

Euclidean quantum field theory on noncommutative spaces

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Wilson's view of quantum field theory

- physical quantities (e.g. fields, action functional, ...) depend on some **scale Λ**
- **effective action** at scale Λ_1 is obtained by integration of degrees of freedom of scale $\Lambda > \Lambda_1$, together with **renormalisation of parameters**
- yields **renormalisation group flow** of effective actions

Q: What should we take for **initial action** at scale Λ_0 ?

A: Take **any** initial action!

Q: What about $\Lambda_0 \rightarrow \infty$?

A: This is **possible, but not necessary!**

At $\Lambda \ll \Lambda_0$ **self-organisation** of effective actions: only **a few renormalisable actions** survive, all others damped with $(\frac{\Lambda}{\Lambda_0})^\omega$

Experimental facts

Value of Λ_0 is an experimental issue

- **gravity** is not renormalisable, thus strongly **suppressed, but existent** at low energy scales
- value of gravitational constant leads to $\Lambda_0 = \mathcal{O}(10^{19} \text{ GeV})$

Dominant (because renormalisable!) low-energy action functional is the *standard model of particle physics*

- standard model has a geometrical interpretation . . .
 . . . which is more beautiful in **noncommutative geometry**
- **spectral action principle** suggests natural effective action at scale $\Lambda_0 = \mathcal{O}(10^{15\dots 18} \text{ GeV})$
prediction of Higgs mass $m_H = \mathcal{O}(180 \text{ GeV})$

There is **no need to discuss scales $> \Lambda_0$**

Why noncommutative quantum field theory?

frequent motivation: **quantisation of gravity**

but: no theoretical/experimental need to go beyond **almost commutative geometry** of the standard model **at finite scale Λ_0**

nonetheless, it could be interesting to investigate more general NCG's to cure problems of **perturbative quantum field theory**

First success stories

- ϕ_4^4 -theory on (noncommutative) Moyal space is
 - renormalisable to all orders [H. Grosse+R.W.]
 - asymptotically safe [H.G.+R.W.; M. Disertori+V. Rivasseau]**its non-perturbative construction is within reach**
- self-dual ϕ_6^3 -theory on Moyal space is **solvable in the genus expansion** [H. Grosse+H. Steinacker]

A simple NCG: The Moyal space

Moyal space

algebra of rapidly decaying functions over D -dimensional Euclidean space with \star -product

$$(a \star b)(x) = \int d^D y \frac{d^D k}{(2\pi)^D} a(x + \frac{1}{2} \Theta \cdot k) b(x + y) e^{iky}$$

where $\Theta = -\Theta^T \in M_D(\mathbb{R})$

- \star -product is associative, noncommutative, and most importantly: **non-local**
- construction of field theories with **non-local interaction**
- This non-locality has serious consequences for the **renormalisation** of the resulting quantum field theory

The UV/IR-mixing problem and its solution

- *observation*: euclidean quantum field theories on Moyal space suffer from **UV/IR mixing** problem which destroys renormalisability if quadratic divergences are present

Theorem (H. Grosse + R.W.)

The quantum field theory defined by the action

$$S = \int d^4x \left(\frac{1}{2} \phi \star (\Delta + \Omega^2 \tilde{x}^2 + \mu^2) \phi + \frac{\lambda}{4!} \phi \star \phi \star \phi \star \phi \right) (x)$$

with $\tilde{x} = 2\Theta^{-1} \cdot x$, ϕ – real, Euclidean metric
is **perturbatively renormalisable to all orders** in λ .

