From the harmonic oscillator as a spectral triple to the vacuum of nc gauge theory
(work in progress)

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(joint with Harald Grosse, arxiv:0709.0095)

Introduction

Theorem (H. Grosse + R.W.)

The quantum field theory defined by the action

$$S = \int d^4x \Big(\frac{1}{2} \phi \star \big(-\Delta + \Omega^2 \tilde{x}^2 + \mu^2 \big) \phi + \frac{\lambda}{4!} \phi \star \phi \star \phi \star \phi \Big)(x)$$

with $\tilde{x} = 2\Theta^{-1} \cdot x$, ϕ – real, Moyal product, Euclidean metric is perturbatively renormalisable to all orders in λ.

Theorem (M. Disertori + R. Gurau + J. Magnen + V. Rivasseau)

The β-function of the above model vanishes to all orders for $\Omega = 1$ and approaches zero for $0 < \Omega < 1$ such that the bare coupling is bounded.

There is noncommutativity, but (so far) no nc manifolds!

Noncommutative manifolds

1. suggested 1996 by A. Connes using insight from the standard model
2. commutative case fully settled (A.Rennie, J.Várilly; 2007)
3. nc manifolds describe Yang-Mills + gravity through spectral action principle (A. Chamseddine, A. Connes, 1996)
4. Moyal spaces are non-compact spectral triples (V.Gayral, J.M.Gracia-Bondía, B.Iochum, T.Schücker, J.Várilly; 2003)
5. spectral action for Moyal space (V.Gayral, B.Iochum; 2004) is Yang-Mills with Moyal product
   problem: spacial regularisation required
6. examples (Podleś quantum sphere, standard model) require different metric and KO dimensions

What about gauge theory on Moyal space?

- not renormalisable without oscillator potential [string theory]
- formulate Moyal gauge theory with sort of oscillator potential via spectral action principle

need Dirac operator with harmonic oscillator spectrum

first guess: $\mathcal{D} = \gamma^\mu(i\partial_\mu + \Omega\tilde{x}_\mu)$

- corresponding Gross-Neveu model is renormalisable to all orders [F. Vignes-Tourneret]
- describes constant magnetic background field, no oscillator (e.g. its spectrum is infinitely degenerate)

no progress with square root of $H = -\Delta + \Omega^2\|\tilde{x}\|^2$

Effective gauge theory with scalar fields

mimic physical interpretation of spectral action (to evaluate a fermionic one-loop calculation) with scalar fields to which external gauge fields couple minimally

same result in x-space [A. de Goursac+J.-C. Wallet+R.W.] and matrix base [H. Grosse+M. Wohlgenannt]:

Effective action

$$S_{eff} = \int d^4x \Big(c_1 \ln\frac{1}{\epsilon} F_{\mu\nu}F^{\mu\nu} + c_2 \ln\frac{1}{\epsilon} (\tilde{X}_\mu \star \tilde{X}^\mu)^2 + \frac{c_3}{\epsilon} \tilde{X}_\mu \star \tilde{X}^\mu \Big)$$

where $\tilde{X}_\mu = (\Theta^{-1})_{\mu\nu} x^\nu + A_\mu$ is a covariant coordinate

- A-linear terms imply that $A = 0$ is not a stable vacuum
- quantisation is always about classical solution; impossibility to solve the field equations obstructed any further progress

The dimension of the harmonic oscillator

simplest quantum-mechanical system:
one-dimensional harmonic oscillator with Hamiltonian
$H = -\frac{d^2}{dx^2} + \omega^2 x^2$ on $L^2(\mathbb{R})$

- configuration space $\mathbb{R} \ni x$ is one-dimensional,
  phase space $\mathbb{R}^2 \ni (x, p)$ is two-dimensional
  first example of a noncommutative geometry: $[x, p] = i$
- What is its spectral dimension?
  H^{-1} has eigenvalues $\mu_n = \frac{1}{\omega(2n+1)}$ so that
  H^{-1} is noncommutative infinitesimal of order one
- H generalises Laplacian $-\Delta$, and dimension is defined via
  Dirac operator $\mathcal{D}$, which is a square root of the Laplacian

