Non-compact spectral triples with finite volume

Towards noncommutative gauge theory

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Spectral triples

see: A. Connes, “On the spectral characterization of manifolds,” 2008

Definition (commutative spectral triple $(\mathcal{A},\mathcal{H},\mathcal{D})$ of dimension $p\in\mathbb{N}$)

…given by a Hilbert space $\mathcal{H}$, a commutative involutive unital algebra $\mathcal{A}$ represented in $\mathcal{H}$, and a selfadjoint operator $\mathcal{D}$ in $\mathcal{H}$ with compact resolvent, with

1. *Dimension:* k^{th} characteristic value of resolvent of $\mathcal{D}$ is $\mathcal{O}(k^{-\frac{1}{p}})$
2. *Order one:* $[[\mathcal{D},f],g]=0 \quad \forall f,g\in\mathcal{A}$
3. *Regularity:* f and $[\mathcal{D},f]$ belong to the domain of δ^k, for any $f\in\mathcal{A}$ and $k\in\mathbb{Z}$, where $\delta T:=[|\mathcal{D}|,T]$
4. *Orientability:* $\exists$ Hochschild p-cycle $\boldsymbol{c}\in Z_p(\mathcal{A},\mathcal{A})$ s.t.$\pi_{\mathcal{D}}(\boldsymbol{c})=1$ for p odd, $\pi_{\mathcal{D}}(\boldsymbol{c})=\gamma$ for p even with $\gamma=\gamma^*$, $\gamma^2=1$, $\gamma\mathcal{D}=-\mathcal{D}\gamma$
5. *Finiteness and absolute continuity:* $\mathcal{H}_\infty:=\cap_k \mathrm{dom}(\mathcal{D}^k)\subset\mathcal{H}$ is finitely generated projective $\mathcal{A}$-module, $\mathcal{H}_\infty=e\mathcal{A}^n$, with $e=e^*=e^2\in M_m(\mathcal{A})$. Hermitian structure $(\xi|a\eta)=\sum_{i=1}^n a\xi_i^*\eta_i\in\mathcal{A}$ satisfies $\langle\xi,\eta\rangle=⨍(\xi|\eta)|\mathcal{D}|^{-p}$

Theorem (Connes, 2008)

Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be a commutative spectral triple and assume that

– all endomorphisms $T \in \mathrm{End}_{\mathcal{A}}(\mathcal{H}_\infty)$ are regular,

*– the Hochschild cycle **c** is antisymmetric.*

Then there exists a compact oriented smooth manifold X such that $\mathcal{A} = C^\infty(X)$ is the algebra of smooth functions on X, and every compact oriented smooth manifold appears in this manner.

It is known since [Milnor, 1964] that there exist isospectral manifolds which are not isometric

Connes' proof identifies the missing piece which in addition to the spectrum of $\mathcal{D}$ characterises the geometry:

It is the analogue of the Cabibbo-Kobayashi-Maskawa-Matrix in QFT which encodes the relative position of two bases in the same Hilbert space.

Spectral triples are interesting for physics!

- equivalence classes of spectral triples describe Yang-Mills theory (inner automorphisms; exist always in nc case) and possibly gravity (outer automorphisms)
- inner fluctuations: $\mathcal{D} \mapsto \mathcal{D}_A = \mathcal{D} + A\,, \quad A = \sum f[\mathcal{D}, g]$
  for almost-commutative manifolds: A=Yang-Mills+Higgs

Spectral action principle [Chamseddine+Connes, 1996]

As an automorphism-invariant object, the (bosonic) action functional of physics has to be a function of the spectrum of $\mathcal{D}_A$, i.e. $S(\mathcal{D}_A) = \mathrm{Tr}(\chi(\mathcal{D}_A))$.

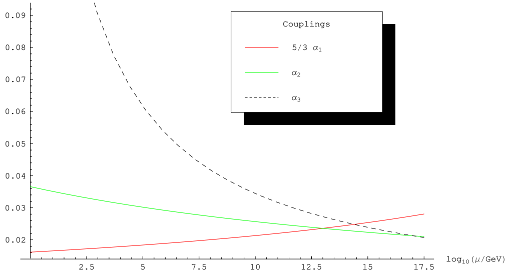
for almost-commutative 4-dim compact manifolds:

