

Quantum field theory on projective modules

Raimar Wulkenhaar*

joint work with Victor Gayral, Jan-Hendrik Jureit, Thomas Krajewski

* Mathematisches Institut der Westfälischen Wilhelms-Universität
Münster, Germany



Introduction

- Classical field theories for fundamental interactions (electroweak, strong, gravitational) of **geometrical origin**
- Quantum field theory for standard model (electroweak+strong) is **renormalisable**
- **Gravity is not renormalisable**

Renormalisation group interpretation

- space-time being smooth manifold \Rightarrow **gravity scaled away**
- weakness of gravity determines **Planck scale where geometry is something different**

promising approach: **noncommutative geometry**
(unifies standard model with gravity [as classical field theories])

Scalar fields and projective modules

- classical picture: **scalar fields** on (space-time) manifold M are **sections of some vector bundle** \mathcal{V} over M

Serre-Swan theorem (M – compact)

space of sections of \mathcal{V} \Leftrightarrow **finitely generated projective module** \mathcal{E} over algebra $C(M)$ of continuous functions

- $\{U_i\}_{i=1,\dots,N}$ – open cover of M
 $|f_i|^2$ – partition of unity, g_{ij} – transition functions on $U_i \cap U_j$
- $e_{ij} := f_i^* g_{ij} f_j \in C(M)$ satisfy $\sum_{j=1}^N e_{ij} e_{jk} = e_{ik}$
 $\mathcal{E} = e(C(M))^N$ projective module over $C(M)$
- generalisation: **A - noncommutative C^* -algebra**
 $e \in M_N(A)$ projection, $\rightarrow \mathcal{E} = eA^N$

Connections

- pass to **smooth level**: \mathcal{A} – Fréchet pre- C^* -algebra of A

- **differential algebra** $\Omega(\mathcal{A}) = \bigoplus_{n \in \mathbb{N}} \Omega^n(\mathcal{A})$, $\Omega^0(\mathcal{A}) = \mathcal{A}$

– differential $d : \Omega^n(\mathcal{A}) \rightarrow \Omega^{n+1}(\mathcal{A})$

– scalar product $\langle \cdot, \cdot \rangle_n$ on $\Omega^n(\mathcal{A})$

(e.g. obtained from spectral triple and Dixmier trace)

- **hermitian structure** $(\cdot, \cdot)_{\mathcal{A}} : \mathcal{E} \times \mathcal{E} \rightarrow \mathcal{A}$

– yields scalar product $\langle \cdot, \cdot \rangle_{\mathcal{A}, n}$ on $\mathcal{E} \otimes_{\mathcal{A}} \Omega^n(\mathcal{A})$

- **connection** $\nabla : \mathcal{E} \rightarrow \mathcal{E} \otimes_{\mathcal{A}} \Omega^1(\mathcal{A})$

– fulfilling $d(\psi, \eta)_{\mathcal{A}} = (\nabla\psi, \eta)_{\mathcal{A}} + (\psi, \nabla\eta)_{\mathcal{A}} \in \Omega^1(\mathcal{A})$

Noncommutative torus

$$\mathcal{A}_\Theta^d := \left\{ \begin{array}{l} \mathbf{a} = \sum_{\gamma \in \mathbb{Z}^d} \mathbf{a}_\gamma U_\gamma : \\ U_\gamma \text{-unitary, } U_\gamma U_{\gamma'} = e^{-i\pi\Theta(\gamma,\gamma')} U_{\gamma+\gamma'}, \{ \mathbf{a}_\gamma \} \in \mathcal{S}(\mathbb{Z}^d) \end{array} \right\}$$

$\Theta = -\Theta^t \in M_d(\mathbb{R})$ defines 2-cocycle on $\mathbb{Z}^d \subset \mathbb{R}^d$

$\Theta_{ij} \in \mathbb{Z}$: \mathcal{A}_Θ^d = algebra of functions on ordinary torus

