

# A spectral triple for Moyal space with harmonic oscillator potential

(work in progress)

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(joint with Harald Grosse, arxiv:0709.0095)

# Introduction

## Theorem (H. Grosse + R.W.)

The quantum field theory defined by the action

$$S = \int d^4x \left( \frac{1}{2} \phi \star (-\Delta + \Omega^2 \tilde{x}^2 + \mu^2) \phi + \frac{\lambda}{4} \phi \star \phi \star \phi \star \phi \right) (x)$$

with  $\tilde{x} = 2\Theta^{-1} \cdot x$ ,  $\phi$  – real, Moyal product, Euclidean metric is **perturbatively renormalisable to all orders** in  $\lambda$ .

## Theorem (M. Disertori + R. Gurau + J. Magnen + V. Rivasseau)

The  $\beta$ -function of the above model **vanishes to all orders** for  $\Omega = 1$  and approaches zero for  $0 < \Omega < 1$  such that the **bare coupling is bounded**.

There is **noncommutativity**, but (so far) **no nc manifolds!**

# Noncommutative manifolds

- 1 suggested 1996 by A. Connes using insight from the **standard model**
- 2 commutative case fully settled (A. Rennie, J. Várilly; 2006)
- 3 nc manifolds describe **Yang-Mills + gravity** through **spectral action principle** (A. Chamseddine, A. Connes, 1996)
- 4 **Moyal spaces are non-compact spectral triples** (V.Gayral, J.M.Gracia-Bondía, B.Iochum, T.Schücker, J.Várilly; 2003)
- 5 spectral action for Moyal space (V.Gayral, B.Iochum; 2004) is Yang-Mills with Moyal product  
problem: **spacial regularisation required**
- 6 examples (Podleś quantum sphere, standard model) require **different metric and KO dimensions**

# What about gauge theory on Moyal space?

- not renormalisable without oscillator potential [string theory]
- formulate Moyal gauge theory with sort of oscillator potential via **spectral action principle**

need Dirac operator with harmonic oscillator spectrum

first guess:  $\mathcal{D} = \gamma^\mu (i\partial_\mu + \Omega \tilde{x}_\mu)$

- corresponding **Gross-Neveu model** is renormalisable to all orders [F. Vignes-Tourneret]
- describes constant magnetic background field, **no oscillator** (e.g. its spectrum is infinitely degenerate)

**no progress with square root of  $H = -\Delta + \Omega^2 \|\tilde{x}\|^2$**

# Effective gauge theory with scalar fields

mimic physical interpretation of spectral action (to evaluate a fermionic one-loop calculation) with scalar fields to which external gauge fields couple minimally

same result in  $x$ -space [A. de Goursac+J.-C. Wallet+R.W.] and matrix base [H. Grosse+M. Wohlgenannt]:

## Effective action

$$S_{\text{eff}} = \int d^4x \left( c_1 \ln \frac{1}{\epsilon} F_{\mu\nu} F^{\mu\nu} + c_2 \ln \frac{1}{\epsilon} (\tilde{X}_\mu \star \tilde{X}^\mu)^2 + \frac{c_3}{\epsilon} \tilde{X}_\mu \star \tilde{X}^\mu \right)$$

where  $\tilde{X}_\mu = (\Theta^{-1})_{\mu\nu} x^\nu + A_\mu$  is a covariant coordinate

- **A-linear terms** imply that  **$A = 0$  is not a stable vacuum**
- **quantisation is always about classical solution**; impossibility to solve the field equations obstructed any further progress

# The dimension of the harmonic oscillator

simplest quantum-mechanical system:

one-dimensional harmonic oscillator with Hamiltonian

$$H = -\frac{d^2}{dx^2} + \omega^2 x^2 \text{ on } L^2(\mathbb{R})$$

- configuration space  $\mathbb{R} \ni x$  is one-dimensional,  
phase space  $\mathbb{R}^2 \ni (x, p)$  is two-dimensional  
first example of a noncommutative geometry:  $[x, p] = i$
- What is its spectral dimension?  
 $H^{-1}$  has eigenvalues  $\mu_n = \frac{1}{\omega(2n+1)}$ ,  $n \in \mathbb{N}$ , so that  
 $H^{-1}$  is noncommutative infinitesimal of order one
- $H$  generalises Laplacian  $-\Delta$ , and dimension is defined via  
Dirac operator  $\mathcal{D}$ , which is a square root of the Laplacian