The additional oscillator potential $\Omega^2 \tilde{x}^2$

- implements **mixing between large and small distance scales**
- results from the renormalisation proof

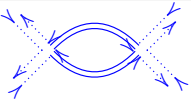
Intuitive renormalisation “proof”

Langmann-Szabo duality

$$\left. \begin{array}{l} \tilde{x} \mapsto p \\ \phi(x) \mapsto \frac{1}{\sqrt{|\det \pi \Theta|}} \hat{\phi}(p) \end{array} \right\} + \text{Fourier transformation}$$

- leaves $\int d^4x (\phi \star \phi \star \phi \star \phi)(x)$ and $\int d^4x (\phi \star \phi)(x)$ invariant

- transforms $\int d^4x (\phi \star \Delta \phi)(x)$ into $\int d^4x (\phi \star \tilde{x}^2 \phi)(x)$

- with  also its **LS-dual** is divergent

- also the **LS-dual** of  is divergent

renormalisation requires $\int d^4x (\phi \star \tilde{x}^2 \phi)(x)$ in initial action

History of the renormalisation proof

- **exact renormalisation group equation in matrix base**
[H. Grosse, R.W. (2004)]
 - simple interaction, complicated propagator
 - power-counting from decay rate and ribbon graph topology
- **multi-scale analysis in matrix base**
[V. Rivasseau, F. Vignes-Tourneret, R.W. (2005)]
 - rigorous bounds for the propagator (requires large Ω)
- **multi-scale analysis in position space**
[R. Gurau, J. Magnen, V. Rivasseau, F. Vignes-Tourneret (2006)]
 - simple propagator (Mehler kernel), oscillating vertex
 - distinction between sum and difference of propagator ends
- **Schwinger parametric representation**
[R. Gurau, V. Rivasseau (2006)]
 - reduction to Symanzik type hyperbolic polynomials

The matrix base of the Moyal plane

- central observation (in 2D):

$$f_{00} := 2e^{-\frac{1}{\theta}(x_1^2+x_2^2)} \quad \Rightarrow \quad f_{00} \star f_{00} = f_{00}$$

- left and right creation operators:

$$f_{mn}(x_1, x_2) = \frac{(x_1 + ix_2)^{\star m}}{\sqrt{m!(2\theta)^m}} \star \left(2e^{-\frac{1}{\theta}(x_1^2+x_2^2)} \right) \star \frac{(x_1 - ix_2)^{\star n}}{\sqrt{n!(2\theta)^n}}$$

$$f_{mn}(\rho, \varphi) = 2(-1)^m \sqrt{\frac{m!}{n!}} e^{i\varphi(n-m)} \left(\sqrt{\frac{2}{\theta}} \rho \right)^{n-m} e^{-\frac{\rho^2}{\theta}} L_m^{n-m} \left(\frac{2}{\theta} \rho^2 \right)$$

- satisfies: $(f_{mn} \star f_{kl})(x) = \delta_{nk} f_{ml}(x)$

$$\int d^2x f_{mn}(x) = \sqrt{\det(2\pi\Theta)} \delta_{mn}$$

- Fourier transformation has the same structure

Extension to four dimensions

(non-vanishing components: $\theta = \Theta_{12} = -\Theta_{21} = \Theta_{34} = -\Theta_{43}$)

$$\phi(x) = \sum_{m_i, n_i \in \mathbb{N}} \phi_{m_2 n_2}^{m_1 n_1} b_{m_2 n_2}^{m_1 n_1}(x), \quad b_{m_2 n_2}^{m_1 n_1}(x) = f_{m_1 n_1}(x^1, x^2) f_{m_2 n_2}(x^3, x^4)$$

non-local \star -product becomes simple *matrix product*

$$S[\phi] = \sqrt{\det(2\pi\Theta)} \sum_{m,n,k,l \in \mathbb{N}^2} \left(\frac{1}{2} \phi_{mn} \Delta_{mn;kl} \phi_{kl} + \frac{\lambda}{4!} \phi_{mn} \phi_{nk} \phi_{kl} \phi_{lm} \right)$$