$\Theta_{ij} \in \mathbb{Q}$: \mathcal{A}_Θ^d = bundle of matrix algebras over ordinary torus

$\Theta_{ij} \notin \mathbb{Q}$: \mathcal{A}_Θ^d = truly noncommutative space

- **Fréchet semi-norms** $p_n(\mathbf{a}) := \sup_{\gamma \in \mathbb{Z}^d} (1 + \|\gamma\|^2)^n |\mathbf{a}_\gamma|$
- **derivations** $\delta_\mu(U_\gamma) = 2i\pi\gamma_\mu U_\gamma \quad \gamma = (\gamma_1, \dots, \gamma_d) \in \mathbb{Z}^d$
- $\Omega^n(\mathcal{A}_\Theta^d) = \{(\omega_{\mu_1, \dots, \mu_n}) \in \Lambda^n((\mathcal{A}_\Theta^d)^d) - \text{completely antisymm.}\}$
- **trace** $\text{Tr}_{\mathcal{A}_\Theta^d} \left(\sum_{\gamma \in \mathbb{Z}^d} \mathbf{a}_\gamma U_\gamma \right) := \mathbf{a}_0$

irrational \mathcal{A}_Θ^d is nc manifold without boundary, $\text{Tr}_{\mathcal{A}_\Theta^d}(\delta_\mu U_\gamma) = 0$

Heisenberg modules

- $G = \mathbb{R}^p \times \mathbb{Z}^q \times F$ – abelian group, \widehat{G} – dual of G
- **Heisenberg group** = central extension of $G \times \widehat{G} \ni (g, \mu)$, acts on Hilbert space $L^2(G, dg) \ni \psi$ by

$$(T_{g,\mu}\psi)(x) := \mu(g)^{\frac{1}{2}} \mu(x) \psi(x + g)$$

- $\Gamma \simeq \mathbb{Z}^d$ **lattice** in $G \times \widehat{G}$; restriction of $G \times \widehat{G}$ to $\Gamma \ni \gamma = (g, \mu)$ yields right action of \mathcal{A}_Θ^d on $L^2(G, dg)$:

$$\psi U_\gamma := T_{g,\mu}\psi$$

Heisenberg cocycle $e^{-2i\pi\Theta(\gamma,\gamma')} := \mu(g')\mu'(g)^{-1}$

- $(G \times \widehat{G})/\Gamma$ compact (enforces d even)
 $\Rightarrow \mathcal{E}_H := \mathcal{S}(G)$ projective module (**Heisenberg module**)

hermitian structure $(\psi, \chi)_{\mathcal{A}_\Theta} := \sum_{\gamma \in \Gamma} \langle \psi, \chi U_\gamma \rangle_{L^2(G, dg)} U_{-\gamma}$

connection from infinitesimal action of $G \times \widehat{G}$

Simplest example: **Schwartz module** $\mathcal{E}_S = \mathcal{S}(\mathbb{R})$ over \mathcal{A}_θ^2

i.e. $d = 2$ and $G = \widehat{G} = \mathbb{R}$, $\Theta = \begin{pmatrix} 0 & \theta \\ -\theta & 0 \end{pmatrix}$

- $(\phi U_\gamma)(x) := e^{i\pi\theta mn} e^{2i\pi nx} \phi(x+m\theta)$, $\gamma = (\theta m, 2\pi n) \in \Gamma$

- hermitian structure

$$(\phi, \chi)_{\mathcal{A}_\theta^2} = \sum_{\gamma \in \Gamma} \left(e^{i\pi\theta mn} \int_{\mathbb{R}} dx \bar{\phi}(x) e^{2i\pi nx} \chi(x+\theta m) \right) U_{-\gamma}$$