- $S(\mathcal{D}_A) = \int_X d\,vol\,(\mathcal{L}_\Lambda + \mathcal{L}_{\text{EH+W}} + \mathcal{L}_{\text{YM}} + \mathcal{L}_{\text{Higgs-kin}} + \mathcal{L}_{\text{Higgs-pot}})$

  for any function of the spectrum (universality of RG)
- structure of the standard model more or less unique

to be precise, an effective action is produced which is spectral at a distiguished scale Λ_{GUT}

- structural relations at Λ_{GUT}: YM couplings $\alpha_3 = \alpha_2$, $\sin^2\theta_W = \frac{3}{8}$
  Higgs coupling $\lambda = \frac{16\pi}{3}\alpha_2$
- connection to experiment: renormalisation group flow:



$\alpha_2, \alpha_3, \theta_W, \lambda$ scale-dependent, slope given by β-functions

- Higgs mass $m_H = \sqrt{\frac{2\lambda}{\pi\alpha_2}}m_W$
  from λ, α_2 evaluated at m_Z one finds $m_H \sim 170\,\text{GeV}$

Message: Geometry of Nature is scale dependent

1. at $E < 10^{-9}\,\text{GeV} \sim 200\,\text{nm}$: Riemannian geometry
2. at $E = 100 GeV \dots \Lambda_{GUT}$: almost-commutative geometry
3. at $E > \Lambda_{GUT}$ truly noncommutative geometry??

We study spectral triples over truly noncommutative algebras and their corresponding spectral actions.

- Simplest example: Moyal space (appears in String Theory)
  $$(f \star g)(x) = \int d^4y \frac{d^4k}{(2\pi)^4} f(x + \tfrac{1}{2}\Theta k) g(x+y) e^{iky}$$
- Naïve replacement of local product by $\star$-product is ill-behaved: not renormalisable (UV/IR-mixing) and incompatible with spectral action principle
- Both problems disappear if we require discrete spectra!

NC ϕ^4_4-theory develops additional marginal coupling

action functional for real-valued field ϕ on $\mathbb{R}^4$:

$$S[\phi] = \int d^4x \Big(\frac{1}{2}\phi \star (-\Delta + \Omega^2 \tilde{x}^2 + \mu^2)\phi + \frac{\lambda}{4}\phi \star \phi \star \phi \star \phi\Big)(x)$$

Moyal product $\star$ defined by Θ and $\tilde{x} := 2\Theta^{-1} \cdot x$

parameters: $\mu^2, \lambda \in \mathbb{R}_+$, $\Omega \in [0,1]$, redef'n $\phi \mapsto \mathcal{Z}\phi$, $\mathcal{Z} \in \mathbb{R}_+$

- renormalisable as formal power series in λ
  [H.Grosse+R.W.; 2004]
  means: well-defined perturbative quantum field theory
- β-function vanishes to all orders in λ for $\Omega = 1$
  [M.Disertori, R.Gurau, J.Magnen + V.Rivasseau; 2006]
  means: model is believed to exist non-perturbatively

Does this model arise from a spectral triple?

The d-dimensional harmonic oscillator

d bosonic and fermionic creation and annihilation operators:

- $[a_\mu, a_\nu] = [a_\mu^\dagger, a_\nu^\dagger] = 0 \qquad [a_\mu, a_\nu^\dagger] = \delta_{\mu\nu} \qquad \mu,\nu{=}1,\dots,d$
  $\{b_\mu, b_\nu\} = \{b_\mu^\dagger, b_\nu^\dagger\} = 0 \qquad \{b_\mu, b_\nu^\dagger\} = \delta_{\mu\nu}$
  realisation: $a_\mu = \frac{1}{\sqrt{2\omega}}(\omega x_\mu + \partial_\mu) = \frac{1}{\sqrt{2\omega}} e^{-h}\partial_\mu e^h$
  $a_\mu^\dagger = \frac{1}{\sqrt{2\omega}}(\omega x_\mu - \partial_\mu) = -\frac{1}{\sqrt{2\omega}} e^{h}\partial_\mu e^{-h}, \qquad h = \frac{\omega}{2}x^2$
- Hilbert space $\mathcal{H} = L^2(\mathbb{R}^d) \otimes \bigwedge(\mathbb{C}^d)$: declare ONB
  $\{(a_1^\dagger)^{n_1} \dots (a_d^\dagger)^{n_d} \otimes (b_1^\dagger)^{s_1} \dots (b_d^\dagger)^{s_d}|0\rangle \;:\; n_\mu \in \mathbb{N}\,, s_\mu \in \{0,1\}\}$
- Dirac operator $$\mathcal{D} = \underbrace{\sqrt{2\omega}\sum_{\mu=1}^d a_\mu \otimes b_\mu^\dagger}_{Q} + \underbrace{\sqrt{2\omega}\sum_{\mu=1}^d a_\mu^\dagger \otimes b_\mu}_{Q^\dagger}$$
  $$= Q + Q^\dagger$$
  $$= \partial^\mu \otimes (b_\mu^\dagger - b_\mu) + \omega x^\mu (b_\mu^\dagger + b_\mu)$$
- algebra $\mathcal{A} = \mathcal{S}(\mathbb{R}^d)$ uniquely determined by smoothness

