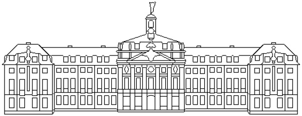
Non-compact spectral triples with finite volume
(work in progress)

Raimar Wulkenhaar

Mathematisches Institut der Westfälischen Wilhelms-Universität
Münster, Germany



Spectral triples

see: A. Connes, "On the spectral characterization of manifolds," 2008

Definition (commutative spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ of dimension $p \in \mathbb{N}$)

...given by a Hilbert space $\mathcal{H}$, a commutative involutive unital algebra $\mathcal{A}$ represented in $\mathcal{H}$, and a selfadjoint operator $\mathcal{D}$ in $\mathcal{H}$ with compact resolvent, with

1. *Dimension:* k^{th} characteristic value of resolvent of $\mathcal{D}$ is $\mathcal{O}(k^{-\frac{1}{p}})$
2. *Order one:* $[[\mathcal{D}, f], g] = 0 \quad \forall f, g \in \mathcal{A}$
3. *Regularity:* f and $[\mathcal{D}, f]$ belong to the domain of δ^k, for any $f \in \mathcal{A}$ and $k \in \mathbb{Z}$, where $\delta T := [|\mathcal{D}|, T]$
4. *Orientability:* $\exists$ Hochschild p-cycle $\boldsymbol{c} \in Z_p(\mathcal{A}, \mathcal{A})$ s.t.$\pi_{\mathcal{D}}(\boldsymbol{c}) = 1$ for p odd, $\pi_{\mathcal{D}}(\boldsymbol{c}) = \gamma$ for p even with $\gamma = \gamma^*$, $\gamma^2 = 1$, $\gamma\mathcal{D} = -\mathcal{D}\gamma$
5. *Finiteness and absolute continuity:* $\mathcal{H}_\infty := \cap_k \mathrm{dom}(\mathcal{D}^k) \subset \mathcal{H}$ is finitely generated projective $\mathcal{A}$-module, $\mathcal{H}_\infty = e\mathcal{A}^n$, with $e = e^* = e^2 \in M_m(\mathcal{A})$. Hermitian structure $(\xi|a\eta) = \sum_{i=1}^n a\xi_i^*\eta_i \in \mathcal{A}$ satisfies $\langle \xi, \eta \rangle = ⨍(\xi|\eta)|\mathcal{D}|^{-p}$

Theorem (Connes, 2008)

Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be a commutative spectral triple and assume that

– all endomorphisms $T \in \mathrm{End}_{\mathcal{A}}(\mathcal{H}_\infty)$ are regular,

*– the Hochschild cycle **c** is antisymmetric.*

Then there exists a compact oriented smooth manifold X such that $\mathcal{A} = C^\infty(X)$ is the algebra of smooth functions on X, and every compact oriented smooth manifold appears in this manner.

Remarks:

- formulated as a problem in [Connes, 1996]
- weaker version which assumes $\mathcal{A} = C^\infty(X)$ and classifies the spin geometry of X was stated in [Connes, 1996]; proof can be found in [Figueroa, Gracia-Bondía & Várilly, 2000]
- reconstruction problem attacked in [Rennie & Várilly, 2006]

Noncommutative generalisation

- Spectral triples generalise to noncommutative algebras, replacing commutativity by $[\mathcal{A}, \mathcal{A}^{op}] = 0$ and order-one by $[[\mathcal{D}, \mathcal{A}], \mathcal{A}^{op}] = 0$
- works well for:
  - almost-commutative manifolds ($C^\infty(X) \otimes$ matrices)
  - isospectral deformations (e.g. nc torus)
- other examples may require modifications of the axioms
- need more examples

Spectral triples are interesting for physics!