$$\begin{aligned} \Delta_{mn;kl} = & \left(\mu^2 + \frac{2}{\theta} (1 + \Omega^2) (m_1 + n_1 + m_2 + n_2 + 2) \right) \delta_{n_1 k_1} \delta_{m_1 l_1} \delta_{n_2 k_2} \delta_{m_2 l_2} \\ & - \frac{2}{\theta} (1 - \Omega^2) \left(\sqrt{k_1 l_1} \delta_{n_1+1, k_1} \delta_{m_1+1, l_1} + \sqrt{m_1 n_1} \delta_{n_1-1, k_1} \delta_{m_1-1, l_1} \right) \delta_{n_2 k_2} \delta_{m_2 l_2} \\ & - \frac{2}{\theta} (1 - \Omega^2) \left(\sqrt{k_2 l_2} \delta_{n_2+1, k_2} \delta_{m_2+1, l_2} + \sqrt{m_2 n_2} \delta_{n_2-1, k_2} \delta_{m_2-1, l_2} \right) \delta_{n_1 k_1} \delta_{m_1 l_1} \end{aligned}$$

important: $\Delta_{mn;kl} = 0$ unless $m-l = n-k$

($SO(2) \times SO(2)$ angular momentum conservation)

- $\Delta_{m,m+h;l+h,l} = \Delta_{ml}^{(h)}$ is band matrix (Jacobi matrix)
- diagonalisation of $\Delta^{(h)}$ yields recursion relation for Meixner polynomials $M_n(x; \beta, c) = {}_2F_1\left(\begin{matrix} -n, -x \\ \beta \end{matrix} \middle| 1-c\right)$

$$\Delta_{\substack{m_1 & m_1+h_1 & \dots & l_1+h_1 & l_1 \\ m_2 & m_2+h_2 & \dots & l_2+h_2 & l_2}} \\ = \sum_{y_1, y_2=0}^{\infty} U_{m_1 y_1}^{(h_1)} U_{m_2 y_2}^{(h_2)} (\mu^2 + \frac{4\Omega}{\theta} (2y_1 + 2y_2 + h_1 + h_2 + 2)) U_{y_1 l_1}^{(h_1)} U_{y_2 l_2}^{(h_2)}$$

with

$$U_{ny}^{(h)} = \sqrt{\binom{h+n}{n} \binom{h+y}{y}} \left(\frac{1-\Omega}{1+\Omega}\right)^{n+y} \left(\frac{2\sqrt{\Omega}}{1+\Omega}\right)^{h+1} {}_2F_1\left(\begin{matrix} -n, -y \\ 1+h \end{matrix} \middle| \frac{4\Omega}{(1+\Omega)^2}\right)$$

- closed formula for propagator $G^{(h)} = (\Delta^{(h)})^{-1}$ thanks to

$$\sum_{x=0}^{\infty} \frac{(h+x)!}{x!h!} a^x {}_2F_1\left(\begin{matrix} -m, -x \\ 1+h \end{matrix} \middle| b\right) {}_2F_1\left(\begin{matrix} -l, -x \\ 1+h \end{matrix} \middle| b\right) \\ = \frac{(1-(1-b)a)^{m+l}}{(1-a)^{h+m+l+1}} {}_2F_1\left(\begin{matrix} -m, -l \\ 1+h \end{matrix} \middle| \frac{ab^2}{(1-(1-b)a)^2}\right), \quad a < 1$$

The propagator

$$\begin{aligned}
 G_{\substack{m_1 & m_1+h_1 & l_1+h_1 & l_1 \\ m_2 & m_2+h_2 & l_2+h_2 & l_2}} &= \frac{\theta}{8\Omega} \sum_{u_1=0}^{\min(m_1, l_1)} \sum_{u_2=0}^{\min(m_2, l_2)} \int_0^1 dt \frac{t^{\frac{\mu^2\theta}{8\Omega} + \alpha} (1-t)^\beta}{\left(1 - \frac{(1-\Omega)^2}{(1+\Omega)^2} t\right)^{2+2\alpha+\beta}} \\
 &\times \left(\frac{1-\Omega}{1+\Omega}\right)^\beta \left(\frac{4\Omega}{(1+\Omega)^2}\right)^{2+2\alpha} \prod_{i=1}^2 \frac{\sqrt{m_i! l_i! (m_i+h_i)! (l_i+h_i)!}}{(m_i-u_i)! (l_i-u_i)! (h_i+u_i)! u_i!} \\
 &= \frac{\theta}{2(1+\Omega)^2} \sum_{u_1=0}^{\min(m_1, l_1)} \sum_{u_2=0}^{\min(m_2, l_2)} {}_2F_1\left(1+\beta, \frac{\mu^2\theta}{8\Omega} - \alpha \mid \frac{(1-\Omega)^2}{(1+\Omega)^2}\right) \\
 &\times \left(\frac{1-\Omega}{1+\Omega}\right)^\beta B\left(1+\frac{\mu^2\theta}{8\Omega} + \alpha, 1+\beta\right) \prod_{i=1}^2 \frac{\sqrt{m_i! l_i! (m_i+h_i)! (l_i+h_i)!}}{(m_i-u_i)! (l_i-u_i)! (h_i+u_i)! u_i!}
 \end{aligned}$$

