The harmonic oscillator, its noncommutative dimension and the vacuum of nc gauge theory

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(joint work with Harald Grosse, arxiv:0709.0095)

Wilson's view of quantum field theory

physical quantities (e.g. fields, action functional, ...) depend on some scale Λ

- effective action at scale Λ_1 is obtained by integration of degrees of freedom of scales Λ with $\Lambda_1 < \Lambda \leq \Lambda_0$, together with renormalisation of parameters
- yields renormalisation group flow of effective actions $(\Lambda_0 \searrow \Lambda_1)$

Q: What should we take for initial action at scale Λ_0?

A: Take any initial action!

Q: What about $\Lambda_0 \to \infty$?

A: This is possible, but not necessary!
At $\Lambda \ll \Lambda_0$ self-organisation of effective actions: only a few renormalisable actions survive, all others damped with $\left(\frac{\Lambda}{\Lambda_0}\right)^{\omega}$

Experimental facts

Value of Λ_0 is an experimental issue

- gravity is not renormalisable, thus strongly suppressed, but existent at low energy scales
- value of gravitational constant leads to $\Lambda_0 = \mathcal{O}(10^{19}\,\mathrm{GeV})$

Dominant (because renormalisable!) low-energy action functional is the standard model of particle physics

- standard model has a geometrical interpretation ...
  ...which is more beautiful in noncommutative geometry
- spectral action principle suggests natural effective action at scale $\Lambda_0 = \mathcal{O}(10^{15\ldots18}\,\mathrm{GeV})$
  prediction of Higgs mass $m_H = \mathcal{O}(180\,\mathrm{GeV})$

There is no need to discuss scales $> \Lambda_0$

Why noncommutative quantum field theory?

1. if Λ_0 is finite, the space with distinguished scale has some particular structure which we want to understand

   NCG-formulation of the standard model suggests that this understanding is possible through noncommutative geometry
2. methods developed in noncommutative field theory might help to cure problems of *perturbative* quantum field theory

First success stories

1. ϕ^4_4-theory on (noncommutative) Moyal space is
   - renormalisable to all orders [H. Grosse+R.W.]
   - asymptotically safe [H.G.+R.W.; M. Disertori+V. Rivasseau] to all orders [Disertori+Gurau+Magnen+Rivasseau]

   its non-perturbative construction is within reach
2. self-dual ϕ^3_6-theory on Moyal space is solvable in the genus expansion [H. Grosse+H. Steinacker]

Renormalisable QFT on Moyal space

Observation: Euclidean quantum field theories on Moyal space suffer from UV/IR mixing problem which destroys renormalisability if quadratic divergences are present

Theorem (H. Grosse + R.W.)

The quantum field theory defined by the action

$$S = \int d^4x \Big(\frac{1}{2}\phi \star \big(-\Delta + \Omega^2 \tilde{x}^2 + \mu^2\big)\phi + \frac{\lambda}{4!}\phi \star \phi \star \phi \star \phi\Big)(x)$$

with $\tilde{x} = 2\Theta^{-1} \cdot x$*,* ϕ *– real, Euclidean metric is perturbatively renormalisable to all orders in* λ*.*

The additional oscillator potential $\Omega^2 \tilde{x}^2$ implements mixing between large and small distance scales

What about gauge theory on Moyal space?

not renormalisable without oscillator potential [string theory]

- formulate Moyal gauge theory with sort of oscillator potential
- powerful strategy in NCG: spectral action principle [A. Connes, A. Chamseddine+A. Connes]

need Dirac operator with harmonic oscillator spectrum

first guess: $\mathcal{D} = \gamma^\mu(i\partial_\mu + \Omega\tilde{x}_\mu)$

- corresponding Gross-Neveu model is renormalisable to all orders [F. Vignes-Tourneret]
- describes constant magnetic background field, no oscillator (e.g. its spectrum is infinitely degenerate)

no progress with square root of $H = -\Delta + \Omega^2\|\tilde{x}\|^2$

Effective gauge theory with scalar fields

mimic physical interpretation of spectral action (to evaluate a fermionic one-loop calculation) with scalar fields to which external gauge fields couple minimally

same result in x-space [A. de Goursac+J.-C. Wallet+R.W.] and matrix base [H. Grosse+M. Wohlgenannt]:

Effective action

$$S_{eff} = \int d^4x \Big(c_1 \ln\frac{1}{\epsilon} F_{\mu\nu} F^{\mu\nu} + c_2 \ln\frac{1}{\epsilon} (\tilde{X}_\mu \star \tilde{X}^\mu)^2 + \frac{c_3}{\epsilon} \tilde{X}_\mu \star \tilde{X}^\mu \Big)$$

where $\tilde{X}_\mu = (\Theta^{-1})_{\mu\nu} x^\nu + A_\mu$ is a covariant coordinate

- A-linear terms imply that $A = 0$ is not a stable vacuum
- quantisation is always about classical solution; impossibility to solve the field equations obstructed any further progress

The dimension of the harmonic oscillator

simplest quantum-mechanical system:
one-dimensional harmonic oscillator with Hamiltonian
$H = -\frac{d^2}{dx^2} + \omega^2 x^2$ on $L^2(\mathbb{R})$

- configuration space $\mathbb{R} \ni x$ is one-dimensional,
  phase space $\mathbb{R}^2 \ni (x, p)$ is two-dimensional
  first example of a noncommutative geometry: $[x, p] = i$
- What is its spectral dimension?
  H^{-1} has eigenvalues $\mu_n = \frac{1}{\omega(2n+1)}$ so that
  H^{-1} is noncommutative infinitesimal of order one
- H generalises Laplacian $-\Delta$, and dimension is defined via
  Dirac operator $\mathcal{D}$, which is a square root of the Laplacian

Should $\mathcal{D} \sim H^{\frac{1}{2}}$ be of dimension two (phase space dimension)?

in two Clifford dimensions, take

$$\mathcal{D}_2 = i\sigma_1 \frac{d}{dx} + \sigma_2 \omega x = \begin{pmatrix} 0 & i(\frac{d}{dx}+\omega x) \\ i(\frac{d}{dx}-\omega x) & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & i\sqrt{2\omega}a \\ -i\sqrt{2\omega}a^\dagger & 0 \end{pmatrix} \qquad [a, a^\dagger] = 1$$

Can we extend $\mathcal{D}_2$ to a two-dimensional spectral triple?

it depends...

- $\mathcal{D}$ ist Dirac operator on $\mathcal{H}_2 = L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$, with $|\mathcal{D}|^{-2} \in \mathcal{L}^{(1,\infty)}(\mathcal{H}_2)$
- but $f|\mathcal{D}|^{-1} \in \mathcal{L}^{(1,\infty)}(\mathcal{H}_2)$ for $f \in \mathcal{A}$!

we will see: KO-dimension detects phase space,
spectral dimension detects configuration space

Spectral data for harmonic oscillator Moyal space

1. $\mathcal{D}_8 = (i\Gamma^\mu \partial_\mu + \Omega \Gamma^{\mu+4} \tilde{x}_\mu)$ $\quad \mu = 1, \dots, 4; \qquad \tilde{x}_\mu = 2(\Theta^{-1})_{\mu\nu} x^\nu$
   Dirac operator on Hilbert space $\mathcal{H}_8 = L^2(\mathbb{R}^4) \otimes \mathbb{C}^{16}$
   $\mathcal{D}_8^2 = (-\Delta + \Omega^2 \|\tilde{x}\|^2)\, 1 + \Sigma, \qquad \Sigma = -i\Omega(\Theta^{-1})_{\mu\nu}[\Gamma^\mu, \Gamma^{\nu+4}]$
2. $\{\Gamma_1, \dots, \Gamma_8\}$ – generators of 8D-Clifford algebra
   $\Gamma_k \Gamma_l + \Gamma_l \Gamma_k = 2\delta_{kl}$, chirality $\Gamma_9 := \Gamma_1 \cdots \Gamma_8$
3. algebra $\mathcal{A}_8 = \mathbb{C} \oplus \mathbb{R}^4_\Theta$ where $\mathbb{R}^4_\Theta = \mathcal{S}(\mathbb{R}^4)$ with
   $$(f \star g)(x) = \int d^4y \frac{d^4k}{(2\pi)^4} f(x + \tfrac{1}{2}\Theta k) g(x+y) e^{iky}$$
4. $\mathcal{A}_8$ acts on $\mathcal{H}_8$ componentwise by $L_\star(f)\psi = f \star \psi$
   $[\mathcal{D}_8, L_\star(f)] = i(\Gamma^\mu + \Omega \Gamma^{\mu+4}) L_\star(\partial_\mu f)$ is bounded
5. algebra generated by $[\mathcal{D}_8, \pi(\mathcal{A}_8)]$ and $\pi(\mathcal{A}_8)$ locally contains
   volume form $\pi(d^4x) = \frac{1}{(1+\Omega^2)^2} \prod_{\mu=1}^{4} (\Gamma^\mu + \Omega \Gamma^{\mu+4})$, but not Γ_9