- equivalence classes of spectral triples describe Yang-Mills theory (inner automorphisms; exist always in nc case) and possibly gravity (outer automorphisms)
- inner fluctuations: $\mathcal{D} \mapsto \mathcal{D}_A = \mathcal{D} + A\,, \quad A = \sum f[\mathcal{D}, g]$
  for almost-commutative manifolds: A=Yang-Mills+Higgs

Spectral action principle [Chamseddine+Connes, 1996]

As an automorphism-invariant object, the (bosonic) action functional of physics has to be a function of the spectrum of $\mathcal{D}_A$, i.e. $S(\mathcal{D}_A) = \mathrm{Tr}(\chi(\mathcal{D}_A))$.

for almost-commutative 4-dim compact manifolds:

- $S(\mathcal{D}_A) = \int_X d\,vol\,(\mathcal{L}_\Lambda + \mathcal{L}_{\text{EH+W}} + \mathcal{L}_{\text{YM}} + \mathcal{L}_{\text{Higgs-kin}} + \mathcal{L}_{\text{Higgs-pot}})$
- structure of the standard model more or less unique
- prediction $m_{Higgs} = 170\,\text{GeV}$

Let's be careful

- spectral action is established for compact manifolds X
- it gives the action at a preferred (grand unification) scale Λ
- we use the renormalisation group equations (β-functions) to relate it to observation scale
- these β-functions are computed from quantum Yang-Mills-Higgs theory on non-compact $\mathbb{R}^4$

Is it justified to replace $\int_X \mapsto \int_{\mathbb{R}^4}$ without change of Lagrangian?

- Classical physics only sees equations of motion; infinite-volume limit can be performed
- Quantum field theory (β-function!) relies on entire action; infinite-volume limit may not exist! (e.g. if the cosmological constant is non-zero)

Non-compact spectral triples

- equivalently: algebra $\mathcal{A}$ is not unital
- for typical examples, resolvent of $\mathcal{D}$ is not compact
- basic idea: require $a(\mathcal{D}-\lambda)^{-1}$ compact for all $a \in \mathcal{A}$ and all $\lambda \notin sp(\mathcal{D})$

$\rightarrow$ axioms for non-compact spectral triples [Gayral, Gracia-Bondía, Iochum, Schücker & Várilly, 2003]

$\rightarrow$ examples include standard $\mathbb{R}^d$ and Moyal plane

BUT:

- Spectral action does still not exist!
- for Moyal: add regularisation [Gayral & Iochum, 2004]
- this talk: Keep the resolvent of $\mathcal{D}$ compact even for non-unital $\mathcal{A}$. Show that there are examples.

Non-compact spectral triples with finite volume

Definition (commutative case)

…given by a Hilbert space $\mathcal{H}$, a commutative involutive possibly non-unital algebra $\mathcal{A}$ represented in $\mathcal{H}$, and a selfadjoint operator $\mathcal{D}$ in $\mathcal{H}$ with compact resolvent, with

1. *Regularity and dimension spectrum:* $f, [\mathcal{D}, f] \in \mathrm{dom}(\delta^k)$, for any $f \in \mathcal{A}$ and $k \in \mathbb{Z}$, where $\delta T := [\langle\mathcal{D}\rangle, T]$ and $\langle\mathcal{D}\rangle = (\mathcal{D}^2 + 1)^{\frac{1}{2}}$.
   $\zeta_\phi(z) := \mathrm{Tr}(\phi\langle\mathcal{D}\rangle^{-z})$ extends holomorphically to $\mathbb{C} \setminus \mathrm{Sd}$, for any $\phi \in \Psi_0(\mathcal{A})$ (alg. gen. by $\delta^n f$ and $\delta^n[\mathcal{D}, f]$), where Sd is discrete.
2. *Metric dimension:* For $d := \sup\{r \in \mathbb{R} \cap \mathrm{Sd}\}$ and $f \in \mathcal{A}$, the noncommutative integral $⨍ f\langle\mathcal{D}\rangle^{-d}$ is finite and positive for $f > 0$.
3. *Orientability:* Let $\mathcal{B} = \{b \in \mathcal{A}'' \;:\; b, [\mathcal{D}, b] \in \bigcap_{k\in\mathbb{N}} \mathrm{dom}(\mathcal{D}^k)\}$.
   There is a Hochschild cycle $\boldsymbol{c} \in Z(\mathcal{B}, \mathcal{B})$ such that $\gamma = \pi_\mathcal{D}(\boldsymbol{c})$ satisfies $\gamma^2 = 1$ and $\gamma = \gamma^*$.
   $\phi_\gamma(f_0, \dots, f_d) := ⨍ (\gamma f_0[\mathcal{D}, f_1] \cdots [\mathcal{D}, f_d]\langle\mathcal{D}\rangle^{-d})$ defines a non-vanishing Hochschild d-cocycle on $\mathcal{A}$.