Should  $\mathcal{D} \sim H^{\frac{1}{2}}$  be of dimension two (phase space dimension)?

suggests: look for Dirac operator of 1D harmonic oscillator in  
two Clifford dimensions:

$$\begin{aligned} \mathcal{D}_1 &= i\sigma_1 \frac{d}{dx} + \sigma_2 \omega x = \begin{pmatrix} 0 & i\left(\frac{d}{dx} + \omega x\right) \\ i\left(\frac{d}{dx} - \omega x\right) & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & i\sqrt{2\omega} a \\ -i\sqrt{2\omega} a^\dagger & 0 \end{pmatrix} \end{aligned} \quad [a, a^\dagger] = 1$$

Can we extend  $\mathcal{D}_1$  to a two-dimensional spectral triple?

No for metric dimension!

- $\mathcal{D}_1$  is Dirac operator on  $\mathcal{H}_1 = L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$ , with  $(\mathcal{D}_1^2 + 1)^{-1} \in \mathcal{L}^{(1,\infty)}(\mathcal{H}_2)$   
( $\exists$  zero-mode; asymptotics is 2D)
- but dimension drop  $f(\mathcal{D}_1^2 + 1)^{-\frac{1}{2}} \in \mathcal{L}^{(1,\infty)}(\mathcal{H}_1)$  for  $f \in \mathcal{A}$ !

No for KO-dimension!

- KO-dimension is 0 mod 8 (for any dim. of harmonic oscillator)
- chirality is  $\chi_1 = \sigma_3$ , real structure  $\mathcal{J}_1 \psi = \sigma_3 \bar{\psi}$

# Generalisation to $d$ -dimensional harmonic oscillator

$d$  bosonic and fermionic creation and annihilation operators:

$$\begin{aligned} \textcircled{1} \quad [a_\mu, a_\nu] &= [a_\mu^\dagger, a_\nu^\dagger] = 0 & [a_\mu, a_\nu^\dagger] &= \delta_{\mu\nu} & \mu, \nu &= 1, \dots, d \\ \{b_\mu, b_\nu\} &= \{b_\mu^\dagger, b_\nu^\dagger\} = 0 & \{b_\mu, b_\nu^\dagger\} &= \delta_{\mu\nu} \end{aligned}$$

$$\begin{aligned} \textcircled{2} \quad \mathcal{D}_d &= \sqrt{2}\omega \sum_{\mu=1}^d (ia_\mu \otimes b_\mu^\dagger - ia_\mu^\dagger \otimes b_\mu) & (\text{supersymmetry}) \\ &= i \frac{\partial}{\partial x_\mu} \otimes (b_\mu + b_\mu^\dagger) + \omega x^\mu \otimes (ib_\mu^\dagger - ib_\mu) \\ &= i\Gamma_\mu \frac{\partial}{\partial x_\mu} + \omega \Gamma_{d+\mu} x^\mu & \text{Cliff}(\mathbb{C}^{2d}) \end{aligned}$$

$$\textcircled{3} \quad \mathcal{H}_d = L^2(\mathbb{R}^d) \otimes \wedge(\mathbb{C}^d) \text{ (bosonic } \otimes \text{ fermionic Hilbert spaces)}$$

$$\chi_d = 1 \otimes \prod_{\mu=1}^d (b_\mu^\dagger b_\mu - b_\mu b_\mu^\dagger) = (-i)^d (-1)^{\frac{d(d+1)}{2}} \Gamma_1 \cdots \Gamma_{2d}$$

$$\begin{aligned} \textcircled{4} \quad \mathcal{D}_n^2 &= \omega \sum_{\mu=1}^d (\{a_\mu, a_\mu^\dagger\} \otimes 1 + 1 \otimes \{b_\mu^\dagger, b_\mu\}) = H_d \otimes 1 + \omega \otimes \Sigma_d \\ H_d &= -\frac{\partial^2}{\partial x_\mu \partial x^\mu} + \omega^2 x_\mu x^\mu & \Sigma_d &= \sum_{\mu=1}^d [b_\mu^\dagger, b_\mu] \end{aligned}$$