Trace theorem

Dixmier trace $Tr_\omega(\mathrm{a}|\mathcal{D}_8|^{-n}) = \lim_{s\to 1}(s-1)\,Tr(\mathrm{a}|\mathcal{D}_8|^{-ns})$

for $a \in \mathbb{C}$, $f \in \mathbb{R}^4_\Theta$:

$$Tr_\omega(a|\mathcal{D}_8|^{-8} + L_\star(f)|\mathcal{D}_8|^{-4}) = \frac{tr_{c\ell}(1)}{192}\Big(\frac{\theta}{2\Omega}\Big)^4 a + \frac{tr_{c\ell}(1)}{32\pi^2(1+\Omega^2)^2}\int d^4x\, f(x)$$

hermitian form $(\;,\;)_{\mathcal{A}_8} : \mathcal{H}_8^\infty \times \mathcal{H}_8^\infty \to \mathbb{R}^4_\Theta$ with $(\psi,\psi')_{\mathcal{A}_8} = \sum_{i=1}^{16} \overline{\psi_i} \star \psi'_i$ and Dixmier trace give scalar product for smooth spinors: $\langle\psi,\psi'\rangle_{\mathcal{H}_8} = Tr_\omega\big((\psi,\psi')_{\mathcal{A}_8}|\mathcal{D}_8|^{-4}\big)$

Summary:

configuration space dimension (4)	phase space dimension (8)
domain of algebra + Hilbert space volume form	Clifford algebra dimension table } KO-dim.
spectrum of $\mathcal{D}$ on fields	[spectrum of $\mathcal{D}$ on vacuum]

U(1)-Higgs model

tensor $(\mathcal{A}_8, \mathcal{H}_8, \mathcal{D}_8, \Gamma_9)$ with $(\mathbb{C}\oplus\mathbb{C}, \mathbb{C}^2, M\sigma_1)$ [Connes+Lott]