Laplace operator

$\mathcal{D}^2 = \omega \sum_{\mu=1}^{d} (\{a_\mu, a_\mu^\dagger\} \otimes 1 + 1 \otimes [b_\mu^\dagger, b_\mu]) = H \otimes 1 + \omega \otimes \Sigma$

where

$H = -\frac{\partial^2}{\partial x_\mu \partial x^\mu} + \omega^2 x_\mu x^\mu$ – harmonic oscillator hamiltonian

$\Sigma = \sum_{\mu=1}^{d} [b_\mu^\dagger, b_\mu]$ – spin matrix

- $\mathcal{H}_\infty = \bigcap_{k \in \mathbb{N}} \mathrm{dom}(\mathcal{D}^{2k}) = \mathcal{S}(\mathbb{R}^d) \otimes \bigwedge(\mathbb{C}^d) \simeq (\mathcal{S}(\mathbb{R}^d))^{2^d}$ (trivial projector of rank 2^d)
- Hermitian structure takes values in Schwartz class functions: We may choose
  1. the commutative algebra $\mathcal{A} = \mathcal{S}(\mathbb{R}^d)$
     $[\mathcal{D}, f] = \partial^\mu f \otimes (b_\mu^\dagger - b_\mu)$ bounded & order-one
  2. the noncommutative Moyal algebra $\mathcal{A}_\Theta$ which provides an isospectral deformation (work with H. Grosse)
     $[\mathcal{D}, L_\star(f)] = \partial^\mu f \otimes \big(b_\mu^\dagger - b_\mu + \frac{1}{2}\omega\Theta^{\mu\nu}(\mathrm{i}b_\nu^\dagger + \mathrm{i}b_\nu)\big)$

All axioms of spectral triples satisfied, with minor adaptation:

- The metric dimension is different from the asymptotics of eigenvalues of $\mathcal{D}^{-1}$.

  This is taken into account by the dimension spectrum computed from the Mehler kernel

$$e^{-tH}(x,y) = \Big(\frac{\omega}{2\pi\sinh(2\omega t)}\Big)^{\frac{d}{2}} e^{-\frac{\omega}{4}\coth(\omega t)\|x-y\|^2 - \frac{\omega}{4}\tanh(\omega t)\|x+y\|^2}$$

- $\boldsymbol{c}$ takes values in unitisation
  $\mathcal{B} = \{b \in \mathcal{A}'' \;:\; b, [\mathcal{D}, b] \in \bigcap_{k\in\mathbb{N}} \mathrm{dom}(\mathcal{D}^k)\}$

- $\gamma{=}\pi_{\mathcal{D}}(\boldsymbol{c})$ fulfils $\gamma^2{=}1$ and $\gamma{=}\gamma^*$, but not $\gamma{=}1$ or $\gamma\mathcal{D}{=}-\mathcal{D}\gamma$.

  In the reconstruction theorem this is repaired by the dimension spectrum.

U(1)-Higgs model for commutative algebra

tensor $(\mathcal{A}_4, \mathcal{H}_4, \mathcal{D}_4)$ with $(\mathbb{C}\oplus\mathbb{C}, \mathbb{C}^2, M\sigma_1, \sigma_3)$ [Connes+Lott]