## Spectral data for harmonic oscillator Moyal space

$$\textcircled{1} \quad \mathcal{D}_4 = (i\Gamma^\mu \partial_\mu + \Omega \Gamma^{\mu+4} \tilde{\chi}_\mu) \quad \mu = 1, \dots, 4; \quad \tilde{\chi}_\mu = 2(\Theta^{-1})_{\mu\nu} x^\nu$$

$$\mathcal{D}_4^2 = (-\Delta + \Omega^2 \|\tilde{\chi}\|^2) 1 + \tilde{\Omega} \Sigma_4 \quad (\tilde{\Omega} = \frac{2\Omega}{\theta}; \quad \Theta = \theta S; \quad S^2 = -1)$$

$$\textcircled{2} \quad \text{Hilbert space } \mathcal{H}_4 = L^2(\mathbb{R}^4) \otimes \mathbb{C}^{16}$$

smooth spinors  $\mathcal{H}_4^\infty = \bigcap_{k=1}^\infty \text{dom}(\mathcal{D}_4^k)$  are Schwartzian

$$\textcircled{3} \quad \text{algebra } \mathcal{A}_4 = \mathcal{S}(\mathbb{R}^4) \text{ with}$$

$$(f \star g)(x) = \int d^4 y \frac{d^4 k}{(2\pi)^4} f(x + \frac{1}{2}\Theta k) g(x + y) e^{iky}$$

$\mathcal{A}_4$  acts on  $\mathcal{H}_4$  componentwise by  $L_\star(f)\psi = f \star \psi$ ;

$[\mathcal{D}_4, L_\star(f)] = i(\Gamma^\mu + \Omega \Gamma^{\mu+4}) L_\star(\partial_\mu f)$  is bounded

$$\textcircled{4} \quad \chi_4 = \Gamma_9 \quad \mathcal{J}_4 \psi = \chi_4 \bar{\psi}$$

orientation form is  $\pi(\mathbf{c}_4) = \frac{1}{(1+\Omega^2)^2} \prod_{\mu=1}^4 (\Gamma^\mu + \Omega \Gamma^{\mu+4})$ , not  $\Gamma_9$

# The trace theorem

**Dixmier trace** (define  $\langle \mathcal{D} \rangle = (\mathcal{D}^2 + 1)^{\frac{1}{2}}$ )

$$\begin{aligned} \mathrm{Tr}_\omega(L_\star(f)\langle \mathcal{D}_4 \rangle^{-d}) &= \lim_{s \rightarrow 1} (s-1) \mathrm{Tr}(L_\star(f)\langle \mathcal{D}_4 \rangle^{-ds}) \\ &= \lim_{s \rightarrow 1} \frac{s-1}{\Gamma(\frac{ds}{2})} \int_0^\infty dt t^{\frac{ds}{2}-1} \mathrm{Tr}(L_\star(f) e^{-t(H_4 + \tilde{\Omega}\Sigma_4 1_2 + 1)}) \end{aligned}$$

## Mehler and Moyal kernels

$$e^{-H_4 t}(x, y) = \frac{\tilde{\Omega}^2(1 - \tanh^2(\tilde{\Omega}t))^2}{16\pi^2 \tanh^2(\tilde{\Omega}t)} e^{-\frac{\tilde{\Omega}}{4} \frac{|x-y|^2}{\tanh(\tilde{\Omega}t)} - \frac{\tilde{\Omega}}{4} \tanh(\tilde{\Omega}t) |x+y|^2}$$

$$(L_\star(f))(x, y) = \int \frac{d^4 z}{\pi^4 \theta^4} f(z) e^{2i(x \wedge y + y \wedge z + z \wedge x)}$$