- $\mathcal{D} = \mathcal{D}_8 \otimes 1_2 + \Gamma_9 \otimes \sigma_1 M = \begin{pmatrix} \mathcal{D}_8 & M\Gamma_9 \\ M\Gamma_9 & \mathcal{D}_8 \end{pmatrix}$
- selfadjoint fluctuated Dirac operators $\mathcal{D}_A := \mathcal{D} + \sum_i a_i[\mathcal{D}, b_i]$, $a_i, b_i \in \mathcal{A} = \mathcal{A}_8 \oplus \mathcal{A}_8$ are of the form
  $$\mathcal{D}_A = \begin{pmatrix} \mathcal{D}_8 + (\Gamma^\mu + \Omega\Gamma^{\mu+4})L_\star(A_\mu) & \Gamma_9 L_\star(\phi) \\ \Gamma_9 L_\star(\bar{\phi}) & \mathcal{D}_8 + (\Gamma^\mu + \Omega\Gamma^{\mu+4})L_\star(B_\mu) \end{pmatrix}$$
  for $A_\mu = \overline{A_\mu}, B_\mu = \overline{B_\mu}, \phi \in \mathbb{R}^4_\Theta$
- $$\mathcal{D}_A^2 = \begin{pmatrix} (H + L_\star(\phi\star\bar{\phi}))1 + \Sigma + F_A & i(\Gamma^\mu + \Omega\Gamma^{\mu+4})\Gamma_9 L_\star(D_\mu\phi) \\ i(\Gamma^\mu + \Omega\Gamma^{\mu+4})\Gamma_9 L_\star(\overline{D_\mu\phi}) & (H + L_\star(\bar{\phi}\star\phi))1 + \Sigma + F_B \end{pmatrix}$$
- $$D_\mu\phi = \partial_\mu\phi - iA\star\phi + i\phi\star B$$
  $$F_A = \{L_\star(A^\mu), i\partial_\mu + \Omega^2 M_\bullet(\tilde{x}_\mu)\} + (1+\Omega^2)L_\star(A_\mu \star A^\mu) + i\big(\tfrac{1}{4}[\Gamma^\mu, \Gamma^\nu] + \tfrac{1}{4}\Omega^2[\Gamma^{\mu+4}, \Gamma^{\nu+4}] + \Omega\Gamma^\mu\Gamma^{\nu+4}\big)L_\star(F^A_{\mu\nu})$$

Spectral action principle

most general form of bosonic action is $S(\mathcal{D}_A) = Tr\big(\chi(\mathcal{D}_A^2)\big)$

- Laplace transf. + asympt. expansion $e^{-t\mathcal{D}_A^2} = \sum_{n=-d/2}^{\infty} a_n(\mathcal{D}_A^2) t^n$ (d = dimension)

  lead to $S(\mathcal{D}_A) = \sum_{n=-d/2}^{\infty} \chi_n \,\mathrm{Tr}(a_n(\mathcal{D}_A^2))$

  with $\chi_{-n} = \frac{1}{\Gamma(n)} \int_0^\infty ds\, s^{n-1} \chi(s)$ for $n \in \mathbb{N}_+$

  $\chi_k = (-1)^k \chi^{(k)}(0)$ for $k \in \mathbb{N}$
- a_n – Seeley coefficients, must be computed from scratch

Duhamel expansion: $\mathcal{D}_A^2 = H_0 - V$

$$e^{-t(H_0-V)} = e^{-tH_0} - \int_0^t dt_1 \frac{d}{dt_1}\big(e^{-(t-t_1)(H_0-V)} e^{-t_1 H_0}\big)$$

$$= e^{-tH_0} + \int_0^t dt_1 \big(e^{-(t-t_1)(H_0-V)} V e^{-t_1 H_0}\big) \quad \ldots\text{iteration}$$

Position space kernels

Mehler kernel

$$e^{-H_0 t}(x,y) = \frac{\tilde{\Omega}^2(1-\tanh^2(\tilde{\Omega}t))^2}{16\pi^2\tanh^2(\tilde{\Omega}t)} e^{-t\Sigma 1_2 - \frac{\tilde{\Omega}}{4}\frac{|x-y|^2}{\tanh(\tilde{\Omega}t)} - \frac{\tilde{\Omega}}{4}\tanh(\tilde{\Omega}t)|x+y|^2}$$

(with $\tilde{\Omega} = \frac{2\Omega}{\theta}$ $\quad \Theta = i\theta\sigma_2 \otimes 1_2$)

vacuum trace

$$\mathrm{Tr}(e^{-t\mathrm{H}_0}) = \mathrm{tr}_{c\ell}\int d^4x\,(e^{-t\mathrm{H}_0})(x,x) = \frac{1}{8\sinh^4(\tilde{\Omega}t)}\mathrm{tr}_{c\ell}(e^{-t\Sigma})$$

expansion starts with t^{-4} $\Rightarrow$ corresponds to 8-dimensional space

Vertex kernels (with $x \wedge y = x^\mu (\Theta^{-1})_{\mu\nu} y^\nu$)

$$(L_\star(f))(x,y) = \int \frac{d^4z}{\pi^4\theta^4}\, f(z)\, \mathrm{e}^{2i(x\wedge y + y\wedge z + z\wedge x)}$$

$$\big\{L_\star(A^\mu), i\partial_\mu + \Omega^2 M_\bullet(\tilde{x}_\mu)\big\}(x,y)$$
$$= \int \frac{d^4z}{\pi^4\theta^4}\, \big(2\tilde{z}^\mu - (1-\Omega^2)(\tilde{x}^\mu + \tilde{y}^\mu)\big) A_\mu(z)\, \mathrm{e}^{2i(x\wedge y + y\wedge z + z\wedge x)}$$