with $\alpha = \frac{1}{2} \sum_{i=1}^2 (h_i + 2u_i) \geq 0$ $\beta = \sum_{i=1}^2 (m_i + l_i - 2u_i) \geq 0$

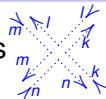
- all matrix elements $G_{mn;kl}$ **non-negative**, all sums **finite**

- $G_{\substack{m & m \\ 0 & 0}}^{(\mu=0)} = \frac{\theta}{2(1+\Omega)^2(m+1)} {}_2F_1\left(1, -m \mid \frac{(1-\Omega)^2}{(1+\Omega)^2}\right) \sim \frac{\theta/8}{\sqrt{\frac{4}{\pi}(m+1)+\Omega^2(m+1)^2}}$

$$G_{\substack{m_1 & m_1 \\ m_2 & m_2}}^{(\mu=0)} = \frac{\theta}{2(1+\Omega)^2(m_1+m_2+1)} \left(\frac{1-\Omega}{1+\Omega}\right)^{m_1+m_2}$$

Ribbon graphs

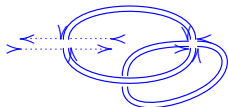
Feynman graphs are **ribbon graphs** with V vertices



and

I edges $\begin{matrix} \xleftarrow{n} & \xrightarrow{k} \\ \xrightarrow{m} & \xleftarrow{l} \end{matrix} = G_{mn;kl}$ and N external legs

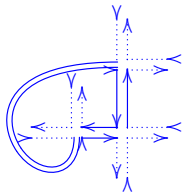
- leads to F faces, B of them with external legs
- ribbon graph can be drawn on **Riemann surface** of genus $g = 1 - \frac{1}{2}(F - I + V)$ with B holes



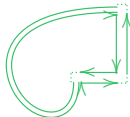
→



$$\begin{array}{ll} F = 1 & g = 1 \\ I = 3 & B = 1 \\ V = 2 & N = 2 \end{array}$$



→

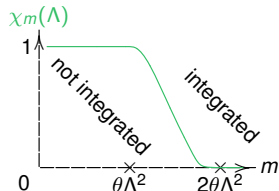


$$\begin{array}{ll} L = 2 & g = 0 \\ I = 3 & B = 2 \\ V = 3 & N = 6 \end{array}$$

First proof: exact renormalisation group equations

QFT defined via **partition function** $Z[J] = \int \mathcal{D}[\phi] e^{-S[\phi] - \text{tr}(\phi J)}$

- Wilson's strategy: integration of field modes ϕ_{mn} with indices $\geq \theta\Lambda^2$ yields **effective action** $L[\phi, \Lambda]$
- variation of cut-off function $\chi(\Lambda)$ with Λ modifies effective action:



exact renormalisation group equation [Polchinski equation]

$$\Lambda \frac{\partial L[\phi, \Lambda]}{\partial \Lambda} = \sum_{m,n,k,l} \frac{1}{2} Q_{mn;kl}(\Lambda) \left(\frac{\partial L[\phi, \Lambda]}{\partial \phi_{mn}} \frac{\partial L[\phi, \Lambda]}{\partial \phi_{kl}} - \frac{1}{V_{\Theta}} \frac{\partial^2 L[\phi, \Lambda]}{\partial \phi_{mn} \partial \phi_{kl}} \right)$$

with $Q_{mn;kl}(\Lambda) = \Lambda \frac{\partial (G_{mn;kl} \chi_{mn;kl}(\Lambda))}{\partial \Lambda}$

$$V_{\Theta} = \sqrt{\det(2\pi\Theta)}$$

- renormalisation = proof that there exists a **regular solution** which depends on only a **finite number of initial data**