Should $\mathcal{D} \sim H^{\frac{1}{2}}$ be of dimension two (phase space dimension)?

suggests: look for Dirac operator of 1D harmonic oscillator in two Clifford dimensions:

$$\mathcal{D}_1 = i\sigma_1 \frac{d}{dx} + \sigma_2 \omega x = \begin{pmatrix} 0 & i(\frac{d}{dx} + \omega x) \\ i(\frac{d}{dx} - \omega x) & 0 \end{pmatrix} = \begin{pmatrix} 0 & i\sqrt{2\omega} a \\ -i\sqrt{2\omega} a^\dagger & 0 \end{pmatrix} \qquad [a, a^\dagger] = 1$$

Can we extend $\mathcal{D}_1$ to a two-dimensional spectral triple?

No for metric dimension!

- $\mathcal{D}_1$ is Dirac operator on $\mathcal{H}_1 = L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$, with $(\mathcal{D}^2 + 1)^{-1} \in \mathcal{L}^{(1,\infty)}(\mathcal{H}_2)$ ($\exists$ zero-mode; asymptotics is 2D)
- but dimension drop $f(\mathcal{D}^2 + 1)^{-\frac{1}{2}} \in \mathcal{L}^{(1,\infty)}(\mathcal{H}_1)$ for $f \in \mathcal{A}$!

No for KO-dimension!

- KO-dimension is zero (for any dim. of harmonic oscillator)
- chirality is $\chi_1 = \sigma_3$, real structure $J_1 \psi = \sigma_3 \overline{\psi}$

Generalisation to d-dimensional harmonic oscillator

d bosonic and fermionic creation and annihilation operators:

- $[a_\mu, a_\nu] = [a^\dagger_\mu, a^\dagger_\nu] = 0 \qquad [a_\mu, a^\dagger_\nu] = \delta_{\mu\nu} \qquad \mu,\nu{=}1,\dots,d$
  $\{b_\mu, b_\nu\} = \{b^\dagger_\mu, b^\dagger_\nu\} = 0 \qquad \{b_\mu, b^\dagger_\nu\} = \delta_{\mu\nu}$
- $\mathcal{D}_d = \sqrt{2}\omega \sum_{\mu=1}^d (ia_\mu \otimes b^\dagger_\mu - ia^\dagger_\mu \otimes b_\mu)$ (*supersymmetry*)
  $= i\frac{\partial}{\partial x_\mu} \otimes (b_\mu + b^\dagger_\mu) + \omega x^\mu \otimes (ib^\dagger_\mu - ib_\mu)$
  $= i\Gamma_\mu \frac{\partial}{\partial x_\mu} + \omega \Gamma_{d+\mu} x^\mu$ $\qquad \mathrm{Cliff}(\mathbb{C}^{2d})$
- $\mathcal{H}_d = L^2(\mathbb{R}^d) \otimes \bigwedge(\mathbb{C}^d) \cong L^2(\mathbb{R}^d) \otimes \mathbb{C}^{2^d} \cong L^2(\mathbb{R}^d) \otimes \mathrm{Cliff}(\mathbb{C}^d)$
  (use $b_\mu|0\rangle = 0$ and $b^\dagger_\mu|0\rangle = (b^\dagger_\mu + b_\mu)|0\rangle = \Gamma_\mu|0\rangle$)

  $\chi_d = 1 \otimes \prod_{\mu=1}^d (b^\dagger_\mu b_\mu - b_\mu b^\dagger_\mu) = (-i)^d (-1)^{\frac{d(d+1)}{2}} \Gamma_1 \cdots \Gamma_{2d}$
- $\mathcal{D}_n^2 = \omega \sum_{\mu=1}^d (\{a_\mu, a^\dagger_\mu\} \otimes 1 + 1 \otimes [b^\dagger_\mu, b_\mu]) = H_d \otimes 1 + \omega \otimes \Sigma_d$
  $H_d = -\frac{\partial^2}{\partial x_\mu \partial x^\mu} + \omega^2 x_\mu x^\mu \qquad \Sigma_d = \sum_{\mu=1}^d [b^\dagger_\mu, b_\mu]$