- covariant derivatives compatible with hermitian structure:

$$(\nabla_1 \phi)(x) = -\frac{2i\pi x}{\theta} \phi(x) \quad (\nabla_2 \phi)(x) = \phi'(x)$$

extended to $\mathcal{S}(\mathbb{R}^{\frac{d}{2}})$ over $\mathcal{A}_\theta^d := (\mathcal{A}_\theta^2)^{\frac{d}{2}}$, interesting case is $d = 4$

Bargmann module

Bargmann space $\mathcal{H}_B = L^2_{hol}(\mathbb{C}, d\mu)$ of holomorphic functions

- scalar product $\langle \phi, \chi \rangle_B := \int d\mu(z, \bar{z}) \bar{\phi}(\bar{z}) \chi(z)$

$$\text{measure } d\mu(z, \bar{z}) = \frac{\omega}{\pi} e^{-\omega|z|^2} d\Re(z) d\Im(z) \quad \omega = \frac{2\pi}{\theta}$$

- projective repr. $(T_v \phi)(z) := e^{-\frac{\omega|v|^2}{2} - \omega \bar{v}z} \phi(z+v)$ of $v \in \mathbb{C}$
satisfies $T_v T_w = e^{\frac{\omega}{2}(\bar{v}w - \bar{w}v)} T_{v+w}$

yields right action $(\phi U_\gamma)(z) := (T_{\tilde{\gamma}} \phi)(z)$ of \mathcal{A}_θ^2 if
 $\tilde{\gamma} = \frac{\theta}{\sqrt{2}}(m + in)$ for $\gamma = (\theta m, 2\pi n) \in \Gamma$

- **Bargmann transform** $B : \mathcal{H}_S \rightarrow \mathcal{H}_B$

$$(B\chi)(z) := \left(\frac{\omega}{\pi}\right)^{\frac{1}{2}} \int_{\mathbb{R}} dx e^{-\frac{\omega}{2}(x^2 - 2\sqrt{2}zx + z^2)} \chi(x)$$

$$(B^{-1}\phi)(x) := \left(\frac{\omega}{\pi}\right)^{\frac{1}{2}} \int_{\mathbb{C}} d\mu(z, \bar{z}) e^{-\frac{\omega}{2}(x^2 - 2\sqrt{2}\bar{z}x + \bar{z}^2)} \phi(z)$$

transports structures from Schwartz to Bargmann module

Scalar field theory

$$\phi \in \mathcal{E} = \mathcal{S}(\mathbb{R}^2)$$

$$\phi^\dagger(\chi) := (\phi, \chi)_{\mathcal{A}_\theta^4}$$

$$\phi^\dagger \in \mathcal{E}^* = \{\mathcal{A}_\theta^4\text{-linear forms on } \mathcal{E}\}$$

Action functional on 2-dim. Schwartz module over \mathcal{A}_θ^4

$$\mathcal{S}[\phi, \phi^\dagger]$$

$$:= \langle \nabla \phi, \nabla \phi \rangle_{\mathcal{A}_\theta^4, 1} + \text{Tr}_{\mathcal{A}_\theta^4} \left(\mu^2 (\phi, \phi)_{\mathcal{A}_\theta^4} + \frac{\lambda}{2} (\phi, \phi)_{\mathcal{A}_\theta^4}^2 \right)$$

$$= \int_{\mathbb{R}^2} dx \bar{\phi}(x) \left(-\frac{\partial^2}{\partial x_i \partial x_i} + \frac{4\pi^2}{\theta^2} x_i x_i + \mu^2 \right) \phi(x)$$

$$+ \frac{\lambda}{2} \int_{\mathbb{R}^2} dx \sum_{m, n \in \mathbb{Z}^2} \bar{\phi}(x+n+\theta m) \phi(x+n) \bar{\phi}(x) \phi(x+\theta m)$$

- non-local interaction
- harmonic oscillator term $x^2 |\phi(x)|^2$ appears automatically
(ensures renormalisation of scalar models on Moyal plane)