Definition (...continued)

4. *Order one:* $[[\mathcal{D}, f], g] = 0 \quad \forall f, g \in \mathcal{B}$.
5. *Finiteness:* $\mathcal{H}_\infty := \bigcap_{k\in\mathbb{N}} \mathrm{dom}(\mathcal{D}^k) \subset \mathcal{H}$ is a finitely-generated projective $\mathcal{A}$-module, i.e. $\mathcal{H}_\infty = e\mathcal{A}^m$ for some $e = e^* = e^2 \in M_m(\mathcal{B})$.

   Hermitian structure $\langle\ ,\ \rangle_\mathcal{A} : \mathcal{H}_\infty \times \mathcal{H}_\infty \to \mathcal{A}$ and inner product $(\ ,\ )$ on $\mathcal{H}_\infty$ are related by $(\xi, \eta) = ⨍(\langle\xi, \eta\rangle_\mathcal{A}\langle\mathcal{D}\rangle^{-d})$.

Remarks:

- If $1 \in \mathcal{A}$, then $\mathcal{B} = \mathcal{A}$ (Lemma 2.1 of [Connes, 2008]) and k^{th} characteristic value of resolvent of $\mathcal{D}$ is $\mathcal{O}(k^{-\frac{1}{d}})$

  The only difference is that we do not demand $\gamma = 1$ for d odd and $\gamma\mathcal{D} = -\mathcal{D}\gamma$ for d even.
- If $1 \notin \mathcal{A}$, then critical summability p of $\langle\mathcal{D}\rangle^{-1}$ may be $> d$, and $⨍ b\langle\mathcal{D}\rangle^{-d}$ might not exist for b in a larger algebra

  The matching $\mathcal{H}_\infty = e\mathcal{A}^m$ of smoothness with integrability is crucial.

The d-dimensional harmonic oscillator

d bosonic and fermionic creation and annihilation operators:

- $[a_\mu, a_\nu] = [a_\mu^\dagger, a_\nu^\dagger] = 0 \qquad [a_\mu, a_\nu^\dagger] = \delta_{\mu\nu} \qquad \mu,\nu{=}1,\dots,d$
  $\{b_\mu, b_\nu\} = \{b_\mu^\dagger, b_\nu^\dagger\} = 0 \qquad \{b_\mu, b_\nu^\dagger\} = \delta_{\mu\nu}$
  realisation: $a_\mu = \frac{1}{\sqrt{2\omega}}(\omega x_\mu + \partial_\mu), \quad a_\mu^\dagger = \frac{1}{\sqrt{2\omega}}(\omega x_\mu - \partial_\mu)$
- Hilbert space $\mathcal{H} = L^2(\mathbb{R}^d) \otimes \bigwedge(\mathbb{C}^d)$: declare ONB
  $\Big\{(a_1^\dagger)^{n_1} \dots (a_d^\dagger)^{n_d} \otimes (b_1^\dagger)^{s_1} \dots (b_d^\dagger)^{s_d}|0\rangle \;:\; n_\mu \in \mathbb{N}\,, s_\mu \in \{0,1\}\Big\}$
- Dirac operator
  $$\mathcal{D}_d = \underbrace{\sqrt{2\omega}\textstyle\sum_{\mu=1}^d a_\mu \otimes b_\mu^\dagger}_{} + \underbrace{\sqrt{2\omega}\textstyle\sum_{\mu=1}^d a_\mu^\dagger \otimes b_\mu}_{}$$
  $$= \mathrm{d} + \mathrm{d}^*$$
  $$= \partial^\mu \otimes (b_\mu^\dagger - b_\mu) + \omega x^\mu (b_\mu^\dagger + b_\mu)$$
- algebra $\mathcal{A} = \mathcal{S}(\mathbb{R}^d)$ acting by pointwise multiplication, uniquely determined by smoothness

