A spectral triple for harmonic oscillator Moyal space

(work in progress)

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Introduction

An interesting quantum field theoretical model

action functional for real-valued field ϕ on $\mathbb{R}^4$:

$$S[\phi] = \int d^4x \Big(\frac{1}{2}\phi \star (-\Delta + \Omega^2 \tilde{x}^2 + \mu^2)\phi + \frac{\lambda}{4}\phi \star \phi \star \phi \star \phi\Big)(x)$$

Moyal product $\star$ defined by Θ and $\tilde{x} := 2\Theta^{-1} \cdot x$
parameters: $\mu^2, \lambda \in \mathbb{R}_+$, $\Omega \in [0,1]$, redef'n $\phi \mapsto \mathcal{Z}\phi$, $\mathcal{Z} \in \mathbb{R}_+$

- renormalisable as formal power series in λ
  [H.Grosse+R.W., 2004]
  means: well-defined perturbative quantum field theory
- β-function vanishes to all orders in λ for $\Omega = 1$
  [M.Disertori, R.Gurau, J.Magnen + V.Rivasseau, 2006]
  means: model is believed to exist non-perturbatively

Noncommutative spin manifolds = spectral triples

- suggested 1996 by A. Connes (axioms of spectral triples) using insight from the standard model of particle physics
- commutative spectral triples are indeed compact spin manifolds [A. Rennie, J. Várilly; 2006]
- equivalence classes of spectral triples describe Yang-Mills theory (inner automorphisms) and gravity (outer automorphisms)
- spectral action principle [A. Chamseddine, A. Connes; 1996] provides action functional = starting point for QFT

  some divergent one-loop Feynman graphs are renormalisable by construction

Spectral triples and Yang-Mills theory for Moyal space

- Moyal spaces are non-compact spectral triples [V.Gayral, J.M.Gracia-Bondía, B.Iochum, T.Schücker, J.Várilly; 2003]
- spectral action for Moyal space [V.Gayral, B.Iochum; 2004] is Yang-Mills with Moyal product
  problem: spacial regularisation required

for renormalisation we need oscillator potential...

Is there a spectral triple associated with a Dirac operator with oscillator spectrum?

- answer is no if all noncommutative dimensions coincide
- examples (Podleś quantum sphere, standard model) require different metric and KO dimensions

answer is yes for independent metric and KO dimensions

Axioms for non-compact spectral triples

Definition (non-compact spectral triple $(\mathcal{A}, \mathcal{B}, \mathcal{H}, \mathcal{D}, J, \chi, c)$)

- a non-unital algebra $\mathcal{A}$ acting faithfully on a Hilbert space $\mathcal{H}$ (via a representation π)
- a preferred unitisation $\mathcal{B}$ of $\mathcal{A}$ acting on $\mathcal{H}$, too
- a densely defined selfadjoint unbounded operator $\mathcal{D}$ on $\mathcal{H}$ s.t. $[\mathcal{D}, \pi(b)]$ extends to a bounded operator $\forall\, b \in \mathcal{B}$
- in the even case: a selfadjoint operator χ on $\mathcal{H}$ satisfying $\chi^2 = 1, \qquad \chi\pi(b) = \pi(b)\chi \;\; \forall\, b \in \mathcal{B}, \qquad \mathcal{D}\chi = -\chi\mathcal{D}$
- in the real case: an antiunitary operator J on $\mathcal{H}$ which satisfies conditions 4 and 5 below

subject to the following conditions 0–6.

The conditions

0. **Compactness.**
The operator $\pi(a)(\mathcal{D}-\lambda)^{-1}$ is compact for all $a \in \mathcal{A}$ and all λ not contained in the spectrum of $\mathcal{D}$.

1. **Regularity and dimension spectrum.**
Both $\pi(b)$ and $[\mathcal{D}, \pi(b)]$ belong for any $b \in \mathcal{B}$ to $\bigcap_{n=1}^{\infty} \mathrm{dom}(\delta^n)$, with $\delta T := [\langle\mathcal{D}\rangle, T]$ and $\langle\mathcal{D}\rangle := (\mathcal{D}^2+1)^{\frac{1}{2}}$.

   For any element α of the algebra $\Psi_0(\mathcal{A})$ generated by $\delta^n\pi(a)$ and $\delta^n[\mathcal{D}, \pi(a)]$, with $a \in \mathcal{A}$, the function $\zeta_\alpha(z) := \mathrm{Tr}(\alpha\langle\mathcal{D}\rangle^{-z})$ extends holomorphically to $\mathbb{C} \setminus \mathrm{Sd}$ for some discrete set $\mathrm{Sd} \subset \mathbb{C}$ (dimension spectrum).

2. **Metric dimension.**
For the metric dimension $d := \sup\{r \in \mathbb{R} \cap \mathrm{Sd}\}$, the Dixmier trace $\mathrm{Tr}_\omega(\pi(a)\langle\mathcal{D}\rangle^{-d})$ is finite for any $a \in \mathcal{A}$ and positive for positive elements of $\mathcal{A}$.

3. **Finiteness.**
The algebras $\mathcal{A}$ and $\mathcal{B}$ are pre-$C^\star$-algebras.

The space $\mathcal{H}^\infty := \bigcap_{k=0}^{\infty} \mathcal{H}^k$, with $\mathcal{H}^k := \mathrm{dom}(\mathcal{D}^k)$ completed with norm $\|\xi\|_k^2 := \|\xi\|^2 + \|\mathcal{D}^k\xi\|^2$, is a finitely generated projective $\mathcal{A}$-module $p\mathcal{A}^m$, for some projector $p = p^2 = p^* \in M_m(\mathcal{B})$.