Spectral data for harmonic oscillator Moyal space

1. $\mathcal{D}_4 = \big(i\Gamma^\mu\partial_\mu + \Omega\Gamma^{\mu+4}\tilde{x}_\mu\big) \qquad \mu = 1,\dots,4; \qquad \tilde{x}_\mu = 2(\Theta^{-1})_{\mu\nu}x^\nu$

   $\mathcal{D}_4^2 = \big(-\Delta + \Omega^2\|\tilde{x}\|^2\big)\,1 + \tilde{\Omega}\Sigma_4 \qquad (\tilde{\Omega} = \frac{2\Omega}{\theta};\ \ \Theta = \theta S;\ \ S^2 = -1)$

2. Hilbert space $\mathcal{H}_4 = L^2(\mathbb{R}^4) \otimes \mathbb{C}^{16}$
   smooth spinors $\mathcal{H}_4^\infty = \bigcap_{k=1}^\infty \mathrm{dom}(\mathcal{D}_4^k)$ are Schwartzian

3. algebra $\mathcal{A}_4 = \mathcal{S}(\mathbb{R}^4)$ with

   $$(f\star g)(x) = \int d^4y\,\frac{d^4k}{(2\pi)^4}\,f(x+\tfrac{1}{2}\Theta k)g(x+y)e^{iky}$$

   $\mathcal{A}_4$ acts on $\mathcal{H}_4$ componentwise by $L_\star(f)\psi = f\star\psi$;
   $[\mathcal{D}_4, L_\star(f)] = i(\Gamma^\mu + \Omega\Gamma^{\mu+4})L_\star(\partial_\mu f)$ is bounded

4. $\chi_4 = \Gamma_9 \qquad J_4\psi = \chi_4\bar{\psi}$

   orientation form is $\pi(\boldsymbol{c}_4) = \frac{1}{(1+\Omega^2)^2}\prod_{\mu=1}^{4}(\Gamma^\mu + \Omega\Gamma^{\mu+4})$, not Γ_9

The trace theorem

Dixmier trace (define $\langle\mathcal{D}\rangle = (\mathcal{D}^2+1)^{\frac{1}{2}}$)

$$\mathrm{Tr}_\omega(L_\star(f)\langle\mathcal{D}_4\rangle^{-d}) = \lim_{s\to 1}(s-1)\mathrm{Tr}(L_\star(f)\langle\mathcal{D}\rangle_4^{-ds})$$
$$= \lim_{s\to 1}\frac{s-1}{\Gamma(\frac{ds}{2})}\int_0^\infty dt\, t^{\frac{ds}{2}-1}\mathrm{Tr}(L_\star(f)e^{-t(H_4+\tilde{\Omega}\Sigma_4 1_2+1)})$$

Mehler and Moyal kernels

$$e^{-H_4 t}(x,y) = \frac{\tilde{\Omega}^2(1-\tanh^2(\tilde{\Omega}t))^2}{16\pi^2\tanh^2(\tilde{\Omega}t)}e^{-\frac{\tilde{\Omega}}{4}\frac{|x-y|^2}{\tanh(\tilde{\Omega}t)}-\frac{\tilde{\Omega}}{4}\tanh(\tilde{\Omega}t)|x+y|^2}$$

$$(L_\star(f))(x,y) = \int\frac{d^4z}{\pi^4\theta^4}f(z)\,\mathrm{e}^{2i(x\wedge y+y\wedge z+z\wedge x)}$$

(with $x\wedge y = x^\mu(\Theta^{-1})_{\mu\nu}y^\nu$)

for $f\in\mathcal{S}(\mathbb{R}^4)$: $\quad Tr_\omega(L_\star(f)\langle\mathcal{D}_4\rangle^{-4}) = \frac{tr_{c\ell}(1)}{32\pi^2(1+\Omega^2)^2}\int d^4z\, f(z)$

scalar product for smooth spinors from Dixmier trace and hermitian structure $(\ ,\ )_{\mathcal{A}_4}:\mathcal{H}_4^\infty\times\mathcal{H}_4^\infty\to\mathcal{A}_4$