Relation to matrix models ($d = 2$)

interaction term $\int_{\mathbb{R}} dx dy dz dt V(x, y, z, t) \bar{\phi}(x) \phi(y) \bar{\phi}(z) \phi(t)$

with $V(x, y, z, t) = \frac{\lambda}{2} \sum_{m, n \in \mathbb{Z}} \delta(y - x - m\theta) \delta(z - x - m\theta - n) \delta(t - x - n)$

local interaction on quotient space $\mathbb{R}/(\mathbb{Z} + \theta\mathbb{Z})$

meaningful for $\theta = \frac{p}{q} \in \mathbb{Q} \Rightarrow$ models of $p \times q$ rectangular matrices

- F = vector bundle over 2-torus \mathbb{T}^2 of radius $\frac{1}{q}$

$\Gamma^\infty(F) := \{ \mathcal{M} : [0, \frac{1}{q}] \times [0, \frac{1}{q}] \rightarrow M(p \times q, \mathbb{C}) \text{ smooth,}$

$$\mathcal{M}(x, y + \frac{1}{q}) = \mathcal{M}(x, y)$$

$$\mathcal{M}(x + \frac{1}{q}, y) = (\Omega_p)^a(qy) \mathcal{M}(x, y) (\Omega_q)^{-b}(-qy) \}$$

$$aq + bp = 1, \quad \Omega_N(y) = \begin{pmatrix} 0 & 1 & & & \\ & \ddots & \ddots & & \\ & & \ddots & \ddots & \\ e^{2i\pi y} & & & 1 & \\ & & & & 0 \end{pmatrix} \in M(N \times N, \mathbb{C})$$

relation to Schwartz module:

$$\rho : \mathcal{S}(\mathbb{R}) \rightarrow \Gamma^\infty(F), \quad \phi \mapsto \mathcal{M}_{kl}(x, y) := \sum_{n \in \mathbb{Z}} \phi\left(x + \frac{kq + lp + npq}{q}\right) e^{-2i\pi nqy}$$

$$\text{inverse } \rho^* : \Gamma^\infty(F) \rightarrow \mathcal{S}(\mathbb{R}), \quad \mathcal{M}(x, y) \mapsto \phi(x) := q \int_0^{\frac{1}{q}} dy \mathcal{M}_{00}(x, y)$$

- scalar product

$$\langle \mathcal{M}, \mathcal{N} \rangle_F := q \int_0^{\frac{1}{q}} dx \int_0^{\frac{1}{q}} dy \operatorname{Tr}(\mathcal{M}^\dagger(x, y) \mathcal{N}(x, y)) = \langle \rho^* \mathcal{M}, \rho^* \mathcal{N} \rangle_{L^2(\mathbb{R})}$$

- induces connection on $\Gamma^\infty(F)$:

$$\nabla_1 = \frac{\partial}{\partial y} - \frac{2i\pi px}{q} + L(A) + R(B), \quad \nabla_2 = \frac{\partial}{\partial x}$$

$$\text{with diagonal matrices } A_{kk} = -2i\pi pk, \quad B_{ll} = -\frac{2i\pi l}{p}$$

Action functional

$$S[\rho^* \mathcal{M}, (\rho^* \mathcal{M})^\dagger]$$

$$= q \int_0^{\frac{1}{q}} dy \int_0^{\frac{1}{q}} dx \operatorname{Tr} \left((\nabla_i \mathcal{M})^\dagger (\nabla_i \mathcal{M}) + \mu^2 \mathcal{M}^\dagger \mathcal{M} + \frac{\lambda}{2} (\mathcal{M}^\dagger \mathcal{M})^2 \right) (x, y)$$

Quantum field theory

Correlation functions = distributions on $\mathcal{E}^{*\otimes N} \otimes \mathcal{E}^{\otimes N}$

$$G_{2N}(\chi_1^\dagger, \dots, \chi_N^\dagger, \psi_1, \dots, \psi_N) \\ = \frac{\int [D\phi][D\phi^\dagger] \text{Tr}_{\mathcal{A}_\theta^d}((\chi_1, \phi)_{\mathcal{A}_\theta^d}) \dots \text{Tr}_{\mathcal{A}_\theta^d}((\phi, \psi_N)_{\mathcal{A}_\theta^d}) e^{-S[\phi, \phi^\dagger]}}{\int [D\phi][D\phi^\dagger] e^{-S[\phi, \phi^\dagger]}}$$