(with  $x \wedge y = x^\mu (\Theta^{-1})_{\mu\nu} y^\nu$ )

for  $f \in \mathcal{S}(\mathbb{R}^4)$ :  $\mathrm{Tr}_\omega(L_\star(f)\langle \mathcal{D}_4 \rangle^{-4}) = \frac{\mathrm{tr}_{\mathrm{cl}}(1)}{32\pi^2(1+\Omega^2)^2} \int d^4 z f(z)$

scalar product for smooth spinors from Dixmier trace and hermitian structure  $(\cdot, \cdot)_{\mathcal{A}_4} : \mathcal{H}_4^\infty \times \mathcal{H}_4^\infty \rightarrow \mathcal{A}_4$

# Orientability

need unitisation  $\tilde{\mathcal{A}}_4$  of  $\tilde{\mathcal{A}}$  which contains plane waves  $u_\mu = e^{ix_\mu}$

- Hochschild 4-cycle

$$\mathbf{c}_4 = \frac{1}{4!(1+\Omega^2)^2} \sum_{\sigma \in \mathcal{S}_4} (-1)^\sigma \times (u_{\sigma(1)} \star u_{\sigma(2)} \star u_{\sigma(3)} \star u_{\sigma(4)})^{-1 \star} \otimes u_{\sigma(1)} \otimes u_{\sigma(2)} \otimes u_{\sigma(3)} \otimes u_{\sigma(4)}$$

- $\pi(\mathbf{c}_4) = \frac{1}{(1+\Omega^2)^2} (\Gamma_1 + \Omega\Gamma_5)(\Gamma_2 + \Omega\Gamma_6)(\Gamma_3 + \Omega\Gamma_7)(\Gamma_4 + \Omega\Gamma_8)$

defines Hochschild 4-cocycle ( $f \equiv L_\star(f)$ )

$$\begin{aligned} \phi_{\pi(\mathbf{c}_4)}(f_0, \dots, f_4) &= \text{Tr}_\omega(\pi(\mathbf{c}_4) f_0 [\mathcal{D}_4, f_1] \cdots [\mathcal{D}_4, f_4] \langle \mathcal{D}_4 \rangle^{-4}) \\ &= \frac{1}{2\pi^2} \int f_0 \star df_1 \wedge_\star df_2 \wedge_\star df_3 \wedge_\star df_4 \end{aligned}$$

Note: A Hochschild 4-cocycle is obtained for  $\pi(\mathbf{c}_4)$  replaced by any bounded operator which commutes with  $\mathcal{A}$ .

*Is  $\phi_{\pi(\mathbf{c}_4)}$  distinguished in the Hochschild cohomology of the non-unital algebra  $\mathcal{A}$ ?*

# U(1)-Higgs model

tensor  $(\mathcal{A}_4, \mathcal{H}_4, \mathcal{D}_4, \chi_4)$  with  $(\mathbb{C} \oplus \mathbb{C}, \mathbb{C}^2, M\sigma_1)$  [Connes+Lott]

- $\mathcal{D} = \mathcal{D}_4 \otimes \mathbf{1}_2 + \Gamma_9 \otimes \sigma_1 M = \begin{pmatrix} \mathcal{D}_4 & M\Gamma_9 \\ M\Gamma_9 & \mathcal{D}_4 \end{pmatrix}$
- selfadjoint **fluctuated Dirac operators**  $\mathcal{D}_A := \mathcal{D} + \sum_i a_i [\mathcal{D}, b_i]$ ,  
 $a_i, b_i \in \mathcal{A} = \mathcal{A}_4 \oplus \mathcal{A}_4$  are of the form
 