The scalar product on $\mathcal{H}^\infty$ is recovered from the hermitian structure $\langle\ ,\ \rangle_\mathcal{A} : \mathcal{H}^\infty \times \mathcal{H}^\infty \to \mathcal{A}$ and the Dixmier trace:

$$(\xi,\eta) = \mathrm{Tr}_\omega\Big(\langle\xi,\eta\rangle_\mathcal{A}\,\langle\mathcal{D}\rangle^{-d}\Big)\,, \qquad \xi,\eta \in \mathcal{H}^\infty$$

4. **Reality.**
   J defines a real structure of KO-dimension $k \in \mathbb{Z}_8$, i.e.

$$J^2 = \varepsilon\,, \qquad J\mathcal{D} = \varepsilon'\mathcal{D}J\,, \qquad J\chi = \varepsilon''\chi J \quad \text{(even case)}$$

k mod 8	0	1	2	3	4	5	6	7
ε	1	1	−1	−1	−1	−1	1	1
ε'	1	−1	1	1	1	−1	1	1
ε''	1		−1		1		−1	

   The action π of $\mathcal{B}$ on $\mathcal{H}$ satisfies $[\pi(b_1), \pi^o(b_2)] = 0$ for all $b_1, b_2 \in \mathcal{B}$, where $\pi^o(b_2) = J\pi(b_2^*)J^{-1}$.

5. **First order.**
   $[[\mathcal{D}, \pi(b_1)], \pi^o(b_2)] = 0$ for all $b_1, b_2 \in \mathcal{B}$.

6. **Orientability.**
$\exists$ Hochschild d-cycle $\boldsymbol{c}$ on $\mathcal{B}$ with values in $\mathcal{B} \otimes \mathcal{B}^o$,

The representation $\pi(\boldsymbol{c})$ defined by

$$\pi((b_{-1} \otimes b_0) \otimes b_1 \otimes \cdots \otimes b_d) \\ := J\pi(b^*_{-1})J^{-1}\pi(b_0)[\mathcal{D}, \pi(b_1)] \cdots [\mathcal{D}, \pi(b_d)]$$

satisfies $\pi(\boldsymbol{c})^2 = 1$.

$\pi(\boldsymbol{c})$ defines the volume form on $\mathcal{A}$, i.e.

$$\phi_{\pi(\boldsymbol{c})}(a_0, \dots, a_d) \\ = \mathrm{Tr}_\omega\big(\pi(\boldsymbol{c}_4)\pi(a_0)[\mathcal{D}, \pi(a_1)] \cdots [\mathcal{D}, \pi(a_d)]\langle\mathcal{D}\rangle^{-d}\big)$$

provides a non-vanishing Hochschild d-cocylcle ϕ_d on $\mathcal{A}$.

The Dirac operator for the harmonic oscillator

Hamiltonian of one-dimensional harmonic oscillator:
$H = -\frac{d^2}{dx^2} + \omega^2 x^2$ on $L^2(\mathbb{R})$

- What is its spectral dimension d?
  (roughly: n^{th} eigenvalue of $|\mathcal{D}|^{-d}$ of order n^{-1})
  H^{-1} has eigenvalues $\mu_n = \frac{1}{\omega(2n+1)}$, $n \in \mathbb{N}$
- H generalises Laplacian $-\Delta$, Dirac operator is a square root, so eigenvalues of $H^{-1} = \mathcal{D}^{-2}$ are of order $\mathcal{O}(\frac{1}{n})$

suggests: $\mathcal{D} \sim H^{\frac{1}{2}}$ seems to be of spectral dimension two!

Construct $\mathcal{D}$ in two Clifford dimensions

$$\mathcal{D}_1 = i\sigma_1 \frac{d}{dx} + \sigma_2 \omega x$$
$$= \begin{pmatrix} 0 & i(\frac{d}{dx} + \omega x) \\ i(\frac{d}{dx} - \omega x) & 0 \end{pmatrix} = \begin{pmatrix} 0 & i\sqrt{2\omega}a \\ -i\sqrt{2\omega}a^\dagger & 0 \end{pmatrix}$$

$[a, a^\dagger] = 1$

Generalisation to d-dimensional harmonic oscillator

d bosonic and fermionic creation and annihilation operators:

1. $[a_\mu, a_\nu] = [a_\mu^\dagger, a_\nu^\dagger] = 0 \qquad [a_\mu, a_\nu^\dagger] = \delta_{\mu\nu} \qquad \mu,\nu=1,\dots,d$
   $\{b_\mu, b_\nu\} = \{b_\mu^\dagger, b_\nu^\dagger\} = 0 \qquad \{b_\mu, b_\nu^\dagger\} = \delta_{\mu\nu}$
2. $\mathcal{D}_d = \sqrt{2}\omega \sum_{\mu=1}^d (ia_\mu \otimes b_\mu^\dagger - ia_\mu^\dagger \otimes b_\mu)$ (*supersymmetry*)
   $= i\frac{\partial}{\partial x_\mu} \otimes (b_\mu + b_\mu^\dagger) + \omega x^\mu \otimes (ib_\mu^\dagger - ib_\mu)$
   $= i\Gamma_\mu \frac{\partial}{\partial x_\mu} + \omega \Gamma_{d+\mu} x^\mu \qquad \mathrm{Cliff}(\mathbb{C}^{2d})$
3. $\mathcal{H}_d = L^2(\mathbb{R}^d) \otimes \bigwedge(\mathbb{C}^d)$ (bosonic $\otimes$ fermionic Hilbert spaces)
   $\chi_d = 1 \otimes \prod_{\mu=1}^d (b_\mu^\dagger b_\mu - b_\mu b_\mu^\dagger) = (-i)^d (-1)^{\frac{d(d+1)}{2}} \Gamma_1 \cdots \Gamma_{2d}$
4. $\mathcal{D}_n^2 = \omega \sum_{\mu=1}^d (\{a_\mu, a_\mu^\dagger\} \otimes 1 + 1 \otimes [b_\mu^\dagger, b_\mu]) = H_d \otimes 1 + \omega \otimes \Sigma_d$
   $H_d = -\frac{\partial^2}{\partial x_\mu \partial x^\mu} + \omega^2 x_\mu x^\mu \qquad \Sigma_d = \sum_{\mu=1}^d [b_\mu^\dagger, b_\mu]$