Orientability

need unitisation $\tilde{\mathcal{A}}_4$ of $\tilde{\mathcal{A}}$ which contains plane waves $u_\mu = e^{ix_\mu}$

- Hochschild 4-cycle

$$\begin{aligned} \boldsymbol{c}_4 &= \frac{1}{4!(1+\Omega^2)^2} \sum_{\sigma\in S_4} (-1)^\sigma \\ &\quad \times \big(u_{\sigma(1)} \star u_{\sigma(2)} \star u_{\sigma(3)} \star u_{\sigma(4)}\big)^{-1_\star} \otimes u_{\sigma(1)} \otimes u_{\sigma(2)} \otimes u_{\sigma(3)} \otimes u_{\sigma(4)} \end{aligned}$$

- $\pi(\boldsymbol{c}_4) = \frac{1}{(1+\Omega^2)^2}(\Gamma_1 + \Omega\Gamma_5)(\Gamma_2 + \Omega\Gamma_6)(\Gamma_3 + \Omega\Gamma_7)(\Gamma_4 + \Omega\Gamma_8)$

  defines Hochschild 4-cocycle $\qquad (f \equiv L_\star(f))$

$$\begin{aligned} \phi_{\pi(\boldsymbol{c}_4)}(f_0, \dots, f_4) &= \mathrm{Tr}_\omega\big(\pi(\boldsymbol{c}_4) f_0 [\mathcal{D}_4, f_1] \cdots [\mathcal{D}_4, f_4] \langle \mathcal{D}_4 \rangle^{-4}\big) \\ &= \frac{1}{2\pi^2} \int f_0 \star df_1 \wedge_\star df_2 \wedge_\star df_3 \wedge_\star df_4 \end{aligned}$$

Note: A Hochschild 4-cocycle is obtained for $\pi(\boldsymbol{c}_4)$ replaced by any bounded operator which commutes with $\mathcal{A}$.

Is $\phi_{\pi(\boldsymbol{c}_4)}$ distinguished in the Hochschild cohomology of the non-unital algebra $\mathcal{A}$?

U(1)-Higgs model

tensor $(\mathcal{A}_4, \mathcal{H}_4, \mathcal{D}_4, \chi_4)$ with $(\mathbb{C}\oplus\mathbb{C}, \mathbb{C}^2, M\sigma_1)$ [Connes+Lott]

- $\mathcal{D} = \mathcal{D}_4 \otimes 1_2 + \Gamma_9 \otimes \sigma_1 M = \begin{pmatrix} \mathcal{D}_4 & M\Gamma_9 \\ M\Gamma_9 & \mathcal{D}_4 \end{pmatrix}$
- selfadjoint fluctuated Dirac operators $\mathcal{D}_A := \mathcal{D} + \sum_i a_i[\mathcal{D}, b_i]$, $a_i, b_i \in \mathcal{A} = \mathcal{A}_4 \oplus \mathcal{A}_4$ are of the form
  $$\mathcal{D}_A = \begin{pmatrix} \mathcal{D}_4 + (\Gamma^\mu + \Omega\Gamma^{\mu+4})L_\star(A_\mu) & \Gamma_9 L_\star(\phi) \\ \Gamma_9 L_\star(\bar\phi) & \mathcal{D}_4 + (\Gamma^\mu + \Omega\Gamma^{\mu+4})L_\star(B_\mu) \end{pmatrix}$$
  for $A_\mu = \overline{A_\mu}, B_\mu = \overline{B_\mu}, \phi \in \mathcal{A}_4$
- $\mathcal{D}_A^2 =$
  $$\begin{pmatrix} (H_4 + L_\star(\phi\star\bar\phi))1 + \Sigma_4 + F_A & i(\Gamma^\mu + \Omega\Gamma^{\mu+4})\Gamma_9 L_\star(D_\mu\phi) \\ i(\Gamma^\mu + \Omega\Gamma^{\mu+4})\Gamma_9 L_\star(\overline{D_\mu\phi}) & (H_4 + L_\star(\bar\phi\star\phi))1 + \Sigma_4 + F_B \end{pmatrix}$$
- $D_\mu\phi = \partial_\mu\phi - iA\star\phi + i\phi\star B$
  $$F_A = \{L_\star(A^\mu), i\partial_\mu + \Omega^2 M_\bullet(\tilde{x}_\mu)\} + (1+\Omega^2)L_\star(A_\mu \star A^\mu) + i\big(\tfrac{1}{4}[\Gamma^\mu, \Gamma^\nu] + \tfrac{1}{4}\Omega^2[\Gamma^{\mu+4}, \Gamma^{\nu+4}] + \Omega\Gamma^\mu\Gamma^{\nu+4}\big)L_\star(F^A_{\mu\nu})$$

