambda b)^* = a^* + \bar{\lambda} b^*$, $(ab)^* = b^* a^*$, $(a^*)^* = a$
- C^*-algebra: Banach $*$-algebra with $\|a^* a\| = \|a\|^2$

The C^*-property is very restrictive:

1. $\| \ \|$ unique: $\|a\|^2 = \sup\{|\lambda| : \ a^* a - \lambda 1 \text{ not invertible in } A\}$
2. $\phi : A \to B$ isomorphism $\Rightarrow$ $\|\phi(a)\| = \|a\|$
3. any C^*-algebra is $*$-isomorphic to a norm-closed subalgebra of $\mathcal{B}(\mathcal{H})$ (bounded operators on Hilbert space)

Standard example: continuous functions

$A = C(X)$ continuous functions on compact Hausdorff space X
with $\|f\| := \sup_{x\in X}|f(x)|$, $\qquad (f^*)(x) := \overline{f(x)}$

Theorem. $C(X)$ is a C^*-algebra

1. norm-closed: Cauchy sequence $\{f_k\}$ in $C(X)$ defines pointwise limit function f; it follows $\|f - f_k\| \to 0$

   $\frac{\epsilon}{3}$-trick proves continuity of f
2. C^*-property: $\exists p \in X$ with $\|f\| = \sup_{x\in X}|f(x)| = |f(p)|$

   $\|f^*f\| = \sup_{x\in X}|f(x)|^2 = |f(p)|^2 = \|f\|^2$

For X is locally compact: continuous functions vanishing at ∞

$$C_0(X) = \{f \in C(X) \;:\; \forall\epsilon > 0\ \exists K \subset X \text{ compact with } |f(x)| < \epsilon\ \forall x \in X \setminus K\}$$

Standard example: bounded operators

$\mathcal{H}$ complex Hilbert space, $\mathcal{B}(\mathcal{H})$ algebra of bounded linear operators on $\mathcal{H}$ with $\|a\| := \sup_{x\in\mathbb{H},\|x\|=1} \|ax\|$

adjoint operator $\langle a^*x, y\rangle = \langle x, ay\rangle$ from Riesz representation theorem, with $\|a^*\| = \|a\|$

Theorem. $\mathcal{B}(\mathcal{H})$ is a C^*-algebra

1. norm-closed: same argument as for $C(X)$
2. C^*-property: from Cauchy-Schwarz

$$0\leq\|ax\|^2=\langle ax, ax\rangle=\langle a^*ax, x\rangle\leq\|a^*ax\|\|x\|\leq\|a^*a\|\|x\|^2$$

- Example: $\mathcal{H} = L^2(X,\mu)$, then a C^*-subalgebra of $\mathcal{B}(\mathcal{H})$ is $L^\infty(X,\mu)$ with $\|[f]\| = \operatorname{ess\,sup}_\mu\{|f(x)| \;:\; x \in X\}$ (in fact even a von Neumann algebra)

  if (X,μ) a manifold: $\mathcal{A} = C^\infty(X) \subset C(X) \subset L^\infty(X,\mu) = \mathcal{A}''$

Spectrum

A - Banach algebra, $1 \in A$

- spectrum of $a \in A$:
  $\mathrm{sp}(a) = \{\lambda \in \mathbb{C} \ : \ a - \lambda 1 \text{ not invertible in } A\}$
  resolvent set $R(a) = \mathbb{C} \setminus sp(a)$
- spectral radius $r(a) = \sup\{|\lambda| \ : \ \lambda \in sp(a)\}$
  spectral radius formula: $r(a) = \lim_{n\to\infty} \|a^n\|^{\frac{1}{n}} \leq \|a\|$
  (equality for normal elements in C^*-algebra)
- $R(a)$ open
  resolvent function $z \mapsto (a - z1)^{-1}$ holomorphic on $R(a)$
  $sp(a)$ nonempty and compact in $\mathbb{C}$

Characters

A - commutative C^*-algebra

character: non-zero $*$-algebra homomorphism $\chi : A \to \mathbb{C}$

$\chi(1) = 1, \quad \|\chi\| = 1$

$X = \mathrm{Spec}(A) = \{\chi : \ \chi \text{ character on } A\}$

- view X as subspace of unit ball A'_1 in dual space A'
- $A' = \{f : A \to \mathbb{C} \text{ linear}\}$ equipped with weak-$*$-topology
  open subsets generated by $\{\hat{a}^{-1}(U) \ : U \subset \mathbb{C} \text{ open}, a \in A\}$
  with $\hat{a}(f) = f(a)$
- weak-$*$-topology on A' separates points
  $A'_1 \subset A'$ is compact (Banach-Alaoglu)
  $X = Spec(A) \subset A'_1$ closed

X with weak-$*$-topology is compact Hausdorff space

attention: A'_1 not compact in norm topology

The Gelfand-Naimark theorem

Gelfand transformation: $\rho : A \ni a \mapsto \hat{a} \in C(\mathrm{Spec}(A))$, with $\hat{a}(\chi) := \chi(a)$, for commutative Banach algebra A

Theorem

Let A be a commutative C^*-algebra and $X = \mathrm{Spec}(A)$. Then $\rho : A \to C(X)$ is an isometric isomorphism.

We outline proof of $\|\hat{a}\| = \|a\|$ (injectivity). Surjectivity follows from Stone-Weierstraß.

- $|\hat{a}(\chi)| = |\chi(a)| \leq \|\chi\|\|a\| = \|a\|$
  $\|\hat{a}\| = \sup_{\chi \in X} |\hat{a}(\chi)| \leq \|a\|$
- If we can prove that for every $\lambda \in sp(a)$ there is a character $\chi \in X$ with $\chi(a) = \lambda$, then

  $\|a\|^2 = \|a^* a\| = r(a^* a) = \chi(a^* a) = \chi(a^*)\chi(a) = |\chi(a)|^2 = |\hat{a}(\chi)|^2$
  so $\|a\| \leq \|\hat{a}|$

ideal: linear subspace $I \subset A$ with $IA \subset I$ and $1 \notin I$
I maximal if not contained in other ideal $\neq A$

1. $\ker \chi$ is ideal (clear) and maximal:
   - $a \in I \setminus \ker \chi \;\Rightarrow\; \chi(a)$ invertible , $\exists b \in A$: $\chi(b)\chi(a) = 1$
   - $ab \in I$ and $ab - 1 \in \ker \chi \subset I$, so $1 \in I$, contradiction
2. every maximal ideal I in commut. alg. A is of that type:
   - A/I is a field; otherwise $\exists 0 \neq b \in A/I$ not invertible.
     Then $Ab \not\ni 1$ is an ideal, and $I + Ab$ is a bigger ideal ($b \notin I$)
   - This field is $\mathbb{C}$: Take $0 \neq b \in A/I$ and $\lambda \in sp(b)$, then $b - \lambda 1$ not invertible, so $b = \lambda 1$.
   - Composition $A \to A/I \to \mathbb{C}$ defines character χ_I, with $I = \ker \chi_I$

Take $\lambda \in sp(a)$. Then $A(a - \lambda 1) \not\ni 1$ is ideal contained in maximal ideal $\ker \chi$.
$1 \in A \quad \Rightarrow \quad \chi(a) = \lambda$

The evaluation map

Given compact Hausdorff space X, then $Y = Spec(C(X))$ homeomorphic to X, because:

1. evaluation map $\epsilon_x : C(X) \to \mathbb{C}, \quad \epsilon_x(f) = f(x)$ character
2. $\epsilon : X \to Y$ injective because $C(X)$ separates points
3. every character is of this type:
   - Take $\chi \in Spec(C(X))$ with $I = \ker \chi$ different from all $\ker \epsilon_x$.
   - For every $x \in X$ there is an $a_x \in I \subset C(X)$ with $a_x(x) \neq 0$.
   - a_x continuous $\quad \Rightarrow a_x$ non-zero on neighbourhood of x
   - X compact $\quad \Rightarrow \exists$ finitely many $x_1, \cdots, x_n \in X$ with $a = \sum_{i=1}^{n} |a_{x_i}|^2 \in I$ non-zero on X
   - $\exists a^{-1} \in A$ with $1 = a^{-1}a \in I$, contradiction
4. continuity of ϵ, ϵ^{-1} by definition of weak-$*$-topology

A dictionary

compact Hausdorff space	*commutative C^*-algebra*
measure space (X, μ)	commut. von Neumann algebra
group	commutative Hopf algebra
vector bundle over X	*finitely generated projective module over C(X)*
vector field	derivation
K-theory	K-theory
de Rham complex	Hochschild cohomology
de Rham cohomology	cyclic homology
differentiable manifold M	*commutative spectral triple*
diffeomorphism of *M*	automorphism of commutative spectral triple
(real/infinitesimal) variable	linear (selfadjoint/compact) operator on Hilbert space
integral	trace

Vector bundles

Let X be a compact Hausdorff space. A locally-trivial vector bundle over X consists of a topological space E and a continuous surjection $p : E \to X$ such that

1. For every $x \in X$, the fibre $E_X = p^{-1}(x) \subset E$ is a finite-dimensional complex vector space
2. For every $x \in X$ there is a neighbourhood U and $m \in \mathbb{N}$ such that $p^{-1}(U)$ is homeomorphic to $\mathbb{C}^m \times U$.

A global section of $E \stackrel{p}{\to} X$ is a continuous mapping $s : X \to E$ with $p \circ s = id_X$.

We let $\Gamma(E, S)$ be the vector space of sections of $E \stackrel{p}{\to} X$. It becomes a module over $C(X)$ by $(sf)(x) := s(x)f(x)$ for $f \in C(X)$.

- A $C(X)$-module $\mathcal{E}$ is finitely generated if there exists a family $\eta_1, \dots, \eta_k \in \mathcal{E}$ such that every $\eta \in \mathcal{E}$ can be (not uniquely) represented as $\eta = \sum_{i=1}^{k} \eta_i f_i$ for $f_i \in C(X)$.
- A $C(X)$-module $\mathcal{E}$ is free if it is homeomorphic to $\mathbb{C}^m \otimes C(X)$ for some $m \in \mathbb{N}$. A family of generators is e.g. $(e_i \otimes 1)_{i=1,\dots,m}$ for any basis (e_i) of $\mathbb{C}^m$.
- The Whitney sum of vector bundles E_1, E_2 over X is the vector bundle

$$E_1 \oplus E_2 = \{(e_1, e_2) \in E_1 \times E_2 \;:\; p_1(e_1) = p_2(e_2)\}$$

- The corresponding direct sum $\mathcal{E}_1 \oplus \mathcal{E}_2$ of $C(X)$-modules $\mathcal{E}_1, \mathcal{E}_2$ becomes a $C(X)$-module by $(\eta_1, \eta_2)f := (\eta_1 f, \eta_2 f)$.
- A $C(X)$-Modul $\mathcal{E}$ is projective if there is another $C(X)$-module $\mathcal{E}'$ such that $\mathcal{E} \oplus \mathcal{E}'$ is a free module.

The Serre-Swan theorem

Theorem (Swan, 1962)

Let X be a compact Hausdorff space. A $C(X)$-module $\mathcal{E}$ is isomorphic to a module $\Gamma(E, X)$ for a locally-trivial vector bundle $E \xrightarrow{p} X$ if and only iff $\mathcal{E}$ is finitely generated and projective.

- The original article

  R. G. Swan, "Vector Bundles and Projective Modules," Trans. Am. Math. Soc. **105** (1962) 264–277

  is a good reference

- The theorem generalises to e.g. $C^\infty(X)$-modules and for some measure space to L^∞-modules

Proof ($\Rightarrow$)

Let $E \stackrel{p}{\to} X$ be a locally-trivial vector bundle.

- For $x \in X$ there is a neighbourhood U such that $p^{-1}(U) \simeq \mathbb{C}^m \times U$.
- By surjectivity there exist $s_{x,1}, \dots, s_{x,m} \in \Gamma(E, X)$ such that $\{pr_1(s_{x,1}(x)), \dots, pr_1(s_{x,m}(x))\}$ form a basis of $\mathbb{C}^m$.
- By continuity there is a neighbourhood $U_x \subset U$ of x such that $\{pr_1(s_{x,1}(y)), \dots, pr_1(s_{x,m}(y))\}$ are also linearly independent for every $y \in U_x$.

This means that the family $\{s_{x,1}, \dots, s_{x,m}\}$ generates $p^{-1}(U_x)$.

X is compact, i.e. covered by finitely many $U_{x_1}, \dots, U_{x_k}$.

Thus finitely many $s_1(x), \dots, s_n(x)$ generate $\Gamma(E, X)$

previous page: finitely many $s_1(x), \ldots, s_n(x)$ generate $\Gamma(E, X)$

- Consider the free C(X)-module $\mathcal{F} = \mathbb{C}^n \otimes C(X) = \Gamma(\mathbb{C}^n \times X, X)$ with n generators $b_1, \ldots, b_n$.
- Define a mapping $e : \mathcal{F} \to \mathcal{E}$ by $e(b_i) := s_i$ and extension by the module structure. This mapping is surjective and continuous.
- The dimension of the image of e is locally constant. Thus, the dimension of the kernel of e is locally constant, too.
- Locally constant dimension is sufficient to reconstruct a vector bundle (Proposition 1 in Swan's paper).
- Both $\mathrm{im}(e) = \mathcal{E}$ and $\ker(e) =: \mathcal{E}'$ are $C(X)$-modules, and $\mathcal{F} = \mathcal{E} \oplus \mathcal{E}'$.

Hence, $\mathcal{E}$ is projective.

Proof ($\Leftarrow$)

Let $\mathcal{E}$ be a finitely generated projective $C(X)$-module, $\mathcal{E} \oplus \mathcal{E}' = \mathcal{F}$ and $\xi = (\eta, \eta') \in \mathcal{F} = \mathbb{C}^n \otimes C(X)$.

- Define $e : \mathcal{F} \to \mathcal{F}$ by $e\xi = (\eta, 0)$.

  Then $e^2 = e$, $(e\xi)f = e(\xi f)$ and $\mathcal{E} \simeq \mathrm{im}(e)$, $\mathcal{E}' \simeq \ker(e)$.
- For $x \in X$ consider maximal ideal $I_x = \{f \in C(X) \;:\; f(x) = 0\}$ in $C(X)$.
- $\mathcal{F} I_x$ consists of elements of $\mathcal{F}$ vanishing in x

  $\mathcal{F}/\mathcal{F} I_x$ equivalence classes with same value in x
- Evaluation $\xi \mapsto \xi(x)$ defines isomorphism $\mathcal{F}/\mathcal{F} I_x \simeq F_x \simeq \mathbb{C}^n$.
- $(e\xi)f = e(\xi f)$

  The evaluation $e\xi \mapsto (e\xi)(x)$ gives an isomorphism $e\mathcal{F}/e\mathcal{F} I_x \simeq E_x$ with a subspace $E_x \subset \mathbb{C}^m$.