Laplace operator

$\mathcal{D}^2 = \omega \sum_{\mu=1}^{d} (\{a_\mu, a_\mu^\dagger\} \otimes 1 + 1 \otimes [b_\mu^\dagger, b_\mu]) = H_d \otimes 1 + \omega \otimes \Sigma_d$

where

$H_d = -\frac{\partial^2}{\partial x_\mu \partial x^\mu} + \omega^2 x_\mu x^\mu$ – harmonic oscillator hamiltonian

$\Sigma_d = \sum_{\mu=1}^{d} [b_\mu^\dagger, b_\mu]$ – spin matrix

- $\mathcal{H}_\infty = \bigcap_{k \in \mathbb{N}} \mathrm{dom}(\mathcal{D}^{2k}) = \mathcal{S}(\mathbb{R}^d) \otimes \bigwedge(\mathbb{C}^d) \simeq (\mathcal{S}(\mathbb{R}^d))^{2^d}$ (trivial projector of rank 2^d)
- Hermitian structure takes values in Schwartz class functions: We may choose
  - the commutative algebra $\mathcal{A} = \mathcal{S}(\mathbb{R}^d)$ (this talk)
  - or the noncommutative Moyal algebra which provides an isospectral deformation (work with H. Grosse)
- $[\mathcal{D}, f] = \partial^\mu f \otimes (b_\mu^\dagger - b_\mu)$, bounded operator order-one is satisfied

The dimension spectrum

Theorem

The spectral triple $(\mathcal{A},\mathcal{H},\mathcal{D})$ of the d-dimensional harmonic oscillator has dimension spectrum $\mathrm{Sd} = d - \mathbb{N}$ and hence metric dimension d.

All residues of $\zeta_\phi(z)$ at $z \in \mathrm{Sd}$ are local, i.e. for $\phi = \delta^{n_1} f_1 \cdots \delta^{n_v} f_v$ they are finite sums of $\int d^dx\; x^{\alpha_0}(\partial^{\alpha_1} f_1)\cdots(\partial^{\alpha_v} f_v)$, where α_i are multi-indices.

Proposition (Trace theorem)

$$⨍ (f\langle\mathcal{D}\rangle^{-d}) = \frac{1}{(4\pi)^{\frac{d}{2}}\Gamma(\frac{d+2}{2})}\int d^dx\; f(x)\,.$$

For appropriate normalisation, the relation in the finiteness condition is satisfied!

Remarks on the computation of $\mathcal{S}_d$

Two identities:

$$(-\pi)^n \delta^n f$$
$$= \int_0^\infty \Big(\prod_{i=1}^n \frac{1}{\langle\mathcal{D}\rangle^2+\lambda_i}\Big)\{\partial_{\mu_1},\dots,\{\partial_{\mu_n},\partial^{\mu_1}\dots\partial^{\mu_n} f\}\dots\}\prod_{j=1}^n \frac{d\lambda_j\,\sqrt{\lambda_j}}{\langle\mathcal{D}\rangle^2+\lambda_j}$$

$$(\langle\mathcal{D}\rangle^2+\lambda)^{-z} = \frac{1}{\Gamma(\frac{z}{2})}\int_0^\infty dt\ t^{\frac{z}{2}-1}\ e^{-t(H+\omega\Sigma+\lambda)}$$

Mehler kernel (in position space)

$$e^{-tH}(x,y) = \Big(\frac{\omega}{2\pi\sinh(2\omega t)}\Big)^{\frac{d}{2}} e^{-\frac{\omega}{4}\coth(\omega t)\|x-y\|^2-\frac{\omega}{4}\tanh(\omega t)\|x+y\|^2}$$