Spectral data for harmonic oscillator Moyal space

1. $\mathcal{D}_4 = \big(i\Gamma^\mu\partial_\mu + \Omega\Gamma^{\mu+4}\tilde{x}_\mu\big)$ $\mu = 1,\dots,4;$ $\tilde{x}_\mu = 2(\Theta^{-1})_{\mu\nu}x^\nu$

   $\mathcal{D}_4^2 = (-\Delta + \Omega^2\|\tilde{x}\|^2)\,1 + \tilde{\Omega}\Sigma_4$ $(\tilde{\Omega} = \frac{2\Omega}{\theta};\ \ \Theta = \theta S;\ \ S^2 = -1)$

2. Hilbert space $\mathcal{H}_4 = L^2(\mathbb{R}^4) \otimes \mathbb{C}^{16}$

3. algebra $\mathcal{A}_4 = \mathcal{S}(\mathbb{R}^4)$ with

   $$(f \star g)(x) = \int d^4y \frac{d^4k}{(2\pi)^4} f(x + \tfrac{1}{2}\Theta k)g(x+y)e^{iky}$$

   $\mathcal{A}_4$ acts on $\mathcal{H}_4$ componentwise by $\pi(f)\psi = L_\star(f)\psi = f \star \psi$

   $\mathcal{B}_4 = \{$ smooth bounded functions on $\mathbb{R}^4$ with all derivatives bounded $\}$ $\star$ extends to $\mathcal{B}_4$

4. $\chi_4 = \Gamma_9$ $J_4\psi = \chi_4\bar{\psi}$

   orientation form is $\pi(c_4) = \frac{1}{(1+\Omega^2)^2}\prod_{\mu=1}^{4}(\Gamma^\mu + \Omega\Gamma^{\mu+4})$, not Γ_9

Verification of the axioms

$L_\star(b)$ and $[\mathcal{D}, L_\star(b)] = i(\Gamma^\mu + \Omega\Gamma^{\mu+4})L_\star(\partial_\mu b)$ are bounded

0. *Compactness*
   $(\mathcal{D}-\lambda)^{-1}$ itself is a compact operator for $\lambda \notin sp(\mathcal{D})$
   (for spectral action one should probably require $(\mathcal{D}-\lambda)^{-1}$ compact)

1. *Regularity*
   - define $\langle D\rangle = (\mathcal{D}^2+1)^{\frac{1}{2}}$, e.g. $A = \frac{1}{\pi}\int_0^\infty \frac{d\lambda}{\sqrt{\lambda}} \frac{A^2}{A^2+\lambda}$
   - $L_\star(b)$ and $[\mathcal{D}, L_\star(b)]$ belong for all n to the domain of δ^n:

   key: $[\langle\mathcal{D}\rangle^2, L^\star(b)] = -L_\star(\Delta b) - 2L_\star(\partial^\mu b)\nabla_\mu^{(\Omega)}$
   $[\nabla_\mu^{(\Omega)}, L_\star(b)] = L_\star(\partial_\mu b)$ $\qquad [\langle\mathcal{D}\rangle^2, \nabla_\mu^{(\Omega)}] = c_{\mu\nu}\nabla^{(\Omega')\nu}$

   $$\delta^n L_\star(b) = \frac{1}{\pi^n}\sum_{k=0}^n \binom{n}{k}(-1)^k \times \int_0^\infty \Big(\prod_{i=1}^k \frac{1}{\langle\mathcal{D}\rangle^2+\lambda_i}\Big) P_b^{n+k}(\nabla)\langle\mathcal{D}\rangle^{-n} \prod_{j=1}^n \frac{d\lambda_j\sqrt{\lambda_j}\langle\mathcal{D}\rangle}{(\langle\mathcal{D}\rangle^2+\lambda_j)^2}$$

   for $P_b^m(\nabla)$ – operator polynomial of order in m in $\nabla^{(\Omega)}$

1. dimension spectrum
   - $\Psi_0(\mathcal{A})$ = algebra generated by $\delta^n L_\star(a)$ and $\delta^n[\mathcal{D}, L_\star(a)]$
     $\Psi_k(\mathcal{A}) := \Psi_0(\mathcal{A})\langle\mathcal{D}\rangle^k$ pseudo-differential op's of order $\leq k$
   - study zeta-function $\zeta_b(z) := Tr(\alpha\langle\mathcal{D}\rangle^{-z})$ for $\alpha \in \Psi_0(\mathcal{A})$

     attention: $Tr(L_\star(f)) = \int_{\mathbb{R}^4} dx\, f(x)$ finite for $f \in \mathcal{A}$, but $L_\star(f)$ is not trace-class! However, $\langle\mathcal{D}\rangle^{-z}$ is trace class for $\mathrm{Re}(z) > 2d$
   - integral kernel for resolvent
     $$\Big(\frac{1}{\langle\mathcal{D}\rangle^2 + \lambda_i}\Big)(x,y) = \int_0^\infty dt\, e^{-t-t\lambda_i - t\tilde{\Omega}\Sigma_d} \cdot e^{-tH_d}(x,y)$$
     with Mehler kernel
     $$e^{-tH_d}(x,y) = \Big(\frac{\tilde{\Omega}}{2\pi\sinh(2\tilde{\Omega}t)}\Big)^{\frac{d}{2}} e^{-\frac{\tilde{\Omega}}{4}\coth(\tilde{\Omega}t)\|x-y\|^2 - \frac{\tilde{\Omega}}{4}\tanh(\tilde{\Omega}t)\|x+y\|^2}$$
     (solves $(\frac{d}{dt} + H_{d,x})e^{-tH_d}(x,y) = 0$, $\lim_{t\to 0} e^{-tH_d}(x,y) = \delta(x-y)$)
   - integral kernel for Moyal multiplication $(d=4)$
     $$(L_\star(a))(x,y) = \frac{1}{\pi^4\theta^4}\int d^4w\, a(w)\, \mathrm{e}^{i\langle x-y, \Theta^{-1}(x+y)\rangle + 2i\langle w, \Theta^{-1}(x-y)\rangle}$$