If we can show that the dimension of E_x is locally constant, then the fibres E_x define a locally-trivial vector bundle.

- Let $m := \dim(E_x)$. There exist m linearly independent continuous sections $s_1, \dots, s_m$ of $\mathbb{C}^n \times X$, i.e. $s_i \in \mathcal{F}$, with $(es_i)(x) = s_i(x)$.
- By continuity, the (es_i) are linearly independent in a neighbourhood U_x of x, i.e. $\dim(E_y) \geq m$ for all $y \in U_x$.
- Same argument for $1 - e$ gives complementary fibre space E'_x of dimension $n - m$. Consequently, $\dim(E'_y) \geq n - m$.
- Total dimension n is constant, thus $\dim(E_x) = m$ locally constant.

Spectral triples

Definition (A. Connes, 1996)

$(\mathcal{A}, \mathcal{H}, D)$ – commutative spectral triple, i.e. $\mathcal{H}$ a Hilbert space, $\mathcal{A}$ a commutative involutive unital algebra represented in $\mathcal{H}$,
D a selfadjoint operator in $\mathcal{H}$ with compact resolvent, p an integer.

1. Dimension: k^{th} characteristic value of resolvent of D is $\mathcal{O}(k^{-\frac{1}{p}})$
2. Order one: $[[D, f], g] = 0 \quad \forall f, g \in \mathcal{A}$
3. Regularity: f and $[D, f]$ belong to the domain of δ^m, for any $f \in \mathcal{A}$ and $m \in \mathbb{Z}$, where $\delta T := [|D|, T]$
4. Orientability: $\exists$ Hochschild p-cycle $c \in Z_p(\mathcal{A}, \mathcal{A})$ s.t. $\pi_D(c) = 1$ for p odd, $\pi_D(c) = \gamma$ for p even with $\gamma = \gamma^*$, $\gamma^2 = 1$, $\gamma D = -D\gamma$
5. Finiteness and absolute continuity: $\mathcal{H}_\infty := \bigcap_m \mathrm{dom}(D^m) \subset \mathcal{H}$ is finitely generated projective $\mathcal{A}$-module, $\mathcal{H}_\infty = e\mathcal{A}^n$, with $e = e^* = e^2 \in M_n(\mathcal{A})$. Hermitian structure $(\xi|a\eta) = \sum_{i=1}^n a\xi_i^* \eta_i \in \mathcal{A}$ satisfies $\langle \xi, \eta \rangle = ⨍ (\xi|\eta)|D|^{-p}$

Reconstruction theorem

Theorem (A. Connes, 2008)

Let $(\mathcal{A}, D, \mathcal{H})$ be a spectral triple, $\mathcal{A}$ commutative and unital. Let the conditions (1)–(5) be realised in stonger form:

1. *All $T \in \mathrm{End}_{\mathcal{A}}(\mathcal{H}_\infty)$ are regular*
2. *The Hochschild cycle is antisymmetric,* $c = \sum_\alpha a^0_\alpha \otimes \sum_{\beta \in S_p} \epsilon(\beta) a^{\beta(1)}_\alpha \otimes \cdots \otimes a^{\beta(p)}_\alpha$

Then there exists a compact oriented differentialble manifold X with $\mathcal{A} = C^\infty(X)$.
Conversely, every compact oriented differentiable manifold arises in this way.

Second theorem: If in addition the multiplicity of $\mathcal{A}''$ in $\mathcal{H}$ is $2^{p/2}$, then $\mathcal{A} = C^\infty(X)$ for a smooth oriented compact spin^c-manifold X. The stronger condition (1) is automatic.

Definition (smooth compact p-dimensional manifold)

...is a compact Hausdorff space X together with a system of local charts (U_α, s_α) such that

- the U_α are open in X and $X = \bigcup_\alpha U_\alpha$
- $s_\alpha : U_\alpha \to s_\alpha(U_\alpha) \subset \mathbb{R}^p$ is a homeomorphism. In particular, $s_\alpha(U_\alpha)$ is open in $\mathbb{R}^p$ and s_α is injective
- $s_\alpha \circ s_\beta^{-1} : s_\beta(U_\alpha \cap U_\beta) \to s_\alpha(U_\alpha \cap U_\beta)$ is smooth

Strategy:

- norm-completion of $\mathcal{A}$ is unital commutative C^*-algebra $A = C(X)$ for compact Hausdorff space $X = Spec(A)$
- build tentative charts (up to injectivity) from c
- prove that there exists restriction to subsets where s_α is injective (very complicated)

Unbounded operators

From (1), D is unbounded linear operator on $\mathcal{H}$ for $p > 0$.

- $\mathrm{dom}(T) \subset \mathcal{H}$ – domain
- $\Gamma(T) = \{(\phi, T\phi) \in \mathcal{H} \times \mathcal{H} \;:\; \phi \in \mathrm{dom}(T)\}$ – graph
- T closed if $\Gamma(T)$ closed in $\mathcal{H} \times \mathcal{H}$

An extension of T is an operator T_1 with $\mathrm{dom}(T_1) \supset \mathrm{dom}(T)$ and $T_1\phi = T\phi$ for $\phi \in \mathrm{dom}(T)$.

- T is closable if a closed extension exists.
- The smallest closed extension is the closure $\bar{T}$.

$T : \mathrm{dom}(T) \to \mathcal{H}$ – linear densely defined operator

- $\mathrm{dom}(T^*) = \{\phi \in \mathcal{H} \ : \ \exists \eta \in \mathcal{H}$ with $\langle T\psi, \phi\rangle = \langle \psi, \eta\rangle$ for all $\psi \in \mathrm{dom}(T)\}$
  Then $T^*\phi := \eta$
- T^* always closed, T closable if $\mathrm{dom}(T^*) \subset \mathcal{H}$ dense, with $\bar{T} = T^{**}$

Definition

A linear densely defined operator T is

- symmetric if $\mathrm{dom}(T) \subset \mathrm{dom}(T^*)$ and $T\phi = T^*\phi$ for all $\phi \in \mathrm{dom}(T)$
- selfadjoint if $T^* = T$, i.e. $\mathrm{dom}(T) = \mathrm{dom}(T^*)$ and T symmetric
- essentially selfadjoint if T is symmetric and $\bar{T}$ selfadjoint

For closed operators one can define a spectral theory.

- resolvent set
  $R(T) = \{\lambda \in \mathbb{C} \;:\; \lambda 1 - T : \mathrm{dom}(T) \to \mathcal{H}$ bijective and $(\lambda 1 - T)^{-1}$ bounded$\}$
- spectrum $sp(T) = \mathbb{C} \setminus R(T)$
  $sp(T)$ is closed, in general non-compact, possibly empty

If T is selfadjoint, then $sp(T) \subset \mathbb{R}$. Further:

- Cayley transformation $T \mapsto U_T := (\mathrm{i}1 - T)(-\mathrm{i}1 - T)^{-1}$ to unitary U_T with $1 - U_T$ injective
- spectral theorem $T = \int_{\mathbb{R}} \lambda \, dE_\lambda$ for some spectral measure dE_λ

Polar decomposition $T = F|T|$ for closed densely defined T:

- $|T|$ positive and selfadjoint, $\mathrm{dom}(|T|) = \mathrm{dom}(T)$, $\ker |T| = \ker T$
- $F : (\ker T)^{\perp} \to \overline{im(T)}$ partial isometry

Compact operators

Let $\mathcal{K} \subset \mathcal{B}(\mathcal{H})$ be the ideal of compact operators on $\mathcal{H}$ and $T \in \mathcal{K}$.

