The matrix base of the Moyal space and interesting results obtained with it

Raimar Wulkenhaar

Mathematisches Institut der Westfälischen Wilhelms-Universität
Münster, Germany

Introduction

- Classical field theories for fundamental interactions (electroweak, strong, gravitational) of geometrical origin
- Quantum field theory for standard model (electroweak+strong) is renormalisable
- Gravity is not renormalisable

Renormalisation group interpretation

- space-time being smooth manifold $\Rightarrow$ gravity scaled away
- weakness of gravity determines Planck scale where geometry is something different

promising approach: noncommutative geometry
(unifies standard model with gravity [as classical field theories])

Can we make sense of renormalisation in NCG?

First step: construct quantum field theories on simple noncommutative geometries, e.g. the Moyal space

Moyal space

algebra of rapidly decaying functions over D-dimensional Euclidean space with $\star$-product

$$(a \star b)(x) = \int d^D y \frac{d^D k}{(2\pi)^D} \, a(x + \tfrac{1}{2}\Theta \cdot k) b(x+y) \, e^{iky}$$

where $\Theta = -\Theta^T \in M_D(\mathbb{R})$

- $\star$-product is associative, noncommutative, and most importantly: non-local
- construction of field theories with non-local interaction
- This non-locality has serious consequences for the renormalisation of the resulting quantum field theory

The UV/IR-mixing problem and its solution

- *observation*: euclidean quantum field theories on Moyal space suffer from UV/IR mixing problem which destroys renormalisability if quadratic divergences are present

Theorem

The quantum field theory defined by the action

$$S = \int d^4x \Big(\frac{1}{2} \phi \star (\Delta + \Omega^2 \tilde{x}^2 + \mu^2) \phi + \frac{\lambda}{4!} \phi \star \phi \star \phi \star \phi \Big)(x)$$

with $\tilde{x} = 2\Theta^{-1} \cdot x$, ϕ – real, Euclidean metric
is perturbatively renormalisable to all orders in λ.

The additional oscillator potential $\Omega^2 \tilde{x}^2$

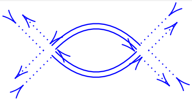
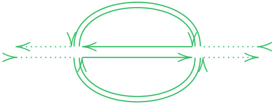
- implements mixing between large and small distance scales
- results from the renormalisation proof

Intuitive renormalisation "proof"

Langmann-Szabo duality

$$\left.\begin{array}{rcl} \tilde{x} & \longmapsto & p \\ \phi(x) & \longmapsto & \frac{1}{\sqrt{|\det \pi\Theta|}}\hat{\phi}(p) \end{array}\right\} + \text{Fourier transformation}$$

- leaves $\int d^4x\,(\phi\star\phi\star\phi\star\phi)(x)$ and $\int d^4x\,(\phi\star\phi)(x)$ invariant
- transforms $\int d^4x\,(\phi\star\Delta\phi)(x)$ into $\int d^4x\,(\phi\star\tilde{x}^2\phi)(x)$

- with  also its LS-dual is divergent
- also the LS-dual of  is divergent

renormalisation requires $\int d^4x\,(\phi\star\tilde{x}^2\phi)(x)$ in initial action

History of the renormalisation proof

- exact renormalisation group equation in matrix base
  [H. Grosse, R.W. (2004)]
  - simple interaction, complicated propagator
  - power-counting from decay rate and ribbon graph topology
- multi-scale analysis in matrix base
  [V. Rivasseau, F. Vignes-Tourneret, R.W. (2005)]
  - rigorous bounds for the propagator (requires large Ω)
- multi-scale analysis in position space
  [R. Gurau, J. Magnen, V. Rivasseau, F. Vignes-Tourneret (2006)]
  - simple propagator (Mehler kernel), oscillating vertex
  - distinction between sum and difference of propagator ends
- Schwinger parametric representation
  [R. Gurau, V. Rivasseau (2006)]
  - reduction to Symanzik type hyperbolic polynomials

The matrix base of the Moyal plane

- central observation (in 2D):
  $f_{00} := 2e^{-\frac{1}{\theta}(x_1^2+x_2^2)} \quad \Rightarrow \quad f_{00} \star f_{00} = f_{00}$
- left and right creation operators:

$$f_{mn}(x_1,x_2) = \frac{(x_1+ix_2)^{\star m}}{\sqrt{m!(2\theta)^m}} \star \Big(2e^{-\frac{1}{\theta}(x_1^2+x_2^2)}\Big) \star \frac{(x_1-ix_2)^{\star n}}{\sqrt{n!(2\theta)^n}}$$

$$f_{mn}(\rho,\varphi) = 2(-1)^m \sqrt{\frac{m!}{n!}} e^{i\varphi(n-m)} \Big(\sqrt{\tfrac{2}{\theta}}\rho\Big)^{n-m} e^{-\frac{\rho^2}{\theta}} L_m^{n-m}\big(\tfrac{2}{\theta}\rho^2\big)$$