- one-vertex trace
$$\int_0^t dt_1\, \mathrm{Tr}(e^{-(t-t_1)H_0} L_\star(f) e^{-t_1 H_0}) = \frac{\tilde{\Omega}^2 t}{16\pi^2(1+\Omega^2)^2 \sinh^2(\tilde{\Omega} t)} \int d^4z\, f(z)\, e^{-\frac{\tilde{\Omega}\tanh(\tilde{\Omega} t)}{1+\Omega^2}|z|^2}$$
- order is t^{-1} as in 4D-standard model
  opposite sign of $t^{-1}, t^0 \Rightarrow$ spontaneous symmetry breaking
- in general: Gaußian integrations using determinant and inverse of $U \otimes 1_2 + V \otimes \sigma_2$ [R. Gurau+V. Rivasseau]

The spectral action

$$S(\mathcal{D}_A) = \frac{\theta^4\chi_{-4}}{8\Omega^4} + \frac{\theta^2\chi_{-2}}{\Omega^2} + \frac{5\chi_0}{6} + \frac{\chi_0}{2\pi^2(1+\Omega^2)^2}\int d^4z\, \Big\{2D_\mu\phi\star\overline{D_\mu\phi}$$
$$+\Big(\frac{(1-\Omega^2)^2}{2} - \frac{(1-\Omega^2)^4}{3(1+\Omega^2)^2}\Big)\big(F^A_{\mu\nu}\star F_A^{\mu\nu} + F^B_{\mu\nu}\star F_B^{\mu\nu}\big)$$
$$+\Big(\phi\star\bar\phi + \frac{4\Omega^2}{1+\Omega^2}\tilde{X}^\mu_A\star\tilde{X}_{A\mu} - \frac{\chi_{-1}}{\chi_0}\Big)^2$$
$$+\Big(\bar\phi\star\phi + \frac{4\Omega^2}{1+\Omega^2}\tilde{X}^\mu_B\star\tilde{X}_{B\mu} - \frac{\chi_{-1}}{\chi_0}\Big)^2$$
$$-2\Big(\frac{4\Omega^2}{1+\Omega^2}\tilde{X}^\mu_0\star\tilde{X}_{0\mu} - \frac{\chi_{-1}}{\chi_0}\Big)^2\Big\}(z) + \mathcal{O}(\chi_1)$$

- deeper entanglement of gauge and Higgs fields: covariant coordinates $\tilde{X}_{A\mu}(z) = (\Theta^{-1})_{\mu\nu}z^\nu + A_\mu(z)$ appear with Higgs field ϕ in unified potential
- non-trivial vacuum for both gauge and Higgs fields; origin is Higgs mechanism with spontaneous symmetry breaking
- invariant under gauge transformations $\phi \mapsto u_A\star\phi\star\overline{u_B}$, $X_{A\mu}\mapsto u_A\star X_{A\mu}\star\overline{u_A}$, $X_{B\mu}\mapsto u_B\star X_{B\mu}\star\overline{u_B}$

The vacuum

Spectral action is translation-invariant!

- choose any origin for your coordinate system

Spectral action is isotropic!