- There is a null sequence of eigenvalues $\mu_i > 0$ of $|T|$
- The eigenspace $E_i = \ker(\mu_i 1 - |T|)$ is finite-dimensional
- $sp(|T|) = \{\mu_i\}_{i \in \mathbb{N}} \cup \{0\}$

$s_k(T) = \inf\{\|T\big|_{E^\perp}\| \;:\; \dim(E) = k\}$ – characteristic value

$s_k(T)$ – k^{th} eigenvalue $\mu_k(|T|)$ if arrangend decreasingly and with multiplicity

Schatten ideal $\mathcal{L}^p$

$$\mathcal{L}^p := \Big\{ T \in K \;:\; \|T\|_p := \Big(\sum_{k=0}^{\infty} (s_k(T))^p\Big)^{\frac{1}{p}} < \infty \Big\}$$

All $(\mathcal{L}^p, \|\;\|_p)$ are Banach spaces.

- $\mathcal{L}^1$ – trace class
  $\mathrm{Tr}(T) = \sum_n \langle\psi_n, T\psi_n\rangle$ for $T \in \mathcal{L}^1$ and $\{\psi_n\}$ ONB
- $\mathcal{L}^2$ – Hilbert-Schmidt class
- $\mathcal{L}^\infty = \mathcal{K}$ with $\|T\|_\infty = \|T\|$

Inequalities

- $\mathrm{Tr}(T) \leq \mathrm{Tr}(|T|) = \|T\|_1$ for $T \in \mathcal{L}^1$
- $\mathrm{Tr}(TS) = \mathrm{Tr}(ST)$ if $TS, ST \in \mathcal{L}^1$
- Hölder: $\|TS\|_1 \leq \|T\|_p \|S\|_q$ for $T \in \mathcal{L}^p$, $S \in \mathcal{L}^q$ with $\frac{1}{p} + \frac{1}{q} = 1$
- $\mathcal{L}^p \subset \mathcal{L}^r$ if $p \leq r$

It turns out that Tr on $\mathcal{L}^1$ is not the right generalisation of the integral.

Dixmier ideal $\mathcal{L}^{1+}$

- partial sums $\sigma_n(T) = \sum_{k=0}^{n-1} s_k(T)$
- $\sigma_n(T) =$
  $\sup\{\|TP_E\|_1 \ : \ P_E$ projector to n-dim. subspace $E \subset \mathcal{H}\ \}$

Lemma

$\sigma_n(T+S) \leq \sigma_n(T) + \sigma_n(S) \leq \sigma_{2n}(T+S)$ *for* $0 \leq T, S \in \mathcal{K}$

Proof: first $\leq$ from norm, second from inequalities

$$\begin{aligned} \|TP_E\|_1 + \|SP_F\|_1 &= \mathrm{Tr}(P_E T P_E) + \mathrm{Tr}(P_F S P_F) \\ &\leq \mathrm{Tr}(P_{E+F} T P_{E+F}) + \mathrm{Tr}(P_{E+F} S P_{E+F}) \\ &= \mathrm{Tr}(P_{E+F}(T+S)P_{E+F}) \end{aligned}$$

Definition (Dixmier-Ideal $\mathcal{L}^{1+} \subset \mathcal{K}$)

$$\mathcal{L}^{1+} := \Big\{ T \in \mathcal{K} \;:\; \|T\|_{1+} := \sup_{n\geq 2} \frac{\sigma_n(T)}{\log n} < \infty \Big\}$$

- $\mathcal{L}^1 \subset \mathcal{L}^{1+} \subset \mathcal{L}^p$ for any $p > 1$

If $\left(\frac{\sigma_n(T)}{\log n}\right)_{n\geq 2}$ convergent, then $T \in \mathcal{L}^{1+}$ is called measurable

Definition (noncommutative integral)

$$⨍ T := \lim_{n\to\infty} \frac{\sigma_n(T)}{\log n}$$

- $⨍$ additive on positive elements, linear by extension
- $\sigma_n(UTU^*) = \sigma_n(T)$:
  $⨍$ is trace on subspace of measurable elements of $\mathcal{L}^{1+}$
- $⨍$ vanishes on $\mathcal{L}^1$
- can be (not uniquely) generalised to Dixmier trace on $\mathcal{L}^{1+}$

Example

$\Delta = -\sum_{\mu=1}^{p} \frac{\partial^2}{\partial x_\mu^2}$ Laplace operator on $\mathbb{T}^p$

- $sp(\Delta) = \{\|k\|^2 \;:\; k \in \mathbb{Z}^p\}$
- Replace Δ by 1 on $\ker\Delta = \mathbb{C}$ $\Rightarrow \tilde{\Delta}^{-\frac{q}{2}}$ compact for $q > 0$
- Determine $s_{n(\|k\|)}(\tilde{\Delta}^{-\frac{q}{2}}) = \|k\|^{-q}$ asymptotically:
  $$n(\|k\|) = \#(\text{lattice points in ball of radius } \|k\| \text{ in } \mathbb{R}^p) = V_p\|k\|^p \quad \text{with } V_p = \frac{\pi^{\frac{p}{2}}}{\Gamma(\frac{p+2}{2})}$$
- $s_n(\tilde{\Delta}^{-\frac{q}{2}}) = \left(\frac{n}{V_p}\right)^{-\frac{q}{p}}$
- $\sigma_n(\tilde{\Delta}^{-\frac{q}{2}}) = \int_1^n du \left(\frac{V_p}{u}\right)^{\frac{q}{p}}, \quad \|\tilde{\Delta}^{-\frac{q}{2}}|_{1+} = \begin{cases} \infty & \text{for } q < p \\ 0 & \text{for } q > p \end{cases}$

$$⨍ \tilde{\Delta}^{-\frac{p}{2}} = \lim_{n\to\infty} \frac{1}{\log n} \int_1^n du \, \frac{V_p}{u} = V_p$$

Hochschild homology

$\mathcal{A}$ unital algebra, $\mathcal{M}$ bimodule over $\mathcal{A}$ and
$C_n(\mathcal{A},\mathcal{M}) = \mathcal{M} \otimes \mathcal{A}^{\otimes n}$

Definition (Hochschild boundary operator)

$b : C_n(\mathcal{A},\mathcal{M}) \to C_{n-1}(\mathcal{A},\mathcal{M})$,

$$\begin{aligned} b(m \otimes a_1 \otimes \dots a_n) \\ := ma_1 \otimes a_2 \otimes \dots \otimes a_n \\ + \sum_{i=1}^{n-1} (-1)^i m \otimes a_1 \otimes \dots \otimes a_{i-1} \otimes a_i a_{i+1} \otimes a_{i+2} \otimes \dots \otimes a_n \\ + (-1)^n a_n m \otimes a_1 \otimes \dots \otimes a_{n-1} \,. \end{aligned}$$

and $b = 0$ for $n = 0$

- $b(m \otimes a_1) = ma_1 - a_1 m$

  $b(m \otimes a_1 \otimes a_2) = ma_1 \otimes a_2 - m \otimes a_1 a_2 + a_2 m \otimes a_1$
- fulfils $b^2 = 0$, e.g.