Spectral action principle

most general form of bosonic action is $S(\mathcal{D}_A) = Tr\big(\chi(\mathcal{D}_A^2)\big)$

- Laplace transf. + asympt. expansion $e^{-t\mathcal{D}_A^2} = \sum_{n=-d/2}^{\infty} a_n(\mathcal{D}_A^2)t^n$

  lead to $S(\mathcal{D}_A) = \sum_{n=-d/2}^{\infty} \chi_n \mathrm{Tr}(a_n(\mathcal{D}_A^2))$ (d = dimension)

  with $\chi_z = \frac{1}{\Gamma(-z)} \int_0^\infty ds\, s^{-z-1}\chi(s)$ for $z \notin \mathbb{N}$

  $\chi_k = (-1)^k \chi^{(k)}(0)$ for $k \in \mathbb{N}$
- a_n – Seeley coefficients, must be computed from scratch

Duhamel expansion: $\mathcal{D}_A^2 = \mathrm{H}_0 - V$

$$e^{-t(\mathrm{H}_0-V)} = e^{-t\mathrm{H}_0} - \int_0^t dt_1 \frac{d}{dt_1}\Big(e^{-(t-t_1)(\mathrm{H}_0-V)} e^{-t_1\mathrm{H}_0}\Big)$$

$$= e^{-t\mathrm{H}_0} + \int_0^t dt_1 \Big(e^{-(t-t_1)(\mathrm{H}_0-V)} V e^{-t_1\mathrm{H}_0}\Big) \qquad \ldots\text{iteration}$$

Position space kernels

Mehler kernel

$$e^{-\mathrm{H}_0 t}(x,y) = \frac{\tilde{\Omega}^2(1-\tanh^2(\tilde{\Omega}t))^2}{16\pi^2\tanh^2(\tilde{\Omega}t)} e^{-t\tilde{\Omega}\Sigma_4 1_2 - \frac{\tilde{\Omega}}{4}\frac{|x-y|^2}{\tanh(\tilde{\Omega}t)} - \frac{\tilde{\Omega}}{4}\tanh(\tilde{\Omega}t)|x+y|^2}$$

(with $\tilde{\Omega} = \frac{2\Omega}{\theta} \quad \Theta = \theta S \quad S^2 = -1$)

vacuum trace

$$\mathrm{Tr}(e^{-t\mathrm{H}_0}) = \mathrm{tr}_{c\ell}\int d^4x\,(e^{-t\mathrm{H}_0})(x,x) = \frac{1}{8\sinh^4(\tilde{\Omega}t)}\mathrm{tr}_{c\ell}(e^{-t\tilde{\Omega}\Sigma_4})$$
$$= \frac{2}{\tanh^4(\tilde{\Omega}t)} = 2(\tilde{\Omega}t)^{-4} + \frac{8}{3}(\tilde{\Omega}t)^{-2} + \frac{52}{45} + \mathcal{O}(t^2)$$

- expansion starts with t^{-4} $\Rightarrow$ corresponds to 8-dim. space
- spectral action is finite, in contrast to pure Moyal at $\Omega = 0$
- view spacial regularisation as part of geometry