- $\mathcal{D} = \mathcal{D}_4 \otimes \sigma_3 + 1 \otimes \sigma_1 M = \begin{pmatrix} \mathcal{D}_4 & M \\ M & -\mathcal{D}_4 \end{pmatrix}$ $\quad \begin{pmatrix} f & 0 \\ 0 & g \end{pmatrix} \in \mathcal{A}$
- selfadjoint fluctuated Dirac operators $\mathcal{D}_A := \mathcal{D} + \sum_i a_i[\mathcal{D}, b_i]$, $a_i, b_i \in \mathcal{A} = \mathcal{A}_4 \oplus \mathcal{A}_4$, are of the form
  $$\mathcal{D}_A = \begin{pmatrix} \mathcal{D}_4 + \mathrm{i}A^\mu \otimes (b_\mu^\dagger - b_\mu) & \phi \otimes 1 \\ \overline{\phi} \otimes 1 & -(\mathcal{D}_4 + \mathrm{i}B^\mu \otimes (b_\mu^\dagger - b_\mu)) \end{pmatrix}$$
  for $A_\mu = \overline{A_\mu}, B_\mu = \overline{B_\mu}, \phi \in \mathcal{A}_4$
- $$\mathcal{D}_A^2 = \begin{pmatrix} (H + |\phi|^2) \otimes 1 + \omega \otimes \Sigma + F_A & D^\mu \phi \otimes (b_\mu^\dagger - b_\mu) \\ -\overline{D^\mu \phi} \otimes (b_\mu^\dagger - b_\mu) & (H + |\phi|^2) \otimes 1 + \omega \otimes \Sigma + F_B \end{pmatrix}$$
- $D_\mu \phi = \partial_\mu \phi + i(A_\mu - B_\mu)\phi$
  $$F_A = (-\{\partial^\mu, A_\mu\} - \mathrm{i}A^\mu A_\mu) \otimes 1 + \tfrac{1}{4} F_A^{\mu\nu} \otimes [b_\mu^\dagger - b_\mu, b_\nu^\dagger - b_\nu]$$

Spectral action principle

most general form of bosonic action is $S(\mathcal{D}_A) = Tr\big(\chi(\mathcal{D}_A^2)\big)$

- Laplace transf. + asympt. expansion $e^{-t\mathcal{D}_A^2} = \sum_{n=-dim/2}^{\infty} a_n(\mathcal{D}_A^2)t^n$

  leads to $S(\mathcal{D}_A) = \sum_{n=-dim/2}^{\infty} \chi_n \mathrm{Tr}(a_n(\mathcal{D}_A^2))$

  with $\chi_z = \frac{1}{\Gamma(-z)} \int_0^\infty ds\, s^{-z-1}\chi(s)$ for $z \notin \mathbb{N}$

  $\chi_k = (-1)^k \chi^{(k)}(0)$ for $k \in \mathbb{N}$
- a_n – Seeley coefficients, must be computed from scratch

Duhamel expansion: $\mathcal{D}_A^2 = \mathrm{H}_0 - V$

$$e^{-t(\mathrm{H}_0-V)} = e^{-t\mathrm{H}_0} - \int_0^t dt_1 \frac{d}{dt_1}\Big(e^{-(t-t_1)(\mathrm{H}_0-V)}e^{-t_1\mathrm{H}_0}\Big)$$

$$= e^{-t\mathrm{H}_0} + \int_0^t dt_1 \Big(e^{-(t-t_1)(\mathrm{H}_0-V)}Ve^{-t_1\mathrm{H}_0}\Big) \qquad \ldots\text{iteration}$$

Vacuum trace

Mehler kernel (in 4D)

$$e^{-t(H+\omega\Sigma)}(x,y) =$$

$$\frac{\omega^2(1-\tanh^2(\omega t))^2}{16\pi^2\tanh^2(\omega t)} e^{-t\omega\Sigma 1_2 - \frac{\omega}{4}\frac{\|x-y\|^2}{\tanh(\omega t)} - \frac{\omega}{4}\tanh(\omega t)\|x+y\|^2}$$

$$\mathrm{tr}(e^{-t\omega\Sigma}) = (2\cosh(\omega t))^d$$

$$\begin{aligned}\mathrm{Tr}(e^{-t(H+\omega\Sigma)\otimes 1_2}) &= 2\mathrm{tr}\Big(\int d^4x\,(e^{-t(\mathrm{H}+\omega\Sigma)})(x,x)\Big)\\ &= \frac{2}{\tanh^4(\omega t)} = 2(\omega t)^{-4} + \frac{8}{3}(\omega t)^{-2} + \frac{52}{45} + \mathcal{O}(t^2)\end{aligned}$$

- Spectral action is finite, in contrast to standard $\mathbb{R}^4$!
- expansion starts with t^{-4} $\Rightarrow$ corresponds to 8-dim. space