  $b^2(m \otimes a_1 \otimes a_2) =$
  $(ma_1 a_2 - a_2 m a_1) - (ma_1 a_2 - a_1 a_2 m) + (a_2 m a_1 - a_1 a_2 m) = 0$

Definition

The homology of the chain complex $(C_*(\mathcal{A}, \mathcal{M}), b)$ is called Hochschild homology and denoted $HH_*(\mathcal{A}, \mathcal{M})$

Means: If $Z_n(\mathcal{A}, \mathcal{M}) \ni c_n$ is the subspace of Hochschild n-cycles $bc_n = 0$, then $HH_n(\mathcal{A}, \mathcal{M}) = Z_n(\mathcal{A}, \mathcal{M}) / bC_{n+1}(\mathcal{A}, \mathcal{M})$

Theorem (Hochschild-Kostant-Rosenberg-Connes)

Let $A = C^\infty(X)$. Then $HH_*(C^\infty(X), C^\infty(X)) \simeq \Omega^*(C^\infty(X))$

remarkable fact: $HH_*(C^\infty(X), C^\infty(X))$ is local, only the diagonal in $(C^\infty(X))^{\otimes(n+1)} \simeq C^\infty(X \times \cdots \times X)$ contributes

- In spectral triple definition, the class $[c] \in HH_p(\mathcal{A}, \mathcal{A})$ of $c \in Z_p(\mathcal{A}, \mathcal{A})$ is the volume form (orientation class). It is local and nowhere vanishing because of $\gamma^2 = 1$, $\gamma = \pi_D(c)$
- $c \in Z_p(\mathcal{A}, \mathcal{A})$ and $\gamma = 1$ for p odd and $\gamma D = -D\gamma$ for p even imply that $|D|^{-p} \in \mathcal{L}^{1+}$ is measurable, i.e. $⨍$ unambiguously defined

Characterisation of the algebra

Main Lemma

$$\mathcal{A} = \{T \in \mathcal{A}'' \;:\; T \in \bigcap_{m>0} \mathrm{dom}(\delta^m)\}$$

- $T \in \mathcal{A}'' \supset \mathcal{A}$ (both commutative) and $T : \mathcal{H}_\infty \to \mathcal{H}_\infty$
  conclusion: $T \in \mathrm{End}_{\mathcal{A}}(\mathcal{H}_\infty)$
- $\mathcal{A}$ is unital $\Rightarrow$ $\mathrm{End}_{\mathcal{A}}(\mathcal{H}_\infty) = eM_n(\mathcal{A})e$
  $T = eTe = (\alpha_{kl})$ with $\alpha_{kl} = (e\xi_k | Te\xi_l) \in \mathcal{A}$
- norm-completion of $\mathcal{A}$ is unital commutative C^*-algebra
  $A = C(X)$ for $X = \mathrm{Spec}(A)$ a compact Hausdorff space
- define positive measure μ on X by

$$\mu(f) := \int\!\!\!\!- f|D|^{-p} \qquad \forall f \in A = C(X)$$

$\mathcal{E} = eA^n = \mathcal{H}_\infty \otimes_\mathcal{A} A$ is finitely generated projective module

- Serre-Swan: $\exists$ a complex locally trivial vector bundle $S \to X$ such that $\mathcal{E} = \Gamma(X, S)$
- Finiteness axiom $\Rightarrow \langle \xi, \eta \rangle = \mu((\xi|\eta))$ for $\xi, \eta \in e\mathcal{A}^n$
- $\mathcal{H}_\infty \subset \mathcal{H}$ dense from selfadjointness of D

  completion of $\mathcal{H}_\infty$ to $\mathcal{H}$ is the same as completion with respect to μ

  $\Rightarrow \mathcal{H} = e(L^2(X, \mu))^n = L^2(X, S, \mu)$
- action of $f \in \mathcal{A}''$ on $\mathcal{H} = L^2(X, S, \mu)$ is diagonal multiplication with $f \in L^\infty(X, \mu)$

  $T = e\,\mathrm{diag}(f, \dots, f) \in eM_n(\mathcal{A}) \quad \Rightarrow \quad f \in \mathcal{A}$

- $\mathrm{tr}(e) = \sum_{i=1}^{n} e_{ii} \in \mathcal{A} \subset A$ is the continuous function which assigns to $\chi \in X$ the dimension of the fibre S_χ of S over χ.
  Thus, $\chi(\mathrm{tr}(e)) \in \{1, \dots, n\}$ (0 can be excluded)
- Let $p_j \in A$ be the projection to the set of connected components X_j of X for which $\dim(S_\chi) = j$. Then, $\sum_{j=1}^{n} p_j = 1$ and $\mathrm{tr}(e) = \sum_{j=1}^{n} j p_j \in \mathcal{A}$.
  Reconstruction $p_k = \prod_{j \in \{1,\dots,n\} \setminus \{k\}} \frac{\mathrm{Tr}(e) - j1}{k - j} \in \mathcal{A}$

Definition (conditional expectation values)

Let $T = (T_{kl}) \in \mathrm{End}_{\mathcal{A}}(\mathcal{H}_\infty) = eM_n(\mathcal{A})e$.

$$E_{\mathcal{A}}(T) := \sum_{k=1}^{n} \frac{p_k}{k} \sum_{j=1}^{n} T_{jj} \in \mathcal{A}$$

examples: $E_{\mathcal{A}}(e) = 1, \qquad E_{\mathcal{A}}(e\,\mathrm{diag}(f, \dots, f)) = f$

Proposition

$\mathcal{A}$ is a Fréchet algebra, i.e. a complete locally convex algebra whose toplogy is defined by the submultiplicative norms $q_k(ab) \leq q_k(a)q_k(b)$,

$$q_k(a) = \|\rho_k(a)\| \,, \quad \rho_k(a) = \begin{pmatrix} a & \delta(a) & \dots & \delta^k(a)/k! \\ 0 & a & \dots & \dots \\ \dots & \dots & a & \delta(a) \\ 0 & \dots & 0 & a \end{pmatrix}$$

Proof:

- If (a_n) Cauchy sequence in $\mathcal{A}$ w.r.t. any q_k, then $(k=0)$ $a_n \to T \in A \subset \mathcal{A}''$
- δ is closed by selfadjointness of D, thus $T \in \mathrm{dom}(\delta)$.
- Inductively $T \in \mathrm{dom}(\delta^m)$, thus $T \in \mathcal{A}$ from Main Lemma.

Proposition

$\mathcal{A}$ is a pre-C^*-algebra, i.e. closed under holomorphic functional calculus. This means: If $a \in \mathcal{A} \subset A$ and f holomorphic in neighbourhood of $sp(a)$ (viewed from A), then

$$f(a) := \frac{1}{2\pi\mathrm{i}} \oint_c dz \frac{f(z)}{z1 - a} \in \mathcal{A}$$

Proof:

- $z \notin sp(a) \quad \Rightarrow \quad (z1 - a)^{-1} \in A \subset \mathcal{A}''$
- $(z1 - a)^{-1}$ locally expanded in power series in $\zeta 1 - a \in \mathcal{A}$
- δ closed $\quad \Rightarrow \quad (z1 - a)^{-1} \in \mathrm{dom}(\delta)$
- Inductively $(z1 - a)^{-1} \in \mathrm{dom}(\delta^m)$, thus $(z1 - a)^{-1} \in \mathcal{A}$ from Main Lemma.
- convergence of Riemann integral in $\mathcal{A}$ by Fréchet

Consequence: $a \in \mathcal{A}$ has the same spectrum in $\mathcal{A}$ and A

$\Rightarrow \quad X = Spec(A) = Spec(\mathcal{A})$

Flows

Main Assumption

Let δ_0 be a continuous $\star$-derivation on the Fréchet pre-C^*-algebra $\mathcal{A}$. Then there exists a unique solution, depending continuously on $(t, a) \in \mathbb{R} \times \mathcal{A}$, of the differential equation

$$\partial_t y(t, a) = \delta_0(y(t, a)) , \quad y(0, a) = a$$

Proposition

Let δ_0 be a continuous $\star$-derivation satisfying the Main Assumtion. Let $F_t(a) := y(t, a)$. Then $F_t \in \mathrm{Aut}(\mathcal{A})$ is a one-parameter family of automorphisms, with smooth dependence on t.