$$\mathcal{D}_A = \begin{pmatrix} \mathcal{D}_4 + (\Gamma^\mu + \Omega \Gamma^{\mu+4}) L_\star(A_\mu) & \Gamma_9 L_\star(\phi) \\ \Gamma_9 L_\star(\bar{\phi}) & \mathcal{D}_4 + (\Gamma^\mu + \Omega \Gamma^{\mu+4}) L_\star(B_\mu) \end{pmatrix}$$
 for  $A_\mu = \overline{A}_\mu, B_\mu = \overline{B}_\mu, \phi \in \mathcal{A}_4$
- $\mathcal{D}_A^2 = \begin{pmatrix} (H_4 + L_\star(\phi \star \bar{\phi})) \mathbf{1} + \Sigma_4 + F_A & i(\Gamma^\mu + \Omega \Gamma^{\mu+4}) \Gamma_9 L_\star(D_\mu \phi) \\ i(\Gamma^\mu + \Omega \Gamma^{\mu+4}) \Gamma_9 L_\star(\overline{D_\mu \phi}) & (H_4 + L_\star(\bar{\phi} \star \phi)) \mathbf{1} + \Sigma_4 + F_B \end{pmatrix}$
- $D_\mu \phi = \partial_\mu \phi - iA \star \phi + i\phi \star B$   
 $F_A = \{L_\star(A^\mu), i\partial_\mu + \Omega^2 M_\bullet(\tilde{X}_\mu)\} + (1 + \Omega^2) L_\star(A_\mu \star A^\mu)$   
 $+ i(\frac{1}{4}[\Gamma^\mu, \Gamma^\nu] + \frac{1}{4}\Omega^2[\Gamma^{\mu+4}, \Gamma^{\nu+4}] + \Omega \Gamma^\mu \Gamma^{\nu+4}) L_\star(F_{\mu\nu}^A)$

# Spectral action principle

most general form of bosonic action is  $S(\mathcal{D}_A) = \text{Tr}(\chi(\mathcal{D}_A^2))$

- Laplace transf. + asympt. expansion  $e^{-t\mathcal{D}_A^2} = \sum_{n=-d/2}^{\infty} a_n(\mathcal{D}_A^2) t^n$   
 lead to  $S(\mathcal{D}_A) = \sum_{n=-d/2}^{\infty} \chi_n \text{Tr}(a_n(\mathcal{D}_A^2))$   
 ( $d = \text{dimension}$ )

with  $\chi_z = \frac{1}{\Gamma(-z)} \int_0^{\infty} ds s^{-z-1} \chi(s)$  for  $z \notin \mathbb{N}$   
 $\chi_k = (-1)^k \chi^{(k)}(0)$  for  $k \in \mathbb{N}$

- $a_n$  – Seeley coefficients, must be computed from scratch

Duhamel expansion:  $\mathcal{D}_A^2 = H_0 - V$

$$e^{-t(H_0 - V)} = e^{-tH_0} - \int_0^t dt_1 \frac{d}{dt_1} (e^{-(t-t_1)(H_0 - V)} e^{-t_1 H_0})$$

$$= e^{-tH_0} + \int_0^t dt_1 (e^{-(t-t_1)(H_0 - V)} V e^{-t_1 H_0}) \quad \dots \text{iteration}$$

# Position space kernels

## Mehler kernel

$$e^{-H_0 t}(x, y) = \frac{\tilde{\Omega}^2(1 - \tanh^2(\tilde{\Omega}t))^2}{16\pi^2 \tanh^2(\tilde{\Omega}t)} e^{-t\tilde{\Omega}\Sigma_4} \frac{1}{2} - \frac{\tilde{\Omega}}{4} \frac{|x-y|^2}{\tanh(\tilde{\Omega}t)} - \frac{\tilde{\Omega}}{4} \tanh(\tilde{\Omega}t) |x+y|^2$$

$$(\text{with } \tilde{\Omega} = \frac{2\Omega}{\theta} \quad \Theta = \theta S \quad S^2 = -1)$$

## vacuum trace

$$\begin{aligned} \text{Tr}(e^{-tH_0}) &= \text{tr}_{cl} \int d^4x (e^{-tH_0})(x, x) = \frac{1}{8 \sinh^4(\tilde{\Omega}t)} \text{tr}_{cl}(e^{-t\tilde{\Omega}\Sigma_4}) \\ &= \frac{2}{\tanh^4(\tilde{\Omega}t)} = 2(\tilde{\Omega}t)^{-4} + \frac{8}{3}(\tilde{\Omega}t)^{-2} + \frac{52}{45} + \mathcal{O}(t^2) \end{aligned}$$

- expansion starts with  $t^{-4} \Rightarrow$  corresponds to 8-dim. space
- spectral action is finite, in contrast to pure Moyal at  $\Omega = 0$
- view spacial regularisation as part of geometry