Vertices

only one-vertex contribution from
$V = \mathrm{diag}(-(A_\mu A^\mu + |\phi|^2) \otimes 1, -(B_\mu B^\mu + |\phi|^2) \otimes 1)$
(no tadpole, in contrast to Moyal where $\{\partial_\mu + \omega x_\mu, \pi(A^\mu)\}$ contributes)

$$\int_0^t dt_1 \,\mathrm{Tr}\big(e^{-(t-t_1)(H+\omega\Sigma)1_2} V e^{-t_1(H+\omega\Sigma)1_2}\big)$$
$$= \frac{\omega^2 t}{\pi^2 \tanh^2(\omega t)} \int d^4x \big(-2|\phi|^2 - A_\mu A^\mu - B_\mu B^\mu\big) e^{-\omega \tanh(\omega t)\|x\|^2}$$

- yields $-\mu^2\phi^2$ for Higgs potential and oscillator potential $+x^2\phi^2$
- A^2, B^2 eliminated by two vertices with $\{\partial^\mu, A_\mu\}, \{\partial^\mu, B_\mu\}$; these also give Yang-Mills with negative sign, exceeded by two curvature-vertices

The spectral action

$$S(\mathcal{D}_A) = \frac{2\chi_{-4}}{\omega^4} + \frac{8\chi_{-2}}{3\omega^2} + \frac{52\chi_0}{45} + \frac{\chi_0}{\pi^2}\int d^4x\,\Big\{ D^\mu\phi\overline{D_\mu\phi} + \frac{5}{12}\big(F^A_{\mu\nu}F_A^{\mu\nu} + F^B_{\mu\nu}F_B^{\mu\nu}\big) + \Big((|\phi|^2)^2 - \frac{2\chi_{-1}}{\chi_0}|\phi|^2 + 2\omega^2\|x\|^2|\phi|^2\Big)\Big\} + \mathcal{O}(\chi_1)$$

- spectral action is finite
- only difference in field equations to infinite volume is additional harmonic oscillator potential for the Higgs
- Yang-Mills is unchanged (in contrast to Moyal)
- vacuum is at $A_\mu = B_\mu = 0$ and (after gauge transformation) $\phi \in \mathbb{R}$, rotationally invariant

Field equations

rescaling $r = 2^{\frac{1}{4}}\sqrt{\omega}\|x\|$, $\phi = \frac{\pi}{\sqrt{2}\chi_0}\varphi$, $\mu^2 = \frac{\chi_{-1}}{\sqrt{8}\,\omega\chi_0}$, $\lambda = \frac{\pi^2}{\sqrt{2}\,\omega\chi_0}$

$$-\varphi''(r) - \frac{3}{r}\varphi'(r) + (r^2 - 4\mu^2)\varphi(r) = -\lambda\varphi^3(r)$$

- inhomogeneous confluent hypergeometric diff. eq.
- expand $\varphi = \frac{2}{\sqrt{\lambda}}\sum_{n=0}^{\infty} c_n\varphi_n$ in terms of eigenfunctions of 4-dim. harmonic oscillator:

$$\varphi_n := e^{-\frac{r^2}{2}} L_n^1(r^2) \qquad \Big(-\frac{d^2}{dr^2} - \frac{3}{r}\frac{d}{dr} + r^2\Big)\varphi_n = 4(n+1)\varphi_n$$

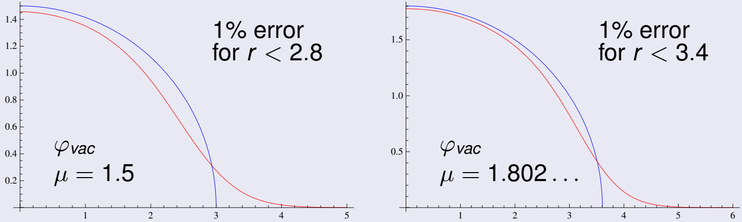
- yields

$$c_n(\mu^2-n-1) = \sum_{k,l,m=0}^{\infty} \frac{c_k c_l c_m}{k!l!m!}\Big(\frac{d^k}{dw^k}\frac{d^l}{dy^l}\frac{d^m}{dz^m}\frac{(1-wy-wz-yz+2wyz)^n}{(2-w-y-z+wyz)^{n+2}}\Big)_{w=y=z=0}$$