- satisfies: $(f_{mn} \star f_{kl})(x) = \delta_{nk} f_{ml}(x)$

$$\int d^2x\, f_{mn}(x) = \sqrt{\det(2\pi\Theta)}\, \delta_{mn}$$

- Fourier transformation has the same structure

Extension to four dimensions

(non-vanishing components: $\theta = \Theta_{12} = -\Theta_{21} = \Theta_{34} = -\Theta_{43}$)

$$\phi(x) = \sum_{m_i, n_i \in \mathbb{N}} \phi_{\substack{m_1 \\ m_2} \substack{n_1 \\ n_2}} b_{\substack{m_1 \\ m_2} \substack{n_1 \\ n_2}}(x), \qquad b_{\substack{m_1 \\ m_2} \substack{n_1 \\ n_2}}(x) = f_{m_1 n_1}(x^1, x^2) f_{m_2 n_2}(x^3, x^4)$$

non-local $\star$-product becomes simple *matrix product*

$$S[\phi] = \sqrt{\det(2\pi\Theta)} \sum_{m,n,k,l \in \mathbb{N}^2} \Big(\frac{1}{2} \phi_{mn} \Delta_{mn;kl} \phi_{kl} + \frac{\lambda}{4!} \phi_{mn}\phi_{nk}\phi_{kl}\phi_{lm} \Big)$$

$$\begin{aligned}
\Delta_{mn;kl} &= \big(\mu^2 + \tfrac{2}{\theta}(1+\Omega^2)(m_1+n_1+m_2+n_2+2)\big) \delta_{n_1 k_1} \delta_{m_1 l_1} \delta_{n_2 k_2} \delta_{m_2 l_2} \\
&- \tfrac{2}{\theta}(1-\Omega^2)\big(\sqrt{k_1 l_1}\, \delta_{n_1+1,k_1} \delta_{m_1+1,l_1} + \sqrt{m_1 n_1}\, \delta_{n_1-1,k_1} \delta_{m_1-1,l_1}\big) \delta_{n_2 k_2} \delta_{m_2 l_2} \\
&- \tfrac{2}{\theta}(1-\Omega^2)\big(\sqrt{k_2 l_2}\, \delta_{n_2+1,k_2} \delta_{m_2+1,l_2} + \sqrt{m_2 n_2}\, \delta_{n_2-1,k_2} \delta_{m_2-1,l_2}\big) \delta_{n_1 k_1} \delta_{m_1 l_1}
\end{aligned}$$

important: $\Delta_{mn;kl} = 0$ unless $m - l = n - k$
($SO(2) \times SO(2)$ angular momentum conservation)

- $\Delta_{m,m+h;l+h,l} = \Delta^{(h)}_{ml}$ is band matrix (Jacobi matrix)
- diagonalisation of $\Delta^{(h)}$ yields recursion relation for Meixner polynomials $M_n(x;\beta,c) = {}_2F_1\left(\begin{smallmatrix}-n,-x\\ \beta\end{smallmatrix}\middle|1-c\right)$

$$\Delta_{\substack{m_1\\ m_2}\substack{m_1+h_1\\ m_2+h_2};\substack{l_1+h_1\\ l_2+h_2}\substack{l_1\\ l_2}} = \sum_{y_1,y_2=0}^{\infty} U^{(h_1)}_{m_1y_1} U^{(h_2)}_{m_2y_2} \Big(\mu^2 + \frac{4\Omega}{\theta}(2y_1+2y_2+h_1+h_2+2)\Big) U^{(h_1)}_{y_1l_1} U^{(h_2)}_{y_2l_2}$$

with

$$U^{(h)}_{ny} = \sqrt{\binom{h+n}{n}\binom{h+y}{y}}\,\Big(\frac{1-\Omega}{1+\Omega}\Big)^{n+y}\Big(\frac{2\sqrt{\Omega}}{1+\Omega}\Big)^{h+1}\,{}_2F_1\left(\begin{matrix}-n,-y\\ 1+h\end{matrix}\middle|\frac{4\Omega}{(1+\Omega)^2}\right)$$