1. dimension spectrum (continued)

$$(\delta^n L_\star(a))(x,y)$$
$$= \frac{1}{(2\sqrt{\pi})^n} \int_0^\infty \prod_{i=1}^n \frac{ds_i dt_i}{(s_i+t_i)^{\frac{3}{2}}} e^{-\sum_{j=1}^n (s_j+t_j)(1+\tilde{\Omega}\Sigma_4)}$$
$$\times \int d^4x'\, d^4y'\; e^{-\sum_{j=1}^n s_j H_4}(x,x')\big((\mathrm{ad}(H_4))^n(L_\star(a))\big)(x',y') e^{-\sum_{j=1}^n t_j H_4}(y',y)$$

with

$$\big((\mathrm{ad}(H_4))^n(L_\star(a))\big)(x,y)$$
$$= \sum \int d^4w\, a^{\nu_1\dots\nu_r}(w)(\tilde{x}+\tilde{y})_{\nu_1}\cdots(\tilde{x}+\tilde{y})_{\nu_r}\, \mathrm{e}^{i\langle x-y,\Theta^{-1}(x+y)\rangle+2i\langle w,\Theta^{-1}(x-y)\rangle}$$

$a^{\nu_1\dots\nu_r} \in \mathcal{A}$ and $r \leq n$

1. dimension spectrum (continued)

$$\mathrm{Tr}\big((\delta^{n_1} L_\star(a_1))\cdots(\delta^{n_p} L_\star(a_p))\langle\mathcal{D}\rangle^{-z}\big)$$
$$= \frac{1}{\Gamma(\frac{z}{2})}\int_0^\infty dt_0\; t_0^{\frac{z}{2}-1}\int d^4x_0\, d^4x_1 \dots d^4x_p$$
$$\times\,(\delta^{n_1} L_\star(a_1))(x_1,x_2)\cdots(\delta^{n_p} L_\star(a_p))(x_p,x_0)(e^{-t_0\mathcal{D}_4^2})(x_0,x_1)\;.$$

is concatenation of Mehler kernels and Moyal kernel polynomials

resembles one-loop Feynman graph, all integrals are Gaußian

1. dimension spectrum (continued)

- result for $p=1$: finite linear combination of integrals

$$I_q(z)=\frac{1}{\Gamma(\frac{z+n_1}{2})}\int_0^\infty dt\, t^{\frac{z+n_1+2q}{2}-1}e^{-t}$$
$$\times\int d^4w\, g_q(w)\Big(\frac{\tanh(\tilde{\Omega}t)}{t}\Big)^q(1+\tanh^2(\tilde{\Omega}t))^h \mathrm{e}^{-\frac{2\Omega\tanh(\tilde{\Omega}t)}{(1+\Omega^2)\theta}|w|^2}$$

for $q\in\{-[\frac{n_1}{2}]-2,-[\frac{n_1}{2}]-1,\dots,n_1-2\}$, $h\in\mathbb{Z}$ and $g_q\in\mathcal{A}$

Taylor expansion: $I_q(z)$ holomorphic for $z+n_1\notin\{2,4,\dots,-2q\}$

$-n_1-2q\leq 4 \quad\Rightarrow\quad \mathrm{Tr}(\delta^n L_\star(a)\langle D\rangle^{-z})$ holomorphic on $\mathbb{C}\setminus(4-\mathbb{N})$

- residues are local, i.e. $\int d^4w\, g(w)$
- general case $p\geq 2$: holomorphic extension to $\mathbb{C}\setminus(4-\mathbb{N})$

  residues are again local, i.e. $\int d^4w\,(g_1\star\cdots\star g_p)(w)$
- relates to locality of counterterms in QFT

  $n_i\in\{0,1\}\quad\Rightarrow\quad$ only finitely many residues

2. *Metric dimension.*
   - $d := \sup\{k \in \mathbb{R} \ : \ k \in Sd\} = 4$ (= oscillator dimension!)
   - Connes' trace theorem (generalised)

$$\mathrm{Tr}_\omega(L_\star(a)\langle\mathcal{D}\rangle^{-d}) = \lim_{s\to 1}(s-1)\mathrm{Tr}(L_\star(a)\langle\mathcal{D}\rangle^{-ds}) = \frac{1}{2\pi^2(1+\Omega^2)^2}\int d^4x\, a(x)$$

3. *Finiteness*
   - smooth spinors $\psi \in \mathcal{H}^\infty := \bigcap_{k\in\mathbb{N}} \mathrm{dom}(\mathcal{D}^k)$ are Schwartzian

     consequence: $\mathcal{H}^\infty = (\mathcal{A})^{16}$ is a free module
   - hermitian structure $\langle\ ,\ \rangle_{\mathcal{A}} : \mathcal{H}^\infty \times \mathcal{H}^\infty \to \mathcal{A}$
   - scalar product from hermitian structure and Dixmier trace: $(\xi,\eta)_{\mathcal{H}^\infty} = \mathrm{Tr}_\omega(\langle\xi,\eta\rangle_{\mathcal{A}}\,\langle\mathcal{D}\rangle^{-4})$