## Vertex kernels (with $x \wedge y = x^\mu (\Theta^{-1})_{\mu\nu} y^\nu$ )

$$(L_\star(f))(x, y) = \int \frac{d^4 z}{\pi^4 \theta^4} f(z) e^{2i(x \wedge y + y \wedge z + z \wedge x)}$$

$$\begin{aligned} & \{L_\star(A^\mu), i\partial_\mu + \Omega^2 M_\bullet(\tilde{x}_\mu)\}(x, y) \\ &= \int \frac{d^4 z}{\pi^4 \theta^4} (2\tilde{z}^\mu - (1 - \Omega^2)(\tilde{x}^\mu + \tilde{y}^\mu)) A_\mu(z) e^{2i(x \wedge y + y \wedge z + z \wedge x)} \end{aligned}$$

- one-vertex trace

$$\begin{aligned} & \int_0^t dt_1 \text{Tr}(e^{-(t-t_1)H_0} L_\star(f) e^{-t_1 H_0}) \\ &= \frac{\tilde{\Omega}^2 t}{16\pi^2 (1 + \Omega^2)^2 \sinh^2(\tilde{\Omega} t)} \int d^4 z f(z) e^{-\frac{\tilde{\Omega} \tanh(\tilde{\Omega} t)}{1 + \Omega^2} |z|^2} \end{aligned}$$

- order is  $t^{-1}$  as in 4D-standard model  
opposite sign of  $t^{-1}$ ,  $t^0 \Rightarrow$  **spontaneous symmetry breaking**
- in general: Gaußian integrations using determinant and inverse of  $U \otimes 1_2 + V \otimes \sigma_2$  [R. Gurau+V. Rivasseau]

# The spectral action

$$\begin{aligned}
 S(\mathcal{D}_A) = & \frac{\theta^4 \chi_{-4}}{8\Omega^4} + \frac{2\theta^2 \chi_{-2}}{3\Omega^2} + \frac{52\chi_0}{45} + \frac{\chi_0}{2\pi^2(1+\Omega^2)^2} \int d^4z \left\{ 2D_\mu \phi \star \overline{D_\mu \phi} \right. \\
 & + \left( \frac{(1-\Omega^2)^2}{2} - \frac{(1-\Omega^2)^4}{3(1+\Omega^2)^2} \right) (F_{\mu\nu}^A \star F_A^{\mu\nu} + F_{\mu\nu}^B \star F_B^{\mu\nu}) \\
 & + \left( \phi \star \bar{\phi} + \frac{4\Omega^2}{1+\Omega^2} \tilde{X}_A^\mu \star \tilde{X}_{A\mu} - \frac{\chi_{-1}}{\chi_0} \right)^2 \\
 & + \left( \bar{\phi} \star \phi + \frac{4\Omega^2}{1+\Omega^2} \tilde{X}_B^\mu \star \tilde{X}_{B\mu} - \frac{\chi_{-1}}{\chi_0} \right)^2 \\
 & \left. - 2 \left( \frac{4\Omega^2}{1+\Omega^2} \tilde{X}_0^\mu \star \tilde{X}_{0\mu} - \frac{\chi_{-1}}{\chi_0} \right)^2 \right\} (z) + \mathcal{O}(\chi_1)
 \end{aligned}$$

deeper entanglement of gauge and Higgs fields

covariant coordinates  $\tilde{X}_{A\mu}(z) = (\Theta^{-1})_{\mu\nu} z^\nu + A_\mu(z)$  appear with Higgs field  $\phi$  in **unified potential**

**potential cannot be restricted to Higgs part** if distinction into discrete and continuous geometries no longer possible