(solves $(\frac{d}{dt}+H_x)e^{-tH}(x,y)=0$, $\lim_{t\to 0} e^{-tH}(x,y)=\delta(x-y)$)

$\mathrm{Tr}(\delta^{n_1} f_1 \dots \delta^{n_v} f_v \langle D\rangle^{-z})$ is concatenation of functions, derivatives and Mehler kernels, integrated over $(\mathbb{R}^d)^v$

- integrate by parts anticommutators of derivatives to Mehler kernels; yields a polynomial factor in x_i, t_i
- Fourier transformation in $f_i(x_i) \mapsto \hat{f}_i(p_i)$
- Gaußian integration in x_i; yields a factor

$$\frac{\exp\Big(-\frac{(p_1+\cdots+p_v)^2}{\omega\tanh(\omega(t_1+\cdots+t_v))}+\sum_{i,j=1}^{v}\frac{2p_ip_j}{\omega}\frac{\sinh(\omega(t_i+\cdots+t_{j-1}))\sinh(\omega(t_j+\cdots+t_{i-1}))}{\sinh(\omega(t_1+\cdots+t_v))}\Big)}{\tanh^{d}(\omega(t_1+\cdots+t_v))}$$

- singularity at $t_1+\cdots+t_v \to 0$ protected for $p_1+\cdots+p_v=0$ (momentum conservation)
- Taylor expansion about $p_v=-(p_1+\cdots+p_{v-1})$, Gaussian integration in p_v:

  singularity is $\frac{t^{\frac{z}{2}}F(t_i)}{\tanh^{\frac{d}{2}}(\omega(t_1+\cdots+t_v))}$, exponential is small
- Taylor expansion in t_i: leading pole at $z=d$, subleading poles for $z=d-\mathbb{N}$ due to δ^{n_i}

  all residues local after Fourier transformation

Orientability

- Plane waves $u_\mu = e^{ix_\mu}$ belong to $\mathcal{B}$ (using formula for $\delta^n f$)

  plausible that $\mathcal{B} = C_b^\infty(\mathbb{R}^d)$, i.e. smooth bounded functions with all derivatives bounded

- Hochschild d-cycle ($b\boldsymbol{c} = 0$)

$$\boldsymbol{c} = \sum_{\sigma\in S_d} \epsilon(\sigma)\frac{\mathrm{i}^{\frac{d(d-1)}{2}}}{d!}(u_1\cdots u_d)^{-1}\otimes u_{\sigma(1)}\otimes\dots u_{\sigma(d)} \in Z_d(\mathcal{B},\mathcal{B})$$

- $\gamma = \pi(\boldsymbol{c}) = \mathrm{i}^{\frac{d(d+1)}{2}}(b_1^\dagger - b_1)\cdots(b_d^\dagger - b_d), \qquad \gamma^2 = 1, \gamma = \gamma^*$

  But γ does not commute/anticommute with $\mathcal{D}$!

- defines Hochschild 4-cocycle on $\mathcal{A}$ (volume form)

$$\phi_\gamma(f_0,\dots,f_d) = ⨍ \big(\gamma f_0[\mathcal{D},f_1]\cdots[\mathcal{D},f_d]\langle\mathcal{D}\rangle^{-d}\big) = \frac{1}{(4\pi)^{\frac{d}{2}}\Gamma(\frac{d+2}{2})}\int f_0\,df_1\wedge\cdots\wedge df_d$$

U(1)-Higgs model

tensor $(\mathcal{A}_4, \mathcal{H}_4, \mathcal{D}_4)$ with $(\mathbb{C} \oplus \mathbb{C}, \mathbb{C}^2, M\sigma_1, \sigma_3)$ [Connes+Lott]