Corollary

Let $\chi \in X = Spec(\mathcal{A})$ be a character, $F_t \in \mathrm{Aut}(\mathcal{A})$ as before. Then $F_t^* \chi \in X$, where $F_t^* \chi(a) := \chi(F_t(a))$.

Main Data

1. p derivations $\delta_1, \dots, \delta_p$ satisfying the Main Assumption, with corresponding flows $F^1_{t_1}, \dots, F^p_{t_p} \in \mathrm{Aut}(\mathcal{A})$,
2. p selfadjoint elements $a^1, \dots, a^p \in \mathcal{A}$,
3. a character $\chi \in X$.

Consider the map $\phi_\chi : \mathbb{R}^p \to \mathbb{R}^p$ with

$$\phi^k_\chi(t_1, \dots, t_p) = \chi\big(F^1_{t_1} \circ \cdots \circ F^p_{t_p}(a^k)\big)$$

It satisfies $(\partial_j \phi^k_\chi)(0) = \chi(\delta_j a^k)$.

Proposition

Given the Main Data (δ_j, a^k, χ) as before, with $\det \chi(\delta_j a^k) \neq 0$.

1. There exists a neighbourhood $Z \subset X = Spec(\mathcal{A})$ of χ and a neighbourhood $W \subset \mathbb{R}^p$ of 0 such that, for any $\kappa \in Z$, the map

   $$W \ni t \mapsto \phi_\kappa(t) \in \mathbb{R}^p$$

   is a diffeomorphism, depending continuously on κ, of W with a neighbourhood $Y_\kappa = \phi_\kappa(W)$ of $(\kappa(a^1), \ldots, \kappa(a^p)) \in \mathbb{R}^p$.

2. The image of any open neighbourhood $U \subset X$ of χ under

   $$U \ni \kappa \mapsto (\kappa(a^1), \ldots, \kappa(a^p)) \in \mathbb{R}^p$$

   contains an open neighbourhood of $(\chi(a^1), \ldots, \chi(a^p))$.

Identifying the derivations

Lemma

One has a finite decomposition $[D,a] = \sum_{j=1}^{m} \delta_j(a)\gamma_j$ for all $a \in \mathcal{A}$, where $\gamma_j \in \mathrm{End}_{\mathcal{A}}(\mathcal{H}_\infty)$, and δ_j are $*$-derivations of $\mathcal{A}$ of the form $\delta_j(a) = i(\eta_j, [D,a]\eta_j)$ for some $\eta_j \in \mathcal{H}_\infty$.

Proof

1. $[D,a]\mathcal{H}_\infty \subset \mathcal{H}_\infty$ from $D[D,a]\xi = F\delta([D,a])\xi + F[D,a]|D|\xi$ etc. and regularity condition, $F = D|D|^{-1}$
2. order-one condition: $b[D,a]\xi = [D,a]b\xi$
   consequence: $[D,a] \in \mathrm{End}_{\mathcal{A}}(\mathcal{H}_\infty) = eM_n(\mathcal{A})e$

   $[D,a] = \sum_{k,l} \alpha_{kl}\varepsilon_{kl}$ with $\alpha_{kl} = (e\xi_k, [D,a]e\xi_l) \in \mathcal{A}$
   with ξ_k units in $\mathcal{A}^n$ and ε_{kl} matrix units in $M_n(\mathcal{A})$

3. Consider $\mathcal{A} \ni a \mapsto L_{kl}(a) := (e\xi_k, [D,a]e\xi_l) \in \mathcal{A}$.

$$\begin{aligned} L_{kl}(ab) &= (e\xi_k, [D,ab]e\xi_l) \\ &= (e\xi_k, [D,a]be\xi_l) + (e\xi_k, a[D,b]e\xi_l) \\ &= (e\xi_k, b[D,a]e\xi_l) + (e\xi_k, a[D,b]e\xi_l) \\ &= b(e\xi_k, [D,a]e\xi_l) + a(e\xi_k, [D,b]e\xi_l) \\ &= L_{kl}(a)b + aL_{kl}(b) \end{aligned}$$

using order-one, $\mathcal{A}$-linearity of Hermitian structure and commutativity of $\mathcal{A}$

Consequence: $[D,a] = \sum_{k,l} L_{kl}(a)\varepsilon_{kl}$ with $L_{kl} \in \mathrm{Der}(\mathcal{A})$

4. achieve $\star$-derivations by polarisation identity

$$\begin{aligned} 2(\xi, T\eta) &= (\xi+\eta, T(\xi+\eta)) - (\xi, T\xi) - (\eta, T\eta) \\ &\quad - i\{(\xi + i\eta, T(\xi+i\eta)) + i(\xi, T\xi) + i(i\eta, i\eta)\} \end{aligned}$$

together with

$$(\xi, [D,a^*]\xi) = -(\xi, [D,a]^*\xi) = -([D,a]\xi, \xi) = -(\xi, [D,a]\xi)^*$$

Evaluation of γ

- Hochschild cycle
  $$c = \sum_\alpha a^0_\alpha \otimes \sum_{\beta\in S_p} \epsilon(\beta) a_\alpha^{\beta(1)} \otimes \cdots \otimes a_\alpha^{\beta(p)} \text{ with } a^i_\alpha = (a^i_\alpha)^*$$
- $$\gamma = \pi_D(c) = \sum_\alpha a^0_\alpha \sum_{\beta\in S_p} \epsilon(\beta)[D, a_\alpha^{\beta(1)}]\cdots[D, a_\alpha^{\beta(p)}] = \sum_\alpha a^0_\alpha T_\alpha$$
- $$T_\alpha = \sum_{j_1,\dots,j_p=1}^m \sum_{\beta\in S_p} \epsilon(\beta)\delta_{j_1}(a_\alpha^{\beta(1)})\cdots\delta_{j_p}(a_\alpha^{\beta(p)})\,\gamma_{j_1}\cdots\gamma_{j_p}$$

$$T_\alpha = \sum_{\beta \in S_p} \epsilon(\beta) \sum_{j_1,\dots,j_p=1}^{m} \delta_{j_1}(a_\alpha^{\beta(1)}) \cdots \delta_{j_p}(a_\alpha^{\beta(p)})\, \gamma_{j_1} \cdots \gamma_{j_p}$$