4. *Reality:* $J\psi = \Gamma_9\bar{\psi}$ defines real structure of *KO*-dimension 0, independent of *d*
5. *First order:* by construction
6. *Orientability*
   - unitisation $\mathcal{B}$ of $\mathcal{A}$ contains plane waves $u_\mu = e^{ix_\mu}$
   - Hochschild 4-cycle on $\mathcal{B}$
     $$\boldsymbol{c}_4 = \frac{1}{4!(1+\Omega^2)^2}\sum_{\sigma\in S_4}(-1)^\sigma \times\big(u_{\sigma(1)}\star u_{\sigma(2)}\star u_{\sigma(3)}\star u_{\sigma(4)}\big)^{-1_\star}\otimes u_{\sigma(1)}\otimes u_{\sigma(2)}\otimes u_{\sigma(3)}\otimes u_{\sigma(4)}$$
   - $\pi(\boldsymbol{c}_4) = \frac{1}{(1+\Omega^2)^2}(\Gamma_1+\Omega\Gamma_5)(\Gamma_2+\Omega\Gamma_6)(\Gamma_3+\Omega\Gamma_7)(\Gamma_4+\Omega\Gamma_8)$
   - Hochschild 4-cocycle on $\mathcal{A}$ is volume form $(a \equiv L_\star(a))$
     $$\phi_{\pi(\boldsymbol{c}_4)}(a_0,\dots,a_4) = \mathrm{Tr}_\omega\big(\pi(\boldsymbol{c}_4)a_0[\mathcal{D}_4,a_1]\cdots[\mathcal{D}_4,a_4]\langle\mathcal{D}\rangle^{-4}\big) = \frac{1}{2\pi^2}\int a_0\star da_1\wedge_\star da_2\wedge_\star da_3\wedge_\star da_4$$

U(1)-Higgs model

tensor $(\mathcal{A}_4, \mathcal{H}_4, \mathcal{D}_4, \chi_4)$ with $(\mathbb{C} \oplus \mathbb{C}, \mathbb{C}^2, M\sigma_1)$ [Connes+Lott]

- $\mathcal{D} = \mathcal{D}_4 \otimes 1_2 + \Gamma_9 \otimes \sigma_1 M = \begin{pmatrix} \mathcal{D}_4 & M\Gamma_9 \\ M\Gamma_9 & \mathcal{D}_4 \end{pmatrix}$
- selfadjoint fluctuated Dirac operators $\mathcal{D}_A := \mathcal{D} + \sum_i a_i[\mathcal{D}, b_i]$, $a_i, b_i \in \mathcal{A} = \mathcal{A}_4 \oplus \mathcal{A}_4$ are of the form
  $$\mathcal{D}_A = \begin{pmatrix} \mathcal{D}_4 + (\Gamma^\mu + \Omega\Gamma^{\mu+4}) L_\star(A_\mu) & \Gamma_9 L_\star(\phi) \\ \Gamma_9 L_\star(\bar{\phi}) & \mathcal{D}_4 + (\Gamma^\mu + \Omega\Gamma^{\mu+4}) L_\star(B_\mu) \end{pmatrix}$$
  for $A_\mu = \overline{A_\mu}, B_\mu = \overline{B_\mu}, \phi \in \mathcal{A}_4$
- $$\mathcal{D}_A^2 = \begin{pmatrix} (H_4 + L_\star(\phi \star \bar{\phi}))1 + \Sigma_4 + F_A & i(\Gamma^\mu + \Omega\Gamma^{\mu+4})\Gamma_9 L_\star(D_\mu\phi) \\ i(\Gamma^\mu + \Omega\Gamma^{\mu+4})\Gamma_9 L_\star(\overline{D_\mu\phi}) & (H_4 + L_\star(\bar{\phi} \star \phi))1 + \Sigma_4 + F_B \end{pmatrix}$$
- $$D_\mu\phi = \partial_\mu\phi - iA \star \phi + i\phi \star B$$
  $$F_A = \{L_\star(A^\mu), i\partial_\mu + \Omega^2 M_\bullet(\tilde{x}_\mu)\} + (1+\Omega^2) L_\star(A_\mu \star A^\mu) + i\big(\tfrac{1}{4}[\Gamma^\mu, \Gamma^\nu] + \tfrac{1}{4}\Omega^2[\Gamma^{\mu+4}, \Gamma^{\nu+4}] + \Omega\Gamma^\mu\Gamma^{\nu+4}\big) L_\star(F^A_{\mu\nu})$$

Spectral action principle

most general form of bosonic action is $S(\mathcal{D}_A) = Tr\big(\chi(\mathcal{D}_A^2)\big)$

- Laplace transf. + asympt. expansion $e^{-t\mathcal{D}_A^2} = \sum_{n=-d/2}^{\infty} a_n(\mathcal{D}_A^2) t^n$ (d = dimension)

  lead to $S(\mathcal{D}_A) = \sum_{n=-d/2}^{\infty} \chi_n \,\mathrm{Tr}(a_n(\mathcal{D}_A^2))$

  with $\chi_z = \frac{1}{\Gamma(-z)} \int_0^\infty ds\, s^{-z-1} \chi(s)$ for $z \notin \mathbb{N}$

  $\chi_k = (-1)^k \chi^{(k)}(0)$ for $k \in \mathbb{N}$
- a_n – Seeley coefficients, must be computed from scratch