- $\mathcal{D} = \mathcal{D}_4 \otimes \sigma_3 + 1 \otimes \sigma_1 M = \begin{pmatrix} \mathcal{D}_4 & M \\ M & -\mathcal{D}_4 \end{pmatrix}$
- selfadjoint fluctuated Dirac operators $\mathcal{D}_A := \mathcal{D} + \sum_i a_i[\mathcal{D}, b_i]$, $a_i, b_i \in \mathcal{A} = \mathcal{A}_4 \oplus \mathcal{A}_4$, are of the form
  $$\mathcal{D}_A = \begin{pmatrix} \mathcal{D}_4 + \mathrm{i}A^\mu \otimes (b^\dagger_\mu - b_\mu) & \phi \otimes 1 \\ \overline{\phi} \otimes 1 & -(\mathcal{D}_4 + \mathrm{i}B^\mu \otimes (b^\dagger_\mu - b_\mu)) \end{pmatrix}$$
  for $A_\mu = \overline{A_\mu}, B_\mu = \overline{B_\mu}, \phi \in \mathcal{A}_4$
- $$\mathcal{D}_A^2 = \begin{pmatrix} (H + |\phi|^2) \otimes 1 + \omega \otimes \Sigma + F_A & D^\mu\phi \otimes (b^\dagger_\mu - b_\mu) \\ -\overline{D^\mu\phi} \otimes (b^\dagger_\mu - b_\mu) & (H + |\phi|^2) \otimes 1 + \omega \otimes \Sigma + F_B \end{pmatrix}$$
- $D_\mu\phi = \partial_\mu\phi + i(A_\mu - B_\mu)\phi$
  $F_A = (-\{\partial^\mu, A_\mu\} - \mathrm{i}A^\mu A_\mu) \otimes 1 + \frac{1}{4} F_A^{\mu\nu} \otimes [b^\dagger_\mu - b_\mu, b^\dagger_\nu - b_\nu]$

Spectral action principle

most general form of bosonic action is $S(\mathcal{D}_A) = Tr\big(\chi(\mathcal{D}_A^2)\big)$

- Laplace transf. + asympt. expansion $e^{-t\mathcal{D}_A^2} = \sum_{n=-dim/2}^{\infty} a_n(\mathcal{D}_A^2)t^n$

  leads to $S(\mathcal{D}_A) = \sum_{n=-dim/2}^{\infty} \chi_n \mathrm{Tr}(a_n(\mathcal{D}_A^2))$

  with $\chi_z = \frac{1}{\Gamma(-z)} \int_0^\infty ds\, s^{-z-1}\chi(s)$ for $z \notin \mathbb{N}$

  $\chi_k = (-1)^k \chi^{(k)}(0)$ for $k \in \mathbb{N}$
- a_n – Seeley coefficients, must be computed from scratch

Duhamel expansion: $\mathcal{D}_A^2 = \mathrm{H}_0 - V$

$$e^{-t(\mathrm{H}_0-V)} = e^{-t\mathrm{H}_0} - \int_0^t dt_1 \,\frac{d}{dt_1}\Big(e^{-(t-t_1)(\mathrm{H}_0-V)}e^{-t_1\mathrm{H}_0}\Big)$$

$$= e^{-t\mathrm{H}_0} + \int_0^t dt_1 \,\Big(e^{-(t-t_1)(\mathrm{H}_0-V)}Ve^{-t_1\mathrm{H}_0}\Big) \qquad \ldots\text{iteration}$$

Vacuum trace

Mehler kernel (in 4D)

$$e^{-t(H+\omega\Sigma)}(x,y) =$$

$$\frac{\omega^2(1-\tanh^2(\omega t))^2}{16\pi^2\tanh^2(\omega t)} e^{-t\omega\Sigma 1_2 - \frac{\omega}{4}\frac{\|x-y\|^2}{\tanh(\omega t)} - \frac{\omega}{4}\tanh(\omega t)\|x+y\|^2}$$

$$\mathrm{tr}(e^{-t\omega\Sigma}) = (2\cosh(\omega t))^d$$

$$\begin{aligned}\mathrm{Tr}(e^{-t(H+\omega\Sigma)\otimes 1_2}) &= 2\mathrm{tr}\Big(\int d^4x\,(e^{-t(\mathrm{H}+\omega\Sigma)})(x,x)\Big)\\ &= \frac{2}{\tanh^4(\omega t)} = 2(\omega t)^{-4} + \frac{8}{3}(\omega t)^{-2} + \frac{52}{45} + \mathcal{O}(t^2)\end{aligned}$$