- closed formula for propagator $G^{(h)} = (\Delta^{(h)})^{-1}$ thanks to

$$\sum_{x=0}^{\infty}\frac{(h+x)!}{x!h!}a^x\,{}_2F_1\left(\begin{matrix}-m,-x\\ 1+h\end{matrix}\middle|b\right){}_2F_1\left(\begin{matrix}-l,-x\\ 1+h\end{matrix}\middle|b\right) = \frac{(1-(1-b)a)^{m+l}}{(1-a)^{h+m+l+1}}\,{}_2F_1\left(\begin{matrix}-m,\,-l\\ 1+h\end{matrix}\middle|\frac{ab^2}{(1-(1-b)a)^2}\right), \qquad a<1$$

The propagator

$$G_{\substack{m_1 \\ m_2}\substack{m_1+h_1 \\ m_2+h_2};\substack{l_1+h_1 \\ l_2+h_2}\substack{l_1 \\ l_2}} = \frac{\theta}{8\Omega} \sum_{u_1=0}^{\min(m_1,l_1)} \sum_{u_2=0}^{\min(m_2,l_2)} \int_0^1 dt \, \frac{t^{\frac{\mu^2\theta}{8\Omega}+\alpha}(1-t)^\beta}{\left(1-\frac{(1-\Omega)^2}{(1+\Omega)^2}t\right)^{2+2\alpha+\beta}}$$

$$\times \left(\frac{1-\Omega}{1+\Omega}\right)^\beta \left(\frac{4\Omega}{(1+\Omega)^2}\right)^{2+2\alpha} \prod_{i=1}^{2} \frac{\sqrt{m_i!l_i!(m_i+h_i)!(l_i+h_i)!}}{(m_i-u_i)!(l_i-u_i)!(h_i+u_i)!u!}$$

$$= \frac{\theta}{2(1+\Omega)^2} \sum_{u_1=0}^{\min(m_1,l_1)} \sum_{u_2=0}^{\min(m_2,l_2)} {}_2F_1\left(\begin{matrix} 1+\beta\,, \ \frac{\mu^2\theta}{8\Omega}-\alpha \\ 2+\frac{\mu^2\theta}{8\Omega}+\alpha+\beta \end{matrix}\middle| \frac{(1-\Omega)^2}{(1+\Omega)^2}\right)$$

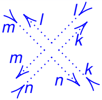
$$\times \left(\frac{1-\Omega}{1+\Omega}\right)^\beta B\left(1+\frac{\mu^2\theta}{8\Omega}+\alpha, 1+\beta\right) \prod_{i=1}^{2} \frac{\sqrt{m_i!l_i!(m_i+h_i)!(l_i+h_i)!}}{(m_i-u_i)!(l_i-u_i)!(h_i+u_i)!u!}$$

with $\alpha = \frac{1}{2}\sum_{i=1}^2 (h_i + 2u_i) \geq 0 \qquad \beta = \sum_{i=1}^2 (m_i + l_i - 2u_i) \geq 0$

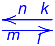
- all matrix elements $G_{mn;kl}$ non-negative, all sums finite
- $G^{(\mu=0)}_{\substack{m \\ 0}\substack{m \\ 0};\substack{m \\ 0}\substack{m \\ 0}} = \frac{\theta}{2(1+\Omega)^2(m+1)} \, {}_2F_1\left(\begin{matrix} 1, -m \\ m+2 \end{matrix}\middle| \frac{(1-\Omega)^2}{(1+\Omega)^2}\right) \sim \frac{\theta/8}{\sqrt{\frac{4}{\pi}(m+1)+\Omega^2(m+1)^2}}$

  $G^{(\mu=0)}_{\substack{m_1 \\ m_2}\substack{m_1 \\ m_2};\substack{0 \\ 0}\substack{0 \\ 0}} = \frac{\theta}{2(1+\Omega)^2(m_1+m_2+1)} \left(\frac{1-\Omega}{1+\Omega}\right)^{m_1+m_2}$

Ribbon graphs

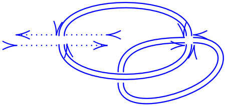
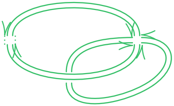
Feynman graphs are ribbon graphs with V vertices 

and

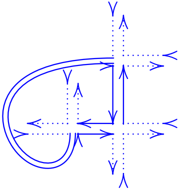
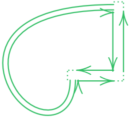
I edges 