Duhamel expansion: $\mathcal{D}_A^2 = \mathrm{H}_0 - V$

$$e^{-t(\mathrm{H}_0 - V)} = e^{-t\mathrm{H}_0} - \int_0^t dt_1 \frac{d}{dt_1}\Big(e^{-(t-t_1)(\mathrm{H}_0-V)} e^{-t_1\mathrm{H}_0}\Big)$$

$$= e^{-t\mathrm{H}_0} + \int_0^t dt_1 \Big(e^{-(t-t_1)(\mathrm{H}_0-V)} V e^{-t_1\mathrm{H}_0}\Big) \qquad \ldots\text{iteration}$$

Position space kernels

Mehler kernel

$$e^{-\mathrm{H}_0 t}(x,y) = \frac{\tilde{\Omega}^2(1-\tanh^2(\tilde{\Omega}t))^2}{16\pi^2\tanh^2(\tilde{\Omega}t)} e^{-t\tilde{\Omega}\Sigma_4 1_2 - \frac{\tilde{\Omega}}{4}\frac{|x-y|^2}{\tanh(\tilde{\Omega}t)} - \frac{\tilde{\Omega}}{4}\tanh(\tilde{\Omega}t)|x+y|^2}$$

(with $\tilde{\Omega} = \frac{2\Omega}{\theta}$ $\quad \Theta = \theta S \quad S^2 = -1$)

vacuum trace

$$\begin{aligned}\mathrm{Tr}(e^{-t\mathrm{H}_0}) &= \mathrm{tr}_{c\ell}\int d^4x\,(e^{-t\mathrm{H}_0})(x,x) = \frac{1}{8\sinh^4(\tilde{\Omega}t)}\mathrm{tr}_{c\ell}(e^{-t\tilde{\Omega}\Sigma_4}) \\ &= \frac{2}{\tanh^4(\tilde{\Omega}t)} = 2(\tilde{\Omega}t)^{-4} + \frac{8}{3}(\tilde{\Omega}t)^{-2} + \frac{52}{45} + \mathcal{O}(t^2)\end{aligned}$$

- expansion starts with t^{-4} $\Rightarrow$ corresponds to 8-dim. space
- spectral action is finite, in contrast to pure Moyal at $\Omega = 0$
- view spacial regularisation as part of geometry

Vertex kernels (with $x \wedge y = x^\mu (\Theta^{-1})_{\mu\nu} y^\nu$)

$$(L_\star(f))(x,y) = \int \frac{d^4z}{\pi^4\theta^4}\, f(z)\, \mathrm{e}^{2i(x\wedge y + y\wedge z + z\wedge x)}$$

$$\big\{L_\star(A^\mu), i\partial_\mu + \Omega^2 M_\bullet(\tilde{x}_\mu)\big\}(x,y)$$
$$= \int \frac{d^4z}{\pi^4\theta^4}\, \big(2\tilde{z}^\mu - (1-\Omega^2)(\tilde{x}^\mu + \tilde{y}^\mu)\big) A_\mu(z)\, \mathrm{e}^{2i(x\wedge y + y\wedge z + z\wedge x)}$$

- one-vertex trace
$$\int_0^t dt_1 \,\mathrm{Tr}\big(e^{-(t-t_1)H_0} L_\star(f) e^{-t_1 H_0}\big)$$
$$= \frac{\tilde{\Omega}^2 t}{16\pi^2(1+\Omega^2)^2 \sinh^2(\tilde{\Omega}t)} \int d^4z\, f(z)\, e^{-\frac{\tilde{\Omega}\tanh(\tilde{\Omega}t)}{1+\Omega^2}|z|^2}$$
- order is t^{-1} as in 4D-standard model
opposite sign of $t^{-1}, t^0 \Rightarrow$ spontaneous symmetry breaking
- in general: Gaußian integrations using determinant and inverse of $U \otimes 1_2 + V \otimes \sigma_2$ [R. Gurau+V. Rivasseau]

The spectral action

$$S(\mathcal{D}_A)=\frac{\theta^4\chi_{-4}}{8\Omega^4}+\frac{2\theta^2\chi_{-2}}{3\Omega^2}+\frac{52\chi_0}{45}+\frac{\chi_0}{2\pi^2(1+\Omega^2)^2}\int d^4z\,\Big\{2D_\mu\phi\star\overline{D_\mu\phi}$$
$$+\Big(\frac{(1-\Omega^2)^2}{2}-\frac{(1-\Omega^2)^4}{3(1+\Omega^2)^2}\Big)\big(F^A_{\mu\nu}\star F_A^{\mu\nu}+F^B_{\mu\nu}\star F_B^{\mu\nu}\big)$$
$$+\Big(\phi\star\bar\phi+\frac{4\Omega^2}{1+\Omega^2}\tilde{X}^\mu_A\star\tilde{X}_{A\mu}-\frac{\chi_{-1}}{\chi_0}\Big)^2$$
$$+\Big(\bar\phi\star\phi+\frac{4\Omega^2}{1+\Omega^2}\tilde{X}^\mu_B\star\tilde{X}_{B\mu}-\frac{\chi_{-1}}{\chi_0}\Big)^2$$
$$-2\Big(\frac{4\Omega^2}{1+\Omega^2}\tilde{X}^\mu_0\star\tilde{X}_{0\mu}-\frac{\chi_{-1}}{\chi_0}\Big)^2\Big\}(z)+\mathcal{O}(\chi_1)$$

deeper entanglement of gauge and Higgs fields

covariant coordinates $\tilde{X}_{A\mu}(z)=(\Theta^{-1})_{\mu\nu}z^\nu+A_\mu(z)$ appear with Higgs field ϕ in unified potential

potential cannot be restricted to Higgs part if distinction into discrete and continuous geometries no longer possible