- Spectral action is finite, in contrast to standard $\mathbb{R}^4$!
- expansion starts with t^{-4} $\Rightarrow$ corresponds to 8-dim. space

Vertices

only one-vertex contribution from
$V = \mathrm{diag}(-(A_\mu A^\mu + |\phi|^2)\otimes 1, -(B_\mu B^\mu + |\phi|^2)\otimes 1)$
(no tadpole, in contrast to Moyal where $\{\partial_\mu + \omega x_\mu, \pi(A^\mu)\}$ contributes)

$$\int_0^t dt_1\ \mathrm{Tr}\big(e^{-(t-t_1)(H+\omega\Sigma)1_2} V e^{-t_1(H+\omega\Sigma)1_2}\big)$$
$$= \frac{\omega^2 t}{\pi^2 \tanh^2(\omega t)} \int d^4x\big(-2|\phi|^2 - A_\mu A^\mu - B_\mu B^\mu\big) e^{-\omega \tanh(\omega t)\|x\|^2}$$

- yields $-\mu^2\phi^2$ for Higgs potential and oscillator potential $+x^2\phi^2$
- A^2, B^2 eliminated by two vertices with $\{\partial^\mu, A_\mu\}, \{\partial^\mu, B_\mu\}$; these also give Yang-Mills with negative sign, exceeded by two curvature-vertices

The spectral action

$$
\begin{aligned}
S(\mathcal{D}_A) = {} & \frac{2\chi_{-4}}{\omega^4} + \frac{8\chi_{-2}}{3\omega^2} + \frac{52\chi_0}{45} \\
& + \frac{\chi_0}{\pi^2}\int d^4x \,\Big\{ D_\mu\phi\overline{D_\mu\phi} + \frac{5}{12}\big(F^A_{\mu\nu}F_A^{\mu\nu} + F^B_{\mu\nu}F_B^{\mu\nu}\big) \\
& \quad + \Big((|\phi|^2)^2 - \frac{2\chi_{-1}}{\chi_0}|\phi|^2 + 2\omega^2\|x\|^2|\phi|^2\Big)\Big\} + \mathcal{O}(\chi_1)
\end{aligned}
$$

- spectral action is finite
- only difference in field equations to infinite volume is additional harmonic oscillator potential for the Higgs
- Yang-Mills is unchanged (in contrast to Moyal)
- vacuum is at $A_\mu = B_\mu = 0$ and (after gauge transformation) $\phi \in \mathbb{R}$, rotationally invariant

Field equations

rescaling $r = 2^{\frac{1}{4}}\sqrt{\omega}\|x\|$, $\phi = \frac{\pi}{\sqrt{2}\chi_0}\varphi$, $\mu^2 = \frac{\chi_{-1}}{\sqrt{8}\,\omega\chi_0}$, $\lambda = \frac{\pi^2}{\sqrt{2}\,\omega\chi_0}$

$$-\varphi''(r) - \frac{3}{r}\varphi'(r) + (r^2 - 4\mu^2)\varphi(r) = -\lambda\varphi^3(r)$$

- inhomogeneous confluent hypergeometric diff. eq.
- expand $\varphi = \frac{2}{\sqrt{\lambda}}\sum_{n=0}^{\infty} c_n\varphi_n$ in terms of eigenfunctions of 4-dim. harmonic oscillator:

$$\varphi_n := e^{-\frac{r^2}{2}} L_n^1(r^2) \qquad \Big(-\frac{d^2}{dr^2} - \frac{3}{r}\frac{d}{dr} + r^2\Big)\varphi_n = 4(n+1)\varphi_n$$