# The spectral action

$$\begin{aligned}
 S(\mathcal{D}_A) = & \frac{\theta^4 \chi_{-4}}{8\Omega^4} + \frac{2\theta^2 \chi_{-2}}{3\Omega^2} + \frac{52\chi_0}{45} + \frac{\chi_0}{2\pi^2(1+\Omega^2)^2} \int d^4z \left\{ 2D_\mu \phi \star \overline{D_\mu \phi} \right. \\
 & + \left( \frac{(1-\Omega^2)^2}{2} - \frac{(1-\Omega^2)^4}{3(1+\Omega^2)^2} \right) (F_{\mu\nu}^A \star F_A^{\mu\nu} + F_{\mu\nu}^B \star F_B^{\mu\nu}) \\
 & + \left( \phi \star \bar{\phi} + \frac{4\Omega^2}{1+\Omega^2} \tilde{X}_A^\mu \star \tilde{X}_{A\mu} - \frac{\chi_{-1}}{\chi_0} \right)^2 \\
 & + \left( \bar{\phi} \star \phi + \frac{4\Omega^2}{1+\Omega^2} \tilde{X}_B^\mu \star \tilde{X}_{B\mu} - \frac{\chi_{-1}}{\chi_0} \right)^2 \\
 & \left. - 2 \left( \frac{4\Omega^2}{1+\Omega^2} \tilde{X}_0^\mu \star \tilde{X}_{0\mu} - \frac{\chi_{-1}}{\chi_0} \right)^2 \right\} (z) + \mathcal{O}(\chi_1)
 \end{aligned}$$

## further remarks

**non-trivial vacuum for both gauge and Higgs fields**; origin is Higgs mechanism with spontaneous symmetry breaking

coefficients in front of Yang-Mills action are **positive** for all  $\Omega \in [0, 1[$  (different for bosonic model)

# The spectral action

$$\begin{aligned}
 S(\mathcal{D}_A) = & \frac{\theta^4 \chi_{-4}}{8\Omega^4} + \frac{2\theta^2 \chi_{-2}}{3\Omega^2} + \frac{52\chi_0}{45} + \frac{\chi_0}{2\pi^2(1+\Omega^2)^2} \int d^4z \left\{ 2D_\mu \phi \star \overline{D_\mu \phi} \right. \\
 & + \left( \frac{(1-\Omega^2)^2}{2} - \frac{(1-\Omega^2)^4}{3(1+\Omega^2)^2} \right) (F_{\mu\nu}^A \star F_A^{\mu\nu} + F_{\mu\nu}^B \star F_B^{\mu\nu}) \\
 & + \left( \phi \star \bar{\phi} + \frac{4\Omega^2}{1+\Omega^2} \tilde{X}_A^\mu \star \tilde{X}_{A\mu} - \frac{\chi_{-1}}{\chi_0} \right)^2 \\
 & + \left( \bar{\phi} \star \phi + \frac{4\Omega^2}{1+\Omega^2} \tilde{X}_B^\mu \star \tilde{X}_{B\mu} - \frac{\chi_{-1}}{\chi_0} \right)^2 \\
 & \left. - 2 \left( \frac{4\Omega^2}{1+\Omega^2} \tilde{X}_0^\mu \star \tilde{X}_{0\mu} - \frac{\chi_{-1}}{\chi_0} \right)^2 \right\} (z) + \mathcal{O}(\chi_1)
 \end{aligned}$$

## further remarks

spectral action is **invariant under gauge transformations**

$$\phi \mapsto u_A \star \phi \star \bar{u}_B, \quad \tilde{X}_{A\mu} \mapsto u_A \star \tilde{X}_{A\mu} \star \bar{u}_A, \quad \tilde{X}_{B\mu} \mapsto u_B \star \tilde{X}_{B\mu} \star \bar{u}_B$$

$$F_{\mu\nu}^A = -i[\tilde{X}_{A\mu}, \tilde{X}_{A\nu}]_\star + (\Theta^{-1})_{\mu\nu}$$