Vertex kernels (with $x \wedge y = x^\mu (\Theta^{-1})_{\mu\nu} y^\nu$)

$$(L_\star(f))(x,y) = \int \frac{d^4z}{\pi^4\theta^4} f(z)\, e^{2i(x\wedge y + y\wedge z + z\wedge x)}$$

$$\{L_\star(A^\mu), i\partial_\mu + \Omega^2 M_\bullet(\tilde{x}_\mu)\}(x,y)$$
$$= \int \frac{d^4z}{\pi^4\theta^4} \big(2\tilde{z}^\mu - (1-\Omega^2)(\tilde{x}^\mu + \tilde{y}^\mu)\big) A_\mu(z)\, e^{2i(x\wedge y + y\wedge z + z\wedge x)}$$

- one-vertex trace
$$\int_0^t dt_1 \,\mathrm{Tr}(e^{-(t-t_1)H_0} L_\star(f) e^{-t_1 H_0})$$
$$= \frac{\tilde{\Omega}^2 t}{16\pi^2(1+\Omega^2)^2 \sinh^2(\tilde{\Omega}t)} \int d^4z\, f(z)\, e^{-\frac{\tilde{\Omega}\tanh(\tilde{\Omega}t)}{1+\Omega^2}|z|^2}$$
- order is t^{-1} as in 4D-standard model
opposite sign of $t^{-1}, t^0 \Rightarrow$ spontaneous symmetry breaking
- in general: Gaußian integrations using determinant and inverse of $U \otimes 1_2 + V \otimes \sigma_2$ [R. Gurau+V. Rivasseau]

The spectral action

$$
\begin{aligned}
S(\mathcal{D}_A) = {} & \frac{\theta^4\chi_{-4}}{8\Omega^4} + \frac{2\theta^2\chi_{-2}}{3\Omega^2} + \frac{52\chi_0}{45} + \frac{\chi_0}{2\pi^2(1+\Omega^2)^2}\int d^4z\,\Big\{2D_\mu\phi\star\overline{D_\mu\phi} \\
& + \Big(\frac{(1-\Omega^2)^2}{2} - \frac{(1-\Omega^2)^4}{3(1+\Omega^2)^2}\Big)\big(F^A_{\mu\nu}\star F_A^{\mu\nu} + F^B_{\mu\nu}\star F_B^{\mu\nu}\big) \\
& + \Big(\phi\star\bar\phi + \frac{4\Omega^2}{1+\Omega^2}\tilde X_A^\mu\star\tilde X_{A\mu} - \frac{\chi_{-1}}{\chi_0}\Big)^2 \\
& + \Big(\bar\phi\star\phi + \frac{4\Omega^2}{1+\Omega^2}\tilde X_B^\mu\star\tilde X_{B\mu} - \frac{\chi_{-1}}{\chi_0}\Big)^2 \\
& - 2\Big(\frac{4\Omega^2}{1+\Omega^2}\tilde X_0^\mu\star\tilde X_{0\mu} - \frac{\chi_{-1}}{\chi_0}\Big)^2\Big\}(z) + \mathcal{O}(\chi_1)
\end{aligned}
$$

deeper entanglement of gauge and Higgs fields

covariant coordinates $\tilde X_{A\mu}(z) = (\Theta^{-1})_{\mu\nu}z^\nu + A_\mu(z)$ appear with Higgs field ϕ in unified potential

potential cannot be restricted to Higgs part if distinction into discrete and continuous geometries no longer possible