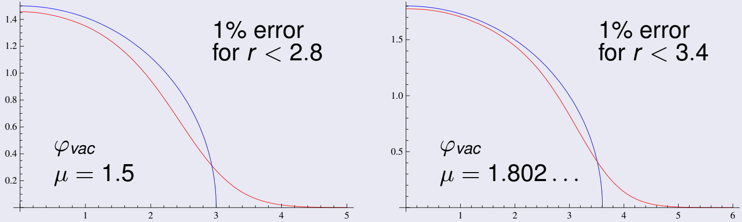
- yields

$$c_n(\mu^2 - n - 1) = \sum_{k,l,m=0}^{\infty} \frac{c_k c_l c_m}{k!l!m!}\left(\frac{d^k}{dw^k}\frac{d^l}{dy^l}\frac{d^m}{dz^m}\frac{(1-wy-wz-yz+2wyz)^n}{(2-w-y-z+wyz)^{n+2}}\right)_{w=y=z=0}$$

- numerical solution: cut-off at N

$-\varphi''(r) - \frac{3}{r}\varphi'(r) + (r^2 - 4\mu^2)\varphi(r) = -\lambda\varphi^3(r)$

$\varphi(r)$ in units of $\frac{2}{\sqrt{\lambda}}$, cutoff at $N = 10$

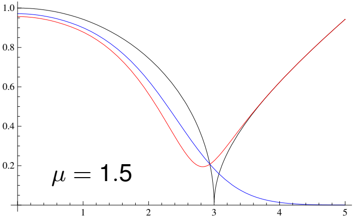


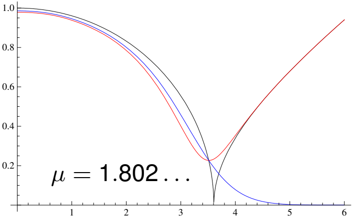
- vacuum solution $\varphi_{vac}(r)$ is smaller than asymptotic curve $\sqrt{\mu^2 - \frac{1}{4}r^2}$ due to its negative curvature and approaches it for $\mu \to \infty$
- $\varphi_{vac} = 0$ for $r \geq 2\mu$, transition is smoothly
- vacuum solution is integrable

Masses

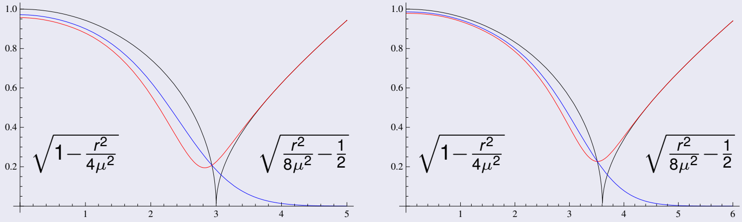
- scale of bare fermion and gauge field masses given by vacuum expectation value $\sqrt{\frac{4\mu^2}{\lambda}}\frac{\varphi_{vac}}{\mu} = \sqrt{\frac{2\chi_{-1}}{\pi^2}}\frac{\varphi_{vac}}{\mu}$
- bare Higgs mass given by difference function

$$\sqrt{\sqrt{2}\omega((r^2-4\mu^2)+12\mu^2\frac{\varphi_{vac}^2}{\mu^2})} = \sqrt{\frac{4\chi_{-1}}{\chi_0}}\frac{\sqrt{\frac{3}{2}\varphi_{vac}^2-\frac{1}{2}\mu^2+\frac{1}{8}r^2}}{\mu}$$





Smooth transition between two phases



1. *Spontaneously broken phase* $\omega^2\|x\|^2 < \frac{\chi_{-1}}{\chi_0}$
   fermions, gauge fields and Higgs are massive, with Higgs mass slightly smaller than NCG-prediction
2. *Unbroken phase* $\omega^2\|x\|^2 \geq \frac{\chi_{-1}}{\chi_0}$
   fermions + gauge fields massless, Higgs remains massive

Mass of gauge fields and fermions dissipates into cosmological constant!