The spectral action

$$
\begin{aligned}
S(\mathcal{D}_A) = {} & \frac{\theta^4\chi_{-4}}{8\Omega^4} + \frac{2\theta^2\chi_{-2}}{3\Omega^2} + \frac{52\chi_0}{45} + \frac{\chi_0}{2\pi^2(1+\Omega^2)^2}\int d^4z\,\Big\{2D_\mu\phi\star\overline{D_\mu\phi} \\
& + \Big(\frac{(1-\Omega^2)^2}{2} - \frac{(1-\Omega^2)^4}{3(1+\Omega^2)^2}\Big)\big(F^A_{\mu\nu}\star F_A^{\mu\nu} + F^B_{\mu\nu}\star F_B^{\mu\nu}\big) \\
& + \Big(\phi\star\bar\phi + \frac{4\Omega^2}{1+\Omega^2}\tilde X_A^\mu \star \tilde X_{A\mu} - \frac{\chi_{-1}}{\chi_0}\Big)^2 \\
& + \Big(\bar\phi\star\phi + \frac{4\Omega^2}{1+\Omega^2}\tilde X_B^\mu \star \tilde X_{B\mu} - \frac{\chi_{-1}}{\chi_0}\Big)^2 \\
& -2\Big(\frac{4\Omega^2}{1+\Omega^2}\tilde X_0^\mu \star \tilde X_{0\mu} - \frac{\chi_{-1}}{\chi_0}\Big)^2\Big\}(z) + \mathcal{O}(\chi_1)
\end{aligned}
$$

further remarks

non-trivial vacuum for both gauge and Higgs fields; origin is Higgs mechanism with spontaneous symmetry breaking

coefficients in front of Yang-Mills action are positive for all $\Omega \in [0,1[$ (different for bosonic model)

The spectral action

$$S(\mathcal{D}_A) = \frac{\theta^4\chi_{-4}}{8\Omega^4} + \frac{2\theta^2\chi_{-2}}{3\Omega^2} + \frac{52\chi_0}{45} + \frac{\chi_0}{2\pi^2(1+\Omega^2)^2}\int d^4z\,\Big\{2D_\mu\phi\star\overline{D_\mu\phi}$$
$$+\Big(\frac{(1-\Omega^2)^2}{2} - \frac{(1-\Omega^2)^4}{3(1+\Omega^2)^2}\Big)\big(F^A_{\mu\nu}\star F^{\mu\nu}_A + F^B_{\mu\nu}\star F^{\mu\nu}_B\big)$$
$$+\Big(\phi\star\bar\phi + \frac{4\Omega^2}{1+\Omega^2}\tilde X^\mu_A\star\tilde X_{A\mu} - \frac{\chi_{-1}}{\chi_0}\Big)^2$$
$$+\Big(\bar\phi\star\phi + \frac{4\Omega^2}{1+\Omega^2}\tilde X^\mu_B\star\tilde X_{B\mu} - \frac{\chi_{-1}}{\chi_0}\Big)^2$$
$$-2\Big(\frac{4\Omega^2}{1+\Omega^2}\tilde X^\mu_0\star\tilde X_{0\mu} - \frac{\chi_{-1}}{\chi_0}\Big)^2\Big\}(z) + \mathcal{O}(\chi_1)$$

further remarks

spectral action is invariant under gauge transformations

$\phi\mapsto u_A\star\phi\star\overline{u_B}$, $\tilde X_{A\mu}\mapsto u_A\star\tilde X_{A\mu}\star\overline{u_A}$, $\tilde X_{B\mu}\mapsto u_B\star\tilde X_{B\mu}\star\overline{u_B}$

$$F^A_{\mu\nu} = -i[\tilde X_{A\mu},\tilde X_{A\nu}]_\star + (\Theta^{-1})_{\mu\nu}$$
$$D_\mu\phi = -i\tilde X_{A\mu}\star\phi + i\phi\star\tilde X_{B\mu}$$

The spectral action

$$S(\mathcal{D}_A) = \frac{\theta^4\chi_{-4}}{8\Omega^4} + \frac{2\theta^2\chi_{-2}}{3\Omega^2} + \frac{52\chi_0}{45} + \frac{\chi_0}{2\pi^2(1+\Omega^2)^2}\int d^4z\,\Big\{2D_\mu\phi\star\overline{D_\mu\phi}$$
$$+\Big(\frac{(1-\Omega^2)^2}{2} - \frac{(1-\Omega^2)^4}{3(1+\Omega^2)^2}\Big)\big(F^A_{\mu\nu}\star F^{\mu\nu}_A + F^B_{\mu\nu}\star F^{\mu\nu}_B\big)$$
$$+\Big(\phi\star\bar\phi + \frac{4\Omega^2}{1+\Omega^2}\tilde X^\mu_A\star\tilde X_{A\mu} - \frac{\chi_{-1}}{\chi_0}\Big)^2$$
$$+\Big(\bar\phi\star\phi + \frac{4\Omega^2}{1+\Omega^2}\tilde X^\mu_B\star\tilde X_{B\mu} - \frac{\chi_{-1}}{\chi_0}\Big)^2$$
$$-2\Big(\frac{4\Omega^2}{1+\Omega^2}\tilde X^\mu_0\star\tilde X_{0\mu} - \frac{\chi_{-1}}{\chi_0}\Big)^2\Big\}(z) + \mathcal{O}(\chi_1)$$

further remarks

spectral action is translation-invariant!