The spectral action

$$
\begin{aligned}
S(\mathcal{D}_A) = {} & \frac{\theta^4\chi_{-4}}{8\Omega^4} + \frac{2\theta^2\chi_{-2}}{3\Omega^2} + \frac{52\chi_0}{45} + \frac{\chi_0}{2\pi^2(1+\Omega^2)^2}\int d^4z\,\Big\{2D_\mu\phi\star\overline{D_\mu\phi} \\
& + \Big(\frac{(1-\Omega^2)^2}{2} - \frac{(1-\Omega^2)^4}{3(1+\Omega^2)^2}\Big)\big(F^A_{\mu\nu}\star F_A^{\mu\nu} + F^B_{\mu\nu}\star F_B^{\mu\nu}\big) \\
& + \Big(\phi\star\bar\phi + \frac{4\Omega^2}{1+\Omega^2}\tilde{X}^\mu_A\star\tilde{X}_{A\mu} - \frac{\chi_{-1}}{\chi_0}\Big)^2 \\
& + \Big(\bar\phi\star\phi + \frac{4\Omega^2}{1+\Omega^2}\tilde{X}^\mu_B\star\tilde{X}_{B\mu} - \frac{\chi_{-1}}{\chi_0}\Big)^2 \\
& -2\Big(\frac{4\Omega^2}{1+\Omega^2}\tilde{X}^\mu_0\star\tilde{X}_{0\mu} - \frac{\chi_{-1}}{\chi_0}\Big)^2\Big\}(z) + \mathcal{O}(\chi_1)
\end{aligned}
$$

further remarks

non-trivial vacuum for both gauge and Higgs fields; origin is Higgs mechanism with spontaneous symmetry breaking

coefficients in front of Yang-Mills action are positive for all $\Omega \in [0,1[$ (different for bosonic model)

The spectral action

$$S(\mathcal{D}_A) = \frac{\theta^4\chi_{-4}}{8\Omega^4} + \frac{2\theta^2\chi_{-2}}{3\Omega^2} + \frac{52\chi_0}{45} + \frac{\chi_0}{2\pi^2(1+\Omega^2)^2}\int d^4z\,\Big\{2D_\mu\phi\star\overline{D_\mu\phi}$$
$$+\Big(\frac{(1-\Omega^2)^2}{2} - \frac{(1-\Omega^2)^4}{3(1+\Omega^2)^2}\Big)\big(F^A_{\mu\nu}\star F^{\mu\nu}_A + F^B_{\mu\nu}\star F^{\mu\nu}_B\big)$$
$$+\Big(\phi\star\bar\phi + \frac{4\Omega^2}{1+\Omega^2}\tilde{X}^\mu_A\star\tilde{X}_{A\mu} - \frac{\chi_{-1}}{\chi_0}\Big)^2$$
$$+\Big(\bar\phi\star\phi + \frac{4\Omega^2}{1+\Omega^2}\tilde{X}^\mu_B\star\tilde{X}_{B\mu} - \frac{\chi_{-1}}{\chi_0}\Big)^2$$
$$-2\Big(\frac{4\Omega^2}{1+\Omega^2}\tilde{X}^\mu_0\star\tilde{X}_{0\mu} - \frac{\chi_{-1}}{\chi_0}\Big)^2\Big\}(z) + \mathcal{O}(\chi_1)$$

further remarks

spectral action is invariant under gauge transformations

$\phi \mapsto u_A\star\phi\star\overline{u_B}$, $\tilde{X}_{A\mu} \mapsto u_A\star\tilde{X}_{A\mu}\star\overline{u_A}$, $\tilde{X}_{B\mu} \mapsto u_B\star\tilde{X}_{B\mu}\star\overline{u_B}$

$$F^A_{\mu\nu} = -i[\tilde{X}_{A\mu},\tilde{X}_{A\nu}]_\star + (\Theta^{-1})_{\mu\nu}$$
$$D_\mu\phi = -i\tilde{X}_{A\mu}\star\phi + i\phi\star\tilde{X}_{B\mu}$$

The spectral action

$$
\begin{aligned}
S(\mathcal{D}_A) = {} & \frac{\theta^4\chi_{-4}}{8\Omega^4} + \frac{2\theta^2\chi_{-2}}{3\Omega^2} + \frac{52\chi_0}{45} + \frac{\chi_0}{2\pi^2(1+\Omega^2)^2}\int d^4z\,\Big\{ 2 D_\mu\phi \star \overline{D_\mu\phi} \\
& + \Big(\frac{(1-\Omega^2)^2}{2} - \frac{(1-\Omega^2)^4}{3(1+\Omega^2)^2}\Big)\big(F^A_{\mu\nu}\star F^{\mu\nu}_A + F^B_{\mu\nu}\star F^{\mu\nu}_B\big) \\
& + \Big(\phi\star\bar\phi + \frac{4\Omega^2}{1+\Omega^2}\tilde{X}^\mu_A \star \tilde{X}_{A\mu} - \frac{\chi_{-1}}{\chi_0}\Big)^2 \\
& + \Big(\bar\phi\star\phi + \frac{4\Omega^2}{1+\Omega^2}\tilde{X}^\mu_B \star \tilde{X}_{B\mu} - \frac{\chi_{-1}}{\chi_0}\Big)^2 \\
& - 2\Big(\frac{4\Omega^2}{1+\Omega^2}\tilde{X}^\mu_0 \star \tilde{X}_{0\mu} - \frac{\chi_{-1}}{\chi_0}\Big)^2\Big\}(z) + \mathcal{O}(\chi_1)
\end{aligned}
$$