- numerical solution: cut-off at N

$-\varphi''(r) - \frac{3}{r}\varphi'(r) + (r^2 - 4\mu^2)\varphi(r) = -\lambda\varphi^3(r)$

$\varphi(r)$ in units of $\frac{2}{\sqrt{\lambda}}$, cutoff at $N = 10$

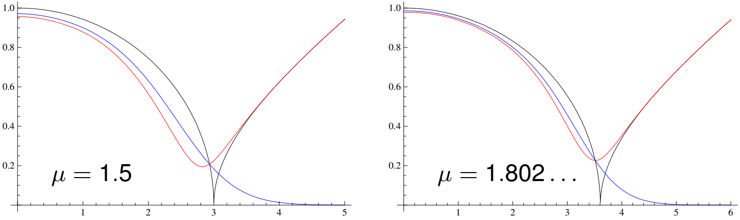


- vacuum solution $\varphi_{vac}(r)$ is smaller than asymptotic curve $\sqrt{\mu^2 - \frac{1}{4}r^2}$ due to its negative curvature and approaches it for $\mu \to \infty$
- $\varphi_{vac} = 0$ for $r \geq 2\mu$, transition is smoothly
- vacuum solution is integrable

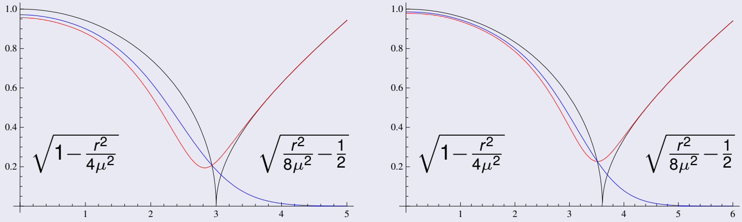
Masses

- scale of bare fermion and gauge field masses given by vacuum expectation value $\sqrt{\frac{4\mu^2}{\lambda}}\frac{\varphi_{vac}}{\mu} = \sqrt{\frac{2\chi_{-1}}{\pi^2}}\frac{\varphi_{vac}}{\mu}$
- bare Higgs mass given by difference function

$$\sqrt{\sqrt{2}\omega((r^2-4\mu^2)+12\mu^2\frac{\varphi_{vac}^2}{\mu^2})} = \sqrt{\frac{4\chi_{-1}}{\chi_0}}\frac{\sqrt{\frac{3}{2}\varphi_{vac}^2-\frac{1}{2}\mu^2+\frac{1}{8}r^2}}{\mu}$$



Smooth transition between two phases



1. *Spontaneously broken phase* $\omega^2\|x\|^2 < \frac{\chi_{-1}}{\chi_0}$
   fermions, gauge fields and Higgs are massive, with Higgs mass slightly smaller than NCG-prediction
2. *Unbroken phase* $\omega^2\|x\|^2 \geq \frac{\chi_{-1}}{\chi_0}$
   fermions + gauge fields massless, Higgs remains massive

Mass of gauge fields and fermions dissipates into cosmological constant!

The spectral action: noncommutative case

$$S(\mathcal{D}_A) = \frac{\theta^4\chi_{-4}}{8\Omega^4} + \frac{2\theta^2\chi_{-2}}{3\Omega^2} + \frac{52\chi_0}{45} + \frac{\chi_0}{2\pi^2(1+\Omega^2)^2}\int d^4x \,\Big\{ 2D_\mu\phi \star \overline{D_\mu\phi}$$
$$+\Big(\frac{(1-\Omega^2)^2}{2} - \frac{(1-\Omega^2)^4}{3(1+\Omega^2)^2}\Big)\big(F^A_{\mu\nu}\star F_A^{\mu\nu} + F^B_{\mu\nu}\star F_B^{\mu\nu}\big)$$
$$+\Big(\phi\star\bar\phi + \frac{4\Omega^2}{1+\Omega^2}X_A^\mu \star X_{A\mu} - \frac{\chi_{-1}}{\chi_0}\Big)^2$$
$$+\Big(\bar\phi\star\phi + \frac{4\Omega^2}{1+\Omega^2}X_B^\mu \star X_{B\mu} - \frac{\chi_{-1}}{\chi_0}\Big)^2$$
$$-2\Big(\frac{4\Omega^2}{1+\Omega^2}X_0^\mu \star X_{0\mu} - \frac{\chi_{-1}}{\chi_0}\Big)^2\Big\}(x) + \mathcal{O}(\chi_1)$$

$$\Theta = \begin{pmatrix} 0 & \theta & 0 & 0 \\ -\theta & 0 & 0 & 0 \\ 0 & 0 & 0 & \theta \\ 0 & 0 & -\theta & 0 \end{pmatrix}$$

$$\omega = \frac{2\Omega}{\theta}$$

deeper entanglement of gauge and Higgs fields

covariant coordinates $X_{A\mu}(x) = (\Theta^{-1})_{\mu\nu}x^\nu + A_\mu(x)$ appear with Higgs field ϕ in unified potential; vacuum is non-trivial!