Second proof: multi-scale analysis

- $$\int_0^1 d\alpha = \sum_{i=1}^{\infty} \int_{M^{-i}}^{M^{1-i}} d\alpha \quad \Rightarrow \quad G_{mn;kl} = \sum_{i=1}^{\infty} G_{mn;kl}^i \quad (\text{slices})$$

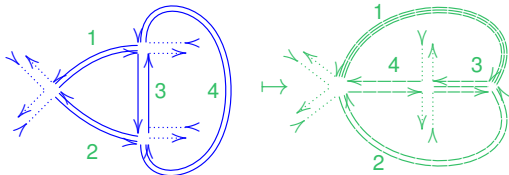
induces **scale attribution** $i_\delta \in \mathbb{N}^+$ for each edge δ of the graph

- estimations:

$$0 \leq G_{mn;kl}^i \leq K_1 M^{-i} e^{-c_1 M^{-i} (\|m\| + \|n\| + \|k\| + \|l\|)} \delta_{m-l, -(k-n)}$$

$$\sum_l \left(\max_{n(l), k(l)} G_{mn;kl}^i \right) \leq K_2 M^{-i} e^{-c_2 M^{-i} \|m\|}$$

- $SO(2) \times SO(2)$
 symmetry
 implemented by
dual graphs
 (vertices \Leftrightarrow faces)



- index-difference** (= angular momentum) conserved at propagators and vertices

index assignment in dual graphs

- given external indices
- reference indices at each internal vertex
- index differences between opposite sides of propagators in the **complement of a maximal tree**

$\Rightarrow \sum_{\text{index differences}} \rightarrow$ factor M^{-i} preserved

$\sum_{\text{reference indices}} \rightarrow$ factor M^{2i} from $\sum_{m \in \mathbb{N}^2} e^{-M^{-i} \|m\|}$

- **power-counting degree of divergence** for dual subgraphs
 $2 \#(\text{inner vertices}) - \#(\text{edges})$

$$= 2(F - B) - I = 4 - 4g - 2V + I - 2B = (2 - \frac{N}{2}) - 2(2g + B - 1)$$

Conclusion

All non-planar graphs and all planar graphs with > 4 external legs are convergent

Renormalisation

Problem: infinitely many planar 2- and 4-leg graphs diverge

Solution: discrete Taylor expansion about reference graphs:

difference expressed in terms of

$$|G_{np;pn} - G_{0p;p0}| \leq K_3 M^{-i} \frac{\|n\|}{M^i} e^{-c_3 \|p\|}$$

put to renormalised value

- similar for all $A_{mn;nk;kl;lm}^{planar}$, $A_{mn;nm}^{planar}$ and $A_{m^1+1, n^1+1; n^1, m^1}^{planar}$

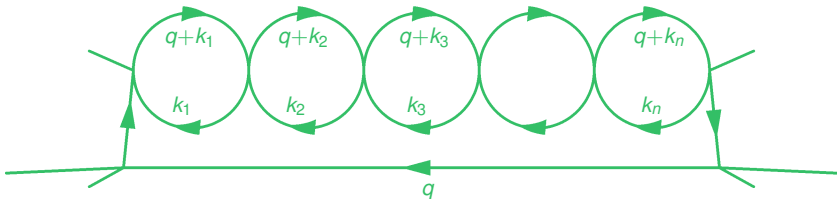
Renormalisation of noncommutative ϕ_4^4 -model to all orders

by normalisation conditions for mass, field amplitude, coupling constant and **oscillator frequency**