further remarks

spectral action is translation-invariant!

$\phi(x) \mapsto \phi(x+a)$ $\quad$ $\tilde{X}_{G\mu}(x) \mapsto \tilde{X}_{G\mu}(x+a)$ $\quad$ for $G \in A, B, 0$

$\tilde{X}_{A\mu}(x)$ entirely, not as $\frac{1}{2}\tilde{x}_\mu + A_\mu(x) \mapsto \frac{1}{2}\tilde{x}_\mu + (\frac{1}{2}\tilde{a}_\mu + A_\mu(x+a))$ (changes class of functions)

The vacuum

gauge transformation:
solution ϕ^{vac} of field equations is real; implies $A_\mu^{vac} = B_\mu^{vac}$

vacuum field equations

$$\frac{1}{g^2}[\tilde{X}_{A\nu}, [\tilde{X}_A^\mu, \tilde{X}_A^\nu]_\star]_\star + 2[\phi, [\tilde{X}_A^\mu, \phi]_\star]_\star = \frac{4\Omega^2}{1+\Omega^2}\Big\{\phi\star\phi + \frac{4\Omega^2}{1+\Omega^2}\tilde{X}_A^\nu \star \tilde{X}_{A\nu} - \eta^2, \tilde{X}_A^\mu\Big\}_\star$$

$$2[\tilde{X}_{A\nu}, [\phi, \tilde{X}_A^\nu]_\star]_\star = \Big\{\phi\star\phi + \frac{4\Omega^2}{1+\Omega^2}\tilde{X}_A^\nu \star \tilde{X}_{A\nu} - \eta^2, \phi\Big\}_\star$$

(with $\frac{1}{4g^2} = \frac{(1-\Omega^2)^2}{2} - \frac{(1-\Omega^2)^4}{3(1+\Omega^2)^2}$ $\qquad \eta^2 = \frac{\chi_{-1}}{\chi_0}$)

Spectral action is translation-invariant, but if for order parameter $\phi\star\phi + \frac{4\Omega^2}{1+\Omega^2}\tilde{X}_A^\nu \star \tilde{X}_{A\nu} \neq \text{const.}$ $\Rightarrow$ $\exists$ distinguished points in $\mathbb{R}^4$

(spontaneous breaking of translation invariance)

Spectral action is isotropic

$\Rightarrow$ choose radial coordinates with any origin

General ansatz for the vacuum solution

$\phi^{vac}(x) = \overline{\phi^{vac}(x)} = f(|x|^2) \qquad \tilde{X}^{vac}_{A\mu} = \tilde{X}^{vac}_{B\mu} = \frac{\tilde{x}_\mu}{2}\xi(|x|^2)$

for radial functions f, ξ

Conjecture (work in progress)

For pure Yang-Mills theory ($\phi = 0$), the order parameter $\tilde{X}_\mu \star \tilde{X}^\mu$ solves an inhomogeneous Bessel differential equation

by-product:

Moyal for radial functions $a(x) = a(r), b(x) = b(r)$, $r = |x|^2$:

$$(a \star b)(x) = \sum_{n=0}^{\infty} \frac{(-1)^n \theta^{2n}}{n!\, r^{\frac{d}{2}-1}} \Big(r^{n+\frac{d}{2}-1} a^{(n)} b^{(n)} \Big)^{(n)}(r)$$