potential cannot be restricted to Higgs part if distinction into discrete and continuous geometries no longer possible

The vacuum

vacuum field equations $(\phi^{vac} = \overline{\phi^{vac}}, \quad A_\mu^{vac} = B_\mu^{vac})$

$$\frac{1}{g^2}[X_{A\nu},[X_A^\mu,X_A^\nu]_\star]_\star + 2[\phi,[X_A^\mu,\phi]_\star]_\star = \frac{4\Omega^2}{1+\Omega^2}\Big\{\phi\star\phi + \frac{4\Omega^2}{1+\Omega^2}X_A^\nu\star X_{A\nu} - \mu^2, X_A^\mu\Big\}_\star$$

$$2[X_{A\nu},[\phi,X_A^\nu]_\star]_\star = \Big\{\phi\star\phi + \frac{4\Omega^2}{1+\Omega^2}X_A^\nu\star X_{A\nu} - \mu^2, \phi\Big\}_\star$$

(with $\frac{1}{4g^2} = \frac{(1-\Omega^2)^2}{2} - \frac{(1-\Omega^2)^4}{3(1+\Omega^2)^2} \qquad \mu^2 = \frac{\chi_{-1}}{\chi_0}$)

spirit of emerging geometry through phase transitions

- $\Omega = 0 \quad \Rightarrow$ solution: $\phi = \mu 1$, $\quad [X_\mu, X_\nu] = \begin{cases} \Theta_{\mu\nu} \\ 0 \end{cases}$

  $\Omega \neq 0$ gives some dynamical geometry
- analytical solution seems impossible

  $\Rightarrow$ need numerical simulations

Matrix representation

- matrix base in two dimensions (radial coordinates):

$$f_{mn}(\rho,\varphi) = 2(-1)^m \sqrt{\frac{m!}{n!}} e^{i\varphi(n-m)} \Big(\sqrt{\tfrac{2}{\theta}}\rho\Big)^{n-m} e^{-\frac{\rho^2}{\theta}} L_m^{n-m}\big(\tfrac{2}{\theta}\rho^2\big)$$

satisfies:
$$(f_{mn} \star f_{kl})(x) = \delta_{nk} f_{ml}(x)$$
$$\int d^2x\, f_{mn}(x) = \sqrt{\det(2\pi\Theta)}\,\delta_{mn}$$

- expansion
$$X^\mu(x) = \sum_{m_i,n_i\in\mathbb{N}} X^\mu_{\substack{m_1\, n_1\\ m_2\, n_2}} f_{m_1 n_1}(x^1,x^2) f_{m_2 n_2}(x^3,x^4)$$
$$\phi(x) = \sum_{m_i,n_i\in\mathbb{N}} \phi_{\substack{m_1\, n_1\\ m_2\, n_2}} f_{m_1 n_1}(x^1,x^2) f_{m_2 n_2}(x^3,x^4)$$

- We can forget the Moyal product:
  Action and field equations correspond to 5-matrix model

Can we use knowledge from lattice gauge theory?

- Action is different, but adaptation seems possible
- Main difficulty is the asymptotics (and its realisation by finite matrices):
  $\phi, A = X - \Theta^{-1} \cdot x$ vanish rapidly at ∞

$$(x_1 + \mathrm{i}x_2)_{\substack{m_1 \; n_1 \\ m_2 \; n_2}} = \sqrt{\theta n_1}\,\delta_{m_1,n_1-1}\delta_{m_2,n_2}$$
$$(x_1 - \mathrm{i}x_2)_{\substack{m_1 \; n_1 \\ m_2 \; n_2}} = \sqrt{\theta m_1}\,\delta_{m_1-1,n_1}\delta_{m_2,n_2}$$

- Choose X_μ or A_μ as variables?
- Minima of action vs. solutions of field equations?
- Dependence on μ, g, λ, Ω? Phase transitions?