- choose radial coordinates
- $\tilde{X}_\mu(x)$ points into radial or tangential direction, its amplitude is only a function of the radius
- ϕ is only a function of the radius

General ansatz for the vacuum solution

$\phi^{vac}(x) = \overline{\phi^{vac}(x)} = f(|x|^2) \qquad \tilde{X}^{vac}_{A\mu} = \tilde{X}^{vac}_{B\mu} = \frac{\tilde{x}_\mu}{2}\xi(|x|^2)$

for radial functions f, ξ

Field equations for f, ξ

insert f, ξ into spectral action, use $\star$-product for radial functions

$$4\xi\star\xi\star f' + 2|x|^2(\xi\star\xi\star f')' = f\star\Big(f\star f + \frac{4|x|^2\Omega^2}{\theta^2(1+\Omega^2)}\xi\star\xi - \eta^2\Big)$$

$$|x|^2(\xi\star\xi)'' + 3(\xi\star\xi)' - \gamma^2|x|^2\xi\star\xi = 4\theta^2 g^2 f'\star f' + \frac{8\Omega^2 g^2}{1+\Omega^2}(f\star f - \eta^2)$$

where $\gamma^2 = \frac{32g^2\Omega^4}{\theta^2(1+\Omega^2)^2}$ $\qquad \frac{1}{4g^2} = \frac{(1-\Omega^2)^2}{2} - \frac{(1-\Omega^2)^4}{3(1+\Omega^2)^2}$ $\qquad \eta^2 = \frac{\chi_{-1}}{\chi_0}$

2[nd] eq. is modified Bessel equation for $\tilde{X}_\mu\star\tilde{X}^\mu = |x|^2(\xi\star\xi)$

$$\tilde{X}_\mu\star\tilde{X}^\mu = -4g^2 K_1(\gamma|x|^2)\int_0^{|x|^2} dr^2\, r^2 I_1(\gamma r^2)\Big(\theta^2 f'\star f' + \frac{2\Omega^2}{1+\Omega^2}(f\star f - \eta^2)\Big)(r^2)$$

$$-4g^2 I_1(\gamma|x|^2)\int_{|x|^2}^{\infty} dr^2\, r^2 K_1(\gamma r^2)\Big(\theta^2 f'\star f' + \frac{2\Omega^2}{1+\Omega^2}(f\star f - \eta^2)\Big)(r^2)$$

to be inserted into first equation, no solution so far

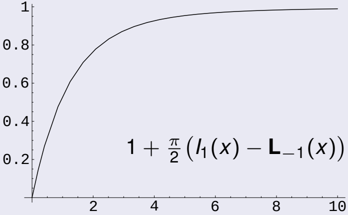
Complete solution for pure Yang-Mills theory

$$(\tilde{X}_\mu \star \tilde{X}^\mu)(x)\Big|^{vac}_{f=0} = \frac{\eta^2(1+\Omega^2)}{4\Omega^2}\Big(1 + \frac{\pi}{2}\big(I_1(\gamma|x|^2) - \mathbf{L}_{-1}(\gamma|x|^2)\big)\Big)$$

with modified Bessel and Struve functions

$$I_\nu(z) = \sum_{k=0}^{\infty} \frac{(\frac{z}{2})^{2k+\nu}}{\Gamma(k+1)\Gamma(k+\nu+1)}$$

$$\mathbf{L}_{-\nu}(z) = \sum_{k=0}^{\infty} \frac{(\frac{z}{2})^{2k-\nu+1}}{\Gamma(k+\frac{3}{2})\Gamma(k-\nu+\frac{3}{2})}$$



order parameter $(\tilde{X}_\mu \star \tilde{X}^\mu)^{vac}$ provides oscillator potential over short distances, which is dynamically generated by spontaneous symmetry breaking

potential is smoothly cut off at large distances, drastically changes multi-scale estimates used in renormalisaton

Next steps

- extract $\tilde{X}_\mu$ from $\tilde{X}_\mu \star \tilde{X}^\mu$ by passage through diagonal matrix base $f_{mm} = 2(-1)^m L_m(\frac{2|x|^2}{\theta}) e^{-\frac{|x|^2}{\theta}}$
- formulate field theory in moving orthonormal frame $\{\frac{x}{|x|}, \frac{\tilde{x}}{|\tilde{x}|}\}$ (for each symplectic pair)
- identify Feynman rules
- compute one-loop graphs
- is this model renormalisable?