Reminder: commutative ϕ_4^4 -theory

perturbatively renormalisable according to BPHZ-theorem

renormalised amplitudes are finite order by order in perturbation theory, but also increasingly larger with the order:



$$\sim \int \frac{d^4 q}{(q^2 + m^2)^3} \left(\ln \frac{q}{m} \right)^n \stackrel{n \rightarrow \infty}{\simeq} c^n n!$$

This is the renormalon problem, it destroys Borel summability of the amplitudes

Constructive renormalisation

avoid Taylor subtraction at 0 and organise perturbation into **power series of effective coupling constants** $(\lambda_i)_{i \in \mathbb{N}}$ at scale Λ_i related by renormalisation group flow

- one finds: $\lambda_{i-1} = \lambda_i - \beta \lambda_i^2$
with the β -function obtained from one-loop bubble graph
- fix normalisation condition at low-energy scale Λ_0 :
solution $\lambda_i = \frac{\lambda_0}{1 - i\beta\lambda_0}$ for $\beta > 0$: pole at finite i
(**Landau ghost**)
- fix bare coupling at high energy scale Λ_N
solution $\lambda_i = \frac{\lambda_N}{1 + (N - i)\beta\lambda_N}$ for $\beta > 0, N \rightarrow \infty$
triviality of ϕ_4^4

similar for QED: almost killed renormalisation theory;

rescue is non-Abelian Yang-Mills theory where β is negative

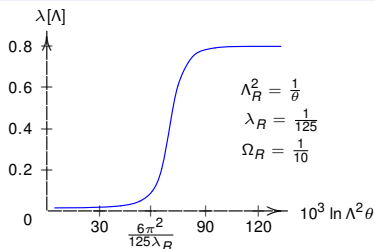
The β -function of noncommutative ϕ_4^4 -theory

one-loop calculation

$$\frac{\lambda[\Lambda]}{\Omega^2[\Lambda]} = \text{const}$$

$$\Omega^2[\Lambda] \leq 1$$

($\lambda[\Lambda]$ diverges in commutative case)



- perturbation theory remains valid at all scales!
- **non-perturbative construction of the model seems possible!**

How does this work?

- four-point function renormalisation with usual sign
- \exists **one-loop wavefunction renormalisation** which compensates four-point function renormalisation for $\Omega \rightarrow 1$

The self-dual model

- $\Omega = 1$ leads to **constant matrix indices** for each face
- **angular momentum ℓ is zero**
exponential decay in $|\ell|$ for general case
 \Rightarrow self-dual model also captures general behaviour
- **powerful techniques from matrix models available**
 - exactly solvable complex scalar model
[E. Langmann, R. Szabo, K. Zarembo, 2003]
 - renormalisation of ϕ_6^3 by relation to Kontsevich model
[H. Grosse, H. Steinacker, 2006]

ingenious idea [M. Disertori, V. Rivasseau (2006)]

compute β -function for $\Omega = 1$

\rightarrow model is **asymptotically safe up to three loops**

(cancellations established by formidable graph calculation)

Asymptotic safety to all orders

[M. Disertorti, R. Gurau, J. Magnen, V. Rivasseau (2006)]

Theorem

$\Gamma^4(0, 0, 0, 0) = \lambda(1 - (\partial\Sigma)(0, 0))^2$ to all orders in λ (up to irr.)
where $(\partial\Sigma)(0, 0) := \Sigma(1, 0) - \Sigma(0, 0)$ Taylor subtraction

Ward identity: $(a - b)$

Schwinger-Dyson equation

Summary

- Renormalisation is **compatible** with noncommutative geometry
- We can renormalise models with **new types of degrees of freedom**, such as dynamical matrix models
- **Equivalence** of renormalisation schemes is confirmed
- Important tools (**multi-scale analysis**) are worked out
- **Rigorous construction** of noncommutative quantum field theories is promising
- **Other models**
 - 1) Gross-Neveu model [F. Vignes-Tourneret (2006)]
 - 2) induced Yang-Mills theory
[A. de Goursac, J.-C. Wallet, R.W.; H. Grosse, M. Wohlgenannt (2007)]