- **generating functional (of general correlation functions)**

$$Z[J, J^\dagger] = \frac{\int [D\phi][D\phi^\dagger] e^{-S[\phi, \phi^\dagger] + \text{Tr}_{\mathcal{A}_\theta^d}((J, \phi)_{\mathcal{A}_\theta^d}) + \text{Tr}_{\mathcal{A}_\theta^d}((\phi, J)_{\mathcal{A}_\theta^d})}}{\int [D\phi][D\phi^\dagger] e^{-S[\phi, \phi^\dagger]}}$$

- **connected** correlations functions: $W[J, J^\dagger] = \log Z[J, J^\dagger]$
- **one-particle irreducible** correlations functions:

$$\Gamma[\varphi, \varphi^\dagger] = \text{Tr}_{\mathcal{A}_\theta^d}((J, \varphi)_{\mathcal{A}_\theta^d}) + \text{Tr}_{\mathcal{A}_\theta^d}((\varphi, J)_{\mathcal{A}_\theta^d}) - W[J, J^\dagger]$$

$$\text{where } \varphi = \frac{\delta W}{\delta J^\dagger}, \quad \varphi^\dagger = \frac{\delta W}{\delta J}$$

Feynman rules for Bargmann module

Action functional for 2-dim. Bargmann module ($z \in \mathbb{C}^2$)

$$S[\phi, \bar{\phi}] = \int d\mu \bar{\phi}(\bar{z})(H\phi)(z) + \int d\mu_1 \dots d\mu_4 V(z_1, \bar{z}_2, z_3, \bar{z}_4) \bar{\phi}(\bar{z}_1) \phi(z_2) \bar{\phi}(\bar{z}_3) \phi(z_4)$$

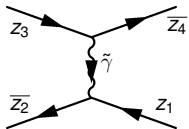
$$H = 2\omega \left(z_i \frac{\partial}{\partial z_i} + \frac{d}{4} \right) + \mu^2$$

$$V(z_1, \bar{z}_2, z_3, \bar{z}_4) = \frac{\lambda}{2} \sum_{\tilde{\gamma}} e^{-\omega|\tilde{\gamma}|^2 + \omega((\bar{z}_2 - \bar{z}_4)\tilde{\gamma} + \tilde{\gamma}(z_3 - z_1) + \bar{z}_2 z_1 + \bar{z}_4 z_3)}$$

- regularised propagator

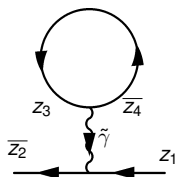
$$\begin{array}{c} z_1 \\ \longrightarrow \\ \bar{z}_2 \end{array} = H_\epsilon^{-1}(z_1, \bar{z}_2) = \frac{1}{2} \int_\epsilon^\infty d\beta e^{-\frac{\beta\mu^2}{2} + \omega\bar{z}_1 z_2} e^{-\beta\omega}$$

- vertex

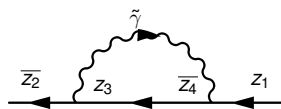


$$= V(z_1, \bar{z}_2, z_3, \bar{z}_4)$$

One-loop two-point function



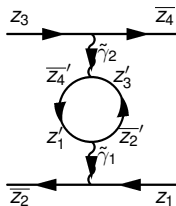
$$= \frac{\lambda}{2\omega} \text{Tr}_{A_\theta^4} [(\varphi_1, \varphi_2)_{A_\theta^4}] \times \left(\frac{1}{\epsilon\omega} + \left(1 - \frac{\mu^2}{2\omega}\right) \ln \frac{1}{\epsilon} \right) + \mathcal{O}(1)$$