- if $j_k = j_l$ for $k \neq l$, then no contribution to the sum
- the remaining j-sum is split into the sum over the subsets $F \subset \{1, \dots, m\}$ with $|F| = p$ and the sum over the permutations σ of F:

$$T_\alpha = \sum_{\beta \in S_p} \epsilon(\beta) \sum_{1 \le j_1 < \dots < j_p \le m} \sum_{\sigma \in S_p} \delta_{\sigma(j_1)}(a_\alpha^{\beta(1)}) \cdots \delta_{\sigma(j_p)}(a_\alpha^{\beta(p)})\, \gamma_{\sigma(j_1)} \cdots \gamma_{\sigma(j_p)}$$

- write $\beta = \beta' \circ \sigma$, with $\epsilon(\beta) = \epsilon(\beta')\epsilon(\sigma)$

$$T_\alpha = \sum_{1 \le j_1 < \dots < j_p \le m} \sum_{\beta' \in S_p} \epsilon(\beta') \sum_{\sigma \in S_p} \epsilon(\sigma) \delta_{\sigma(j_1)}(a_\alpha^{\beta'(\sigma(1))}) \cdots \delta_{\sigma(j_p)}(a_\alpha^{\beta'(\sigma(p))}) \times \gamma_{\sigma(j_1)} \cdots \gamma_{\sigma(j_p)}$$

use commutativity of $\mathcal{A}$ to rearrange the $\delta_{\sigma(j_k)}(a_\alpha^{\beta'(\sigma(k))})$:

$$T_\alpha = \sum_{1 \le j_1 < \dots < j_p \le m} \sum_{\beta' \in S_p} \epsilon(\beta') \delta_{j_1}(a_\alpha^{\beta'(1)}) \cdots \delta_{j_p}(a_\alpha^{\beta'(p)}) \times \sum_{\sigma \in S_p} \epsilon(\sigma) \gamma_{\sigma(j_1)} \cdots \gamma_{\sigma(j_p)}$$

$$= \sum_{1 \le j_1 < \dots < j_p \le m} \det\left(\delta_j a_\alpha^k\right) \sum_{\sigma \in S_p} \epsilon(\sigma) \gamma_{\sigma(j_1)} \cdots \gamma_{\sigma(j_p)}$$

Recall the conditional expectation values $E_\mathcal{A} : \mathrm{End}_\mathcal{A}(\mathcal{H}_\infty) \to \mathcal{A}$ from the projections $p_j \in \mathcal{A}$ of $\mathrm{Tr}(e) = \sum_{j=1}^n j p_j$ by

$$E_\mathcal{A}(T) := \sum_{k=1}^n \frac{p_k}{k} \sum_{j=1}^n T_{jj} , \qquad T = (T_{ij}) \in eM_n(\mathcal{A})e$$

Tentative charts

Definition

1. $\rho_\alpha := i^{\frac{p(p+1)}{2}} E_{\mathcal{A}}(\gamma T_\alpha)$
   (no γ for p odd, by construction $\rho_\alpha = \rho_\alpha^* \in \mathcal{A}$)
2. $U_\alpha := \{\chi \in X \ : \ \rho_\alpha(\chi) \neq 0\} \subset X$
3. $s_\alpha(\chi) := \big(\chi(a_\alpha^1), \dots, \chi(a_\alpha^p)\big)$

Proposition

1. The U_α form an open cover of X.
2. Suppose all derivation of the form $\delta_j(a) = \mathrm{i}(\eta_j | [D, a]\eta_j)$, for $\eta_j \in \mathcal{H}_\infty$, satisfy the Main Assumption. Let $\chi \in U_\alpha$. Then:

 i) There exist p derivations $\delta_1, \dots, \delta_p$ with $\det(\chi(\delta_j a_\alpha^k)) \neq 0$

 ii) The map $s_\alpha : U_\alpha \to \mathbb{R}^p$ is continuous and open.

Remarks on the main assumption

By investigation of maps $t \mapsto \gamma_t(a) = e^{\mathrm{i}t|D|} a e^{-\mathrm{i}t|D|}$ one proves

Theorem

For any $h = h^* \in \mathcal{A}$ the commutator $[D, h]$ vanishes where h reaches its maximum.

More precisely, for any sequence $b_n \in \mathcal{A}$, with $\|b_n\| \leq 1$ and support tending to $\{\chi\}$, where χ is a character such that $|\chi(h)|$ is maximal, one has $\|[D, h]b_n\| \to 0$.

It follows:

Proposition

The derivations $\pm\delta_j$, with $\delta_j(a) = \mathrm{i}(\eta_j|[D, a]\eta_j)$, are dissipative, i.e.

$$\|a + \lambda\delta_j(a)\| \geq \|a\| \quad \forall a \in \mathcal{A}\ ,\ \lambda \in \mathbb{R}$$

As a byproduct, one obtains

Proposition

For $h = h^* \in \mathcal{A}$ one has, with norm convergence,

$$\lim_{\tau\to\infty} \frac{e^{\mathrm{i}\tau h}|D|e^{-\mathrm{i}\tau h}}{\tau}\xi = |[D,h]|\xi\;, \qquad \xi \in \mathrm{dom}(D)$$

- this implies $[|D,h|,[D,a]] = 0$ for all $a, h \in \mathcal{A}$ with $h = h^*$
- $[D,a][D,b] + [D,b][D,a]$ commute with $\mathcal{A}$ and $[D,\mathcal{A}]$
- under the strong regularity assumption, the Clifford algebra is recovered: $[D,a][D,b] + [D,b][D,a] \in \mathcal{A}$

Using dissipativity of δ_0, Sobolev estimates and the Hille-Yosida-Lumer-Phillips Theorem, one shows that $U(t) = e^{t\delta_0}$ is a one-parameter group of isometries of the C^*-algebra A.

It is further shown that this group preserves smoothness

Local bound for multiplicity

The continuous open map $s_\alpha : U_\alpha \to \mathbb{R}^p$ is not injective. Show that at most finitely many $\chi_i \in U_\alpha$ map to the same point:

- the measure $F_t^*(\mu)$ is strongly equivalent to μ
- $s_\alpha(\mu)$ is locally equivalent to the Lebesgue measure on $\mathbb{R}^p$
- $\#\{s_\alpha^{-1}(y)\} \leq \Sigma(y)$ (Lebesgue almost everywhere), where $\Sigma(y)$ is joint spectral multiplicity of action of a_α^k on $\mathcal{H}$
- By results of Voiculescu, this is controlled by a norm on $\mathcal{L}^{(p,1)}$

Theorem

Let $V \subset U_\alpha$ be an open set with $\overline{V} \subset U_\alpha$. Then there exists $m < \infty$ such that

$$\#\{s_\alpha^{-1}(y) \cap V\} < m \quad \forall y \in s_\alpha(V) .$$

Proposition

Let $V \subset U_\alpha$ be an open set with $\overline{V} \subset U_\alpha$.

There exists a dense open subset $Y \subset s_\alpha(V)$ such that every point of $s_\alpha^{-1}(Y) \cap V$ has a neighbourhood N in X such that the restriction of s_α to N is an homeomorphism with an open set of $\mathbb{R}^p$.