$\phi(x)\mapsto\phi(x+a)$ $\quad\tilde X_{G\mu}(x)\mapsto\tilde X_{G\mu}(x+a)$ $\quad$ for $G\in A,B,0$

$\tilde X_{A\mu}(x)$ entirely, not as $\frac{1}{2}\tilde x_\mu + A_\mu(x)\mapsto\frac{1}{2}\tilde x_\mu + (\frac{1}{2}\tilde a_\mu + A_\mu(x+a))$
(changes class of functions)

The vacuum

vacuum field equations

$$\frac{1}{g^2}[\tilde{X}_{A\nu},[\tilde{X}^\mu_A,\tilde{X}^\nu_A]_\star]_\star + 2[\phi,[\tilde{X}^\mu_A,\phi]_\star]_\star = \frac{4\Omega^2}{1+\Omega^2}\Big\{\phi\star\phi + \frac{4\Omega^2}{1+\Omega^2}\tilde{X}^\nu_A\star\tilde{X}_{A\nu} - \eta^2, \tilde{X}^\mu_A\Big\}_\star$$

$$2[\tilde{X}_{A\nu},[\phi,\tilde{X}^\nu_A]_\star]_\star = \Big\{\phi\star\phi + \frac{4\Omega^2}{1+\Omega^2}\tilde{X}^\nu_A\star\tilde{X}_{A\nu} - \eta^2, \phi\Big\}_\star$$

(with $\frac{1}{4g^2} = \frac{(1-\Omega^2)^2}{2} - \frac{(1-\Omega^2)^4}{3(1+\Omega^2)^2} \qquad \eta^2 = \frac{\chi_{-1}}{\chi_0}$)

use gauge transformation:

solution ϕ^{vac} of field equations is real

implies $A^{vac}_\mu = B^{vac}_\mu$

The vacuum

vacuum field equations

$$\frac{1}{g^2}[\tilde{X}_{A\nu},[\tilde{X}_A^\mu,\tilde{X}_A^\nu]_\star]_\star + 2[\phi,[\tilde{X}_A^\mu,\phi]_\star]_\star = \frac{4\Omega^2}{1+\Omega^2}\Big\{\phi\star\phi+\frac{4\Omega^2}{1+\Omega^2}\tilde{X}_A^\nu\star\tilde{X}_{A\nu}-\eta^2,\tilde{X}_A^\mu\Big\}_\star$$

$$2[\tilde{X}_{A\nu},[\phi,\tilde{X}_A^\nu]_\star]_\star = \Big\{\phi\star\phi+\frac{4\Omega^2}{1+\Omega^2}\tilde{X}_A^\nu\star\tilde{X}_{A\nu}-\eta^2,\phi\Big\}_\star$$

spirit of emerging geometry (O'Connor et al)

$\Omega = 0 \quad\Rightarrow$ solution: $[X_\mu, X_\nu] = \Theta_{\mu\nu} = const$, $\phi = \eta = const$

$\Omega \neq 0$ gives some dynamical geometry

The vacuum

vacuum field equations

$$\frac{1}{g^2}[\tilde{X}_{A\nu},[\tilde{X}_A^\mu,\tilde{X}_A^\nu]_\star]_\star + 2[\phi,[\tilde{X}_A^\mu,\phi]_\star]_\star = \frac{4\Omega^2}{1+\Omega^2}\Big\{\phi\star\phi+\frac{4\Omega^2}{1+\Omega^2}\tilde{X}_A^\nu\star\tilde{X}_{A\nu}-\eta^2,\tilde{X}_A^\mu\Big\}_\star$$

$$2[\tilde{X}_{A\nu},[\phi,\tilde{X}_A^\nu]_\star]_\star = \Big\{\phi\star\phi+\frac{4\Omega^2}{1+\Omega^2}\tilde{X}_A^\nu\star\tilde{X}_{A\nu}-\eta^2,\phi\Big\}_\star$$

Spectral action is translation-invariant, but if for order parameter $\phi\star\phi+\frac{4\Omega^2}{1+\Omega^2}\tilde{X}_A^\nu\star\tilde{X}_{A\nu}\neq$ const. $\Rightarrow$ $\exists$ distinguished points in $\mathbb{R}^4$

(spontaneous breaking of translation invariance)

The vacuum

vacuum field equations

$$\frac{1}{g^2}[\tilde{X}_{A\nu},[\tilde{X}_A^\mu,\tilde{X}_A^\nu]_\star]_\star + 2[\phi,[\tilde{X}_A^\mu,\phi]_\star]_\star = \frac{4\Omega^2}{1+\Omega^2}\Big\{\phi\star\phi+\frac{4\Omega^2}{1+\Omega^2}\tilde{X}_A^\nu\star\tilde{X}_{A\nu}-\eta^2,\tilde{X}_A^\mu\Big\}_\star$$

$$2[\tilde{X}_{A\nu},[\phi,\tilde{X}_A^\nu]_\star]_\star = \Big\{\phi\star\phi+\frac{4\Omega^2}{1+\Omega^2}\tilde{X}_A^\nu\star\tilde{X}_{A\nu}-\eta^2,\phi\Big\}_\star$$

Extension to $U(n)$-models:

$SU(n)$-part has vanishing vacuum expectation value

H. Steinacker: interpret $U(1)$-part as gravity

The vacuum

vacuum field equations

$$\frac{1}{g^2}[\tilde{X}_{A\nu},[\tilde{X}^\mu_A,\tilde{X}^\nu_A]_\star]_\star + 2[\phi,[\tilde{X}^\mu_A,\phi]_\star]_\star = \frac{4\Omega^2}{1+\Omega^2}\Big\{\phi\star\phi+\frac{4\Omega^2}{1+\Omega^2}\tilde{X}^\nu_A\star\tilde{X}_{A\nu}-\eta^2,\tilde{X}^\mu_A\Big\}_\star$$

$$2[\tilde{X}_{A\nu},[\phi,\tilde{X}^\nu_A]_\star]_\star = \Big\{\phi\star\phi+\frac{4\Omega^2}{1+\Omega^2}\tilde{X}^\nu_A\star\tilde{X}_{A\nu}-\eta^2,\phi\Big\}_\star$$

Can we solve the vacuum equations?

analytically impossible $\Rightarrow$ need numerical simulations