$$D_\mu \phi = -i\tilde{X}_{A\mu} \star \phi + i\phi \star \tilde{X}_{B\mu}$$

# The spectral action

$$\begin{aligned}
 S(\mathcal{D}_A) = & \frac{\theta^4 \chi_{-4}}{8\Omega^4} + \frac{2\theta^2 \chi_{-2}}{3\Omega^2} + \frac{52\chi_0}{45} + \frac{\chi_0}{2\pi^2(1+\Omega^2)^2} \int d^4z \left\{ 2D_\mu \phi \star \overline{D_\mu \phi} \right. \\
 & + \left( \frac{(1-\Omega^2)^2}{2} - \frac{(1-\Omega^2)^4}{3(1+\Omega^2)^2} \right) (F_{\mu\nu}^A \star F_A^{\mu\nu} + F_{\mu\nu}^B \star F_B^{\mu\nu}) \\
 & + \left( \phi \star \bar{\phi} + \frac{4\Omega^2}{1+\Omega^2} \tilde{X}_A^\mu \star \tilde{X}_{A\mu} - \frac{\chi_{-1}}{\chi_0} \right)^2 \\
 & + \left( \bar{\phi} \star \phi + \frac{4\Omega^2}{1+\Omega^2} \tilde{X}_B^\mu \star \tilde{X}_{B\mu} - \frac{\chi_{-1}}{\chi_0} \right)^2 \\
 & \left. - 2 \left( \frac{4\Omega^2}{1+\Omega^2} \tilde{X}_0^\mu \star \tilde{X}_{0\mu} - \frac{\chi_{-1}}{\chi_0} \right)^2 \right\} (z) + \mathcal{O}(\chi_1)
 \end{aligned}$$

## further remarks

**spectral action is translation-invariant!**

$$\phi(x) \mapsto \phi(x+a) \quad \tilde{X}_{G\mu}(x) \mapsto \tilde{X}_{G\mu}(x+a) \quad \text{for } G \in A, B, 0$$

$\tilde{X}_{A\mu}(x)$  entirely, not as  $\frac{1}{2}\tilde{X}_\mu + A_\mu(x) \mapsto \frac{1}{2}\tilde{X}_\mu + (\frac{1}{2}\tilde{a}_\mu + A_\mu(x+a))$   
(changes class of functions)

# The vacuum

gauge transformation:

solution  $\phi^{vac}$  of field equations is **real**; implies  $A_\mu^{vac} = B_\mu^{vac}$

vacuum field equations

$$\begin{aligned} \frac{1}{g^2} [\tilde{X}_{A\nu}, [\tilde{X}_A^\mu, \tilde{X}_A^\nu]_\star]_\star + 2[\phi, [\tilde{X}_A^\mu, \phi]_\star]_\star \\ = \frac{4\Omega^2}{1+\Omega^2} \left\{ \phi \star \phi + \frac{4\Omega^2}{1+\Omega^2} \tilde{X}_A^\nu \star \tilde{X}_{A\nu} - \eta^2, \tilde{X}_A^\mu \right\}_\star \\ 2[\tilde{X}_{A\nu}, [\phi, \tilde{X}_A^\nu]_\star]_\star = \left\{ \phi \star \phi + \frac{4\Omega^2}{1+\Omega^2} \tilde{X}_A^\nu \star \tilde{X}_{A\nu} - \eta^2, \phi \right\}_\star \\ \left( \text{with } \frac{1}{4g^2} = \frac{(1-\Omega^2)^2}{2} - \frac{(1-\Omega^2)^4}{3(1+\Omega^2)^2} \quad \eta^2 = \frac{\chi-1}{\chi_0} \right) \end{aligned}$$

Spectral action is translation-invariant, but if for order parameter

$\phi \star \phi + \frac{4\Omega^2}{1+\Omega^2} \tilde{X}_A^\nu \star \tilde{X}_{A\nu} \neq \text{const.} \Rightarrow \exists$  distinguished points in  $\mathbb{R}^4$

(spontaneous breaking of translation invariance)

## Spectral action is isotropic

⇒ choose radial coordinates with any origin

### General ansatz for the vacuum solution

$$\phi^{vac}(x) = \overline{\phi^{vac}(x)} = f(|x|^2) \quad \tilde{X}_{A\mu}^{vac} = \tilde{X}_{B\mu}^{vac} = \frac{\tilde{x}_\mu}{2} \xi(|x|^2)$$

for radial functions  $f, \xi$

### Conjecture (work in progress)

For pure Yang-Mills theory ( $\phi = 0$ ), the order parameter  $\tilde{X}_\mu \star \tilde{X}^\mu$  solves an inhomogeneous Bessel differential equation

by-product:

Moyal for radial functions  $a(x) = a(r), b(x) = b(r), r = |x|^2$ :

$$(a \star b)(x) = \sum_{n=0}^{\infty} \frac{(-1)^n \theta^{2n}}{n! r^{\frac{d}{2}-1}} \left( r^{n+\frac{d}{2}-1} a^{(n)} b^{(n)} \right)^{(n)}(r)$$