$= G_{mn;kl}$ and N external legs

- leads to F faces, B of them with external legs
- ribbon graph can be drawn on Riemann surface of genus $g = 1 - \frac{1}{2}(F - I + V)$ with B holes

 $\mapsto$ 

$F = 1$	$g = 1$
$I = 3$	$B = 1$
$V = 2$	$N = 2$

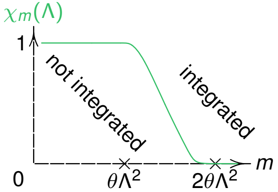
 $\mapsto$ 

$L = 2$	$g = 0$
$I = 3$	$B = 2$
$V = 3$	$N = 6$

First proof: exact renormalisation group equations

QFT defined via partition function $Z[J] = \int \mathcal{D}[\phi]\, e^{-S[\phi]-\mathrm{tr}(\phi J)}$

- Wilson's strategy: integration of field modes ϕ_{mn} with indices $\geq \theta\Lambda^2$ yields effective action $L[\phi,\Lambda]$
- variation of cut-off function $\chi(\Lambda)$ with Λ modifies effective action:



exact renormalisation group equation [Polchinski equation]

$$\Lambda\frac{\partial L[\phi,\Lambda]}{\partial\Lambda} = \sum_{m,n,k,l}\frac{1}{2}Q_{mn;kl}(\Lambda)\left(\frac{\partial L[\phi,\Lambda]}{\partial\phi_{mn}}\frac{\partial L[\phi,\Lambda]}{\partial\phi_{kl}} - \frac{1}{V_\Theta}\frac{\partial^2 L[\phi,\Lambda]}{\partial\phi_{mn}\,\partial\phi_{kl}}\right)$$

with $Q_{mn;kl}(\Lambda) = \Lambda\frac{\partial(G_{mn;kl}\,\chi_{mn;kl}(\Lambda))}{\partial\Lambda}$ $\qquad V_\Theta = \sqrt{\det(2\pi\Theta)}$

- renormalisation = proof that there exists a regular solution which depends on only a finite number of initial data

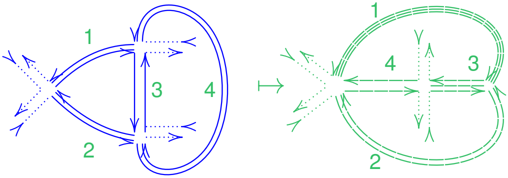
Second proof: multi-scale analysis

- propagator cut into slices: $G_{mn;kl} = \sum_{i=1}^{\infty} G^i_{mn;kl}$
  estimations:

$$0 \leq G^i_{mn;kl} \leq K_1 M^{-i} \, \mathrm{e}^{-c_1 M^{-i}(\|m\|+\|n\|+\|k\|+\|l\|)} \, \delta_{m-l,-(k-n)}$$

$$\sum_l \Big(\max_{n(l),k(l)} G^i_{mn;kl} \Big) \leq K_2 M^{-i} \, \mathrm{e}^{-c_2 M^{-i}\|m\|}$$

- induces scale attribution $i_\delta \in \mathbb{N}^+$ for each edge δ of the graph
- $SO(2) \times SO(2)$ symmetry implemented by dual graphs (vertices $\Leftrightarrow$ faces)



- index-difference (= angular momentum) conserved at propagators and vertices

index assignment in dual graphs

- given external indices
- reference indices at each internal vertex
- index differences between opposite sides of propagators in the complement of a maximal tree

$\Rightarrow$ $\sum_{\text{index differences}} \to$ factor M^{-i} preserved

$\sum_{\text{reference indices}} \to$ factor M^{2i} from $\sum_{m\in\mathbb{N}^2} \mathrm{e}^{-M^{-i}\|m\|}$

- power-counting degree of divergence for dual subgraphs
  $2\,\#(\text{inner vertices}) - \#(\text{edges})$
  $= 2(F-B)-I = 4-4g-2V+I-2B = (2-\frac{N}{2}) - 2(2g+B-1)$