Stratety:

- $m_1 := \sup_{y \in s_\alpha(V)} \#\{s_\alpha^{-1}(y) \cap V\}$, with $0 < m_1 < \infty$
- $Y_1 := \{y \in s_\alpha(V) \ : \ \#\{s_\alpha^{-1}(y) \cap V\} = m_1$
- Y_1 is open
- $\exists V_1, \dots, V_{m_1} \subset V$, with V_i open and mutually disjoint, with $s_\alpha(V_i) = Y_1$

Reconstruction Theorem

Lemma

For every point $\chi \in X$ there exist

- p real elements $x^\mu \in \mathcal{A}$,
- a smooth family $\tau_t \in \mathrm{Aut}(\mathcal{A})$, $t \in \mathbb{R}^p$, $\tau_0 = id$,

such that

1. The x^μ give a homeomorphism of a neighbourhood of χ with an open set in $\mathbb{R}^p$.
2. The map $t \mapsto h(t) = \tau_t^* \chi$ is a homeomorphism of a neighbourhood W of 0 in $\mathbb{R}^p$ with a neighbourhood of χ.
3. The map $\psi = \chi \circ h : W \to \mathbb{R}^p$ is a local diffeomorphism.

Proposition

The algebra $\mathcal{A}$ is locally the algebra of restrictions of smooth functions on $\mathbb{R}^p$ to a bounded open set of $\mathbb{R}^p$.

($\Rightarrow$) C^∞-functional calculus

Given selfadjoint $x_1, \dots, x_p \in \mathcal{A}$ and smooth function $f : \mathrm{R}^p \to \mathbb{C}$ defined on a neighbourhood of the joint spectrum of the x_j. Then

$$f_{(x_1,\dots,x_p)} := \frac{1}{(2\pi)^p} \int_{\mathbb{R}^p} d(t_1, \dots t_p)\; \hat{f}(t_1, \dots, t_p) \exp\Big(\mathrm{i} \sum_{j=1}^{p} t_j x_j\Big) \in \mathcal{A}$$

($\Leftarrow$) use the diffeomorphisms $\phi_\kappa : \mathbb{R}^p \supset W \to Y \subset \mathbb{R}^p$

Theorem

Let $(\mathcal{A}, \mathcal{H}, D)$ be a strongly regular spectral triple fulfilling the five conditions, with c antisymmetric. Then there exists an oriented smooth compact manifold X such that $\mathcal{A} = C^\infty(X)$.

- change of charts $x_2 : V_2 \to \mathbb{R}^p$ to $x_1 : V_1 \to \mathbb{R}^p$, with $\chi \in V_1 \cap V_2$
- By previous proposition, there exist p smooth functions $f^\mu \in C^\infty(x_2(V_1 \cap V_2))$ with $x_1^\mu(\chi) = (f^\mu \circ x_2)(\chi)$.
- Let $y = x_2(\chi) \in x_2(V_1 \cap V_2)$. Then

  $$x_1^\mu \circ x_2^{-1}(y) = x_1^\mu(\chi) = f^\mu(y)$$

  i.e. $x_1 \circ x_2^{-1}$ is smooth
- Hochschild cycle c gives a nowhere vanishing section of $\Lambda^p(T^*X)$

The converse

Theorem

An involutive algebra $\mathcal{A}$ is the algebra of smooth functions on an oriented smooth compact manifold if and only if it admits a faithful representation in a pair $(\mathcal{H}, D)$ fulfilling the five conditions of a spectral triple with the Hochschild cycle antisymmetric and the strong regularity.

- $\mathcal{H} = L^2(X, \Lambda^*_{\mathbb{C}})$
- $D = d + d^*$ for codifferential d^* with respect to any metric on X
- $[D, f]\xi = (\partial_\mu f)\gamma^\mu$ with $\gamma^\mu \xi = e_\mu \wedge \xi - i_{e_\mu}\xi$
- $\gamma = \mathrm{i}^{-\frac{p(p+1)}{2}} e^1 \wedge \cdots \wedge e^p$

regularity:

use $(D^2+1)^{\frac{1}{2}} =: \langle D\rangle = \frac{1}{\pi}\int_0^\infty \frac{d\lambda}{\sqrt{\lambda}}\,\frac{\langle D\rangle^2}{\lambda+\langle D\rangle^2}$ to obtain

$$\tilde{\delta}^m T = \frac{1}{\pi^m}\int_0^\infty \Big(\prod_{i=1}^m \frac{1}{\langle D\rangle^2+\lambda_i}\Big)\big((\mathrm{ad}\,D^2)^m T\big)\Big(\prod_{j=1}^m \frac{d\lambda_j\,\sqrt{\lambda_j}}{\langle D\rangle^2+\lambda_j}\Big)$$

$$= \sum_{k=0}^m (-1)^k \binom{m}{k} \frac{1}{\pi^m}\int_0^\infty \Big(\prod_{i=1}^k \frac{1}{\langle D\rangle^2+\lambda_i}\Big)\big((\mathrm{ad}\,D^2)^{m+k} T\big)\langle D\rangle^{-m} \times \prod_{j=1}^m \frac{d\lambda_j\,\sqrt{\lambda_j}\langle D\rangle}{(\langle D\rangle^2+\lambda_j)^2}$$

- D^2 is scalar elliptic operator with principal symbol $g_{\mu\nu}$
- Let P be a ΨDO of order q, then the principal symbol of $[D^2, P]$ of order $q+2$ vanishes

Spin^c-manifolds

Definition

Let (X, g) be a smooth compact Riemannian manifold and $A = C(X)$. Let $B = \Gamma(C\ell(X))$ be the algebra of continuous sections of the Clifford bundle over X, generated by T^*X and g^{-1}.

A Clifford module over X is a finitely generated projective right A-module $\mathcal{E}$ with hermitian structure $(\ |\ ) : \mathcal{E} \times \mathcal{E} \to A$, together with a homomorphism $c : B \to \mathrm{End}_A(\mathcal{E})$, such that $(\xi|c(\lambda)\eta) = (c(\lambda^*)\xi|\eta)$ for $\lambda \in B$ and $\xi, \eta \in \mathcal{E}$.
(This makes $\mathcal{E}$ a B-A-bimodule.)

X is called a spin^c-manifold if there is a B-A-bimodule $\mathcal{E}$ with $\mathrm{End}_A(\mathcal{E}) \simeq B$, and each isomorphism class of such B-A-bimodules is called a spin^c-structure.

Theorem

Let $(\mathcal{A}, \mathcal{H}, D)$ be a spectral triple, with $\mathcal{A}$ commutative and the cycle c antisymmetric. Assume that the multiplicity of the action of $\mathcal{A}''$ in $\mathcal{H}$ is $2^{[\frac{p}{2}]}$. Then there exists a smooth oriented compact spin^c-manifold X such that $A = C^\infty(X)$.

- Starting point: $[([D,a][D,b] + [D,b][D,a]), [D,c]] = 0$ for all $a, b, c \in \mathcal{A}$
- for dimensional reasons: $\mathrm{End}_{\mathcal{H}} = \Gamma^\infty(C\ell_Q(X))$
- $c : (da)(\chi) \to \mathrm{i}[D,a](\chi)$ isomorphism
- $[D,a][D,b] + [D,b][D,a] =: -2g^{-1}(da, db)$ metric
- $\|da\| = \sup_{\chi \in X} \|(\mathrm{grad}\, a)(\chi)\|$ leads to distance formula

$$\mathrm{dist}_g(\chi, \kappa) = \sup\{|a(\chi) - a(\kappa)| \;:\; a \in \mathcal{A}\,,\; \|[\mathcal{D}, a]\| \leq 1\}$$

Summary

- We have sketched the proof of a recent Theorem of A. Connes which establishes a 1:1 correspondence between commutative spectral triples and smooth compact oriented manifolds.
- The purpose was to present parts of the enormous diversity of methods used in Noncommutative Geometry.
- The reconstruction theorem provides strong motivation for noncommutative spectral triples as possible candidates for new forms of geometry in the early universe.