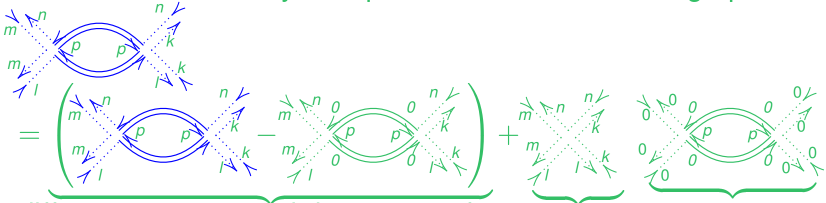
Conclusion

All non-planar graphs and all planar graphs with ≥ 4 external legs are convergent

Renormalisation

Problem: infinitely many planar 2- and 4-leg graphs diverge

Solution: discrete Taylor expansion about reference graphs:



difference expressed in terms of $|G_{np;pn} - G_{0p;p0}| \leq K_3 M^{-i} \frac{\|n\|}{M^i} e^{-c_3\|p\|}$

$=1$

put to renormalised value

- similar for all $A^{planar}_{mn;nk;kl;lm}$ $A^{planar}_{mn;nm}$ and $A^{planar}_{\substack{m^1 \\ m^2}+1\,\substack{n^1 \\ n^2}+1;\,\substack{n^1 \\ n^2}\,\substack{m^1 \\ m^2}}$

Renormalisation of noncommutative ϕ^4_4-model to all orders

by normalisation conditions for mass, field amplitude, coupling constant and oscillator frequency

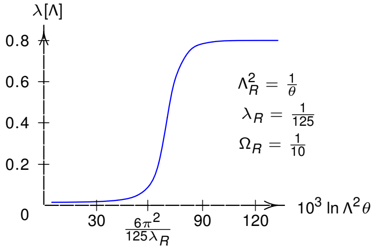
The β-function

one-loop calculation

$$\frac{\lambda[\Lambda]}{\Omega^2[\Lambda]} = \text{const}$$

$$\Omega^2[\Lambda] \leq 1$$

($\lambda[\Lambda]$ diverges in commutative case)



- perturbation theory remains valid at all scales!
- non-perturbative construction of the model seems possible!

How does this work?

- four-point function renormalisation with usual sign
- $\exists$ one-loop wavefunction renormalisation which compensates four-point function renormalisation for $\Omega \to 1$

The self-dual model

- $\Omega = 1$ leads to constant matrix indices for each face
- angular momentum ℓ is zero
  exponential decay in $|\ell|$ for general case
  $\Rightarrow$ self-dual model also captures general behaviour
- powerful techniques from matrix models available
  - exactly solvable complex scalar model
    [E. Langmann, R. Szabo, K. Zarembo, 2003]
  - renormalisation of ϕ^3_6 by relation to Kontsevich model
    [H. Grosse, H. Steinacker, 2006]

ingenious idea [M. Disertori, V. Rivasseau (2006)]

compute β-function for $\Omega = 1$
$\rightarrow$ model is asymptotically safe up to three loops
(cancellations established by formidable graph calculation)

Asymptotic safety to all orders

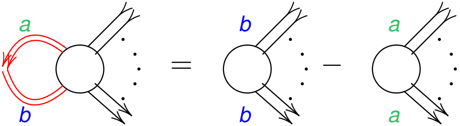
[M. Disertorti, R. Gurau, J. Magnen, V. Rivasseau (2006)]

Theorem

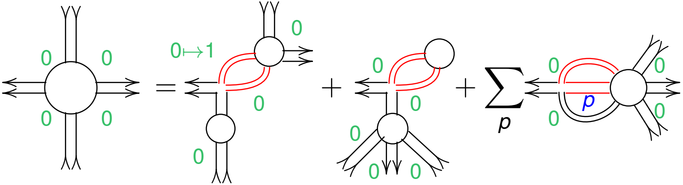
$\Gamma^4(0,0,0,0) = \lambda(1-(\partial\Sigma)(0,0))^2$ *to all orders in* λ *(up to irr.)*

where $(\partial\Sigma)(0,0) := \Sigma(1,0) - \Sigma(0,0)$ *Taylor subtraction*

Ward identity: $(a-b)$



Dyson equation



Summary

- Renormalisation is compatible with noncommutative geometry
- We can renormalise models with new types of degrees of freedom, such as dynamical matrix models
- Equivalence of renormalisation schemes is confirmed
- Important tools (multi-scale analysis) are worked out
- Rigorous construction of noncommutative quantum field theories is promising
- Other models

  1) Gross-Neveu model [F. Vignes-Tourneret (2006)]

  2) induced Yang-Mills theory

  [A. de Goursac, J.-C. Wallet, R.W.; H. Grosse, M. Wohlgenannt (2007)]