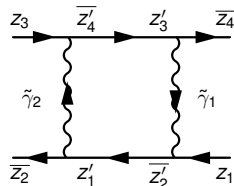
$$= \frac{\lambda}{2\theta^2\omega} \text{Tr}_{A_\theta^4} [(\varphi_1, \varphi_2)_{A_\theta^4}] \times \left(\frac{1}{\epsilon\omega} + \left(1 - \frac{\mu^2}{2\omega}\right) \ln \frac{1}{\epsilon} \right) + \mathcal{O}(1)$$

- the graphs are **dual to each other** (Poisson resummation)
- corresponds to **local mass renormalisation for any $\theta \in \mathbb{R}$**
- no wave function renormalisation

One-loop four-point function: planar sector



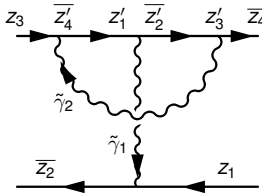
$$= -\frac{\lambda^2}{8\omega^2} \text{Tr}_{\mathcal{A}_\theta^4} [(\varphi_1, \varphi_2)_{\mathcal{A}_\theta^4} (\varphi_3, \varphi_4)_{\mathcal{A}_\theta^4}] \ln \frac{1}{\epsilon} + \mathcal{O}(1)$$



$$= -\frac{\lambda^2}{8\theta^2\omega^2} \text{Tr}_{\mathcal{A}_\theta^4} [(\varphi_1, \varphi_2)_{\mathcal{A}_\theta^4} (\varphi_3, \varphi_4)_{\mathcal{A}_\theta^4}] \ln \frac{1}{\epsilon} + \mathcal{O}(1)$$

- the graphs are **dual to each other**
- corresponds to **local coupling constant renormalisation for any $\theta \in \mathbb{R}$**

One-loop four-point function: non-planar sector



$$\begin{aligned}
 &= -\frac{\lambda^2}{4\theta^2} \sum_{\tilde{\gamma}_1, \tilde{\gamma}_2^*} e^{\omega(\bar{z}_2 z_1 + \bar{z}_4 z_3 + i\tilde{\gamma}_2^* z_3 + \bar{z}_4 i\tilde{\gamma}_2^* - \bar{\gamma}_1 z_1 + \bar{z}_2 \tilde{\gamma}_1)} \\
 &\quad \times \int_{\epsilon}^{\infty} d\beta_1 d\beta_2 \frac{e^{-\frac{\mu^2}{2}(\beta_1 + \beta_2)}}{(1 - e^{-(\beta_1 + \beta_2)\omega})^2} \\
 &\quad \times e^{-\frac{\omega}{1 - e^{-(\beta_1 + \beta_2)\omega}} |\tilde{\gamma}_1 + i\tilde{\gamma}_2^*|^2} \\
 &\quad \times e^{-\frac{\omega}{1 - e^{-(\beta_1 + \beta_2)\omega}} (-\bar{\gamma}_1 i\tilde{\gamma}_2^* (1 - e^{-\beta_2\omega}) + i\tilde{\gamma}_2^* \tilde{\gamma}_1 (1 - e^{-\beta_1\omega}))}
 \end{aligned}$$

- $\tilde{\gamma}_2^*$ appears after **Poisson resummation** in $\tilde{\gamma}_2$
 $\tilde{\gamma}_1 = \frac{\theta}{\sqrt{2}}(m_1 + in_1)$, $\tilde{\gamma}_2^* = \frac{1}{\sqrt{2}}(m_2 + in_2)$ $m_{1,2}, n_{1,2} \in \mathbb{Z}$
- β -integral divergent for $\tilde{\gamma}_1 + i\tilde{\gamma}_2^* = 0$, solutions **depend on θ**

Number-theoretical aspect for $\tilde{\gamma}_1 + i\tilde{\gamma}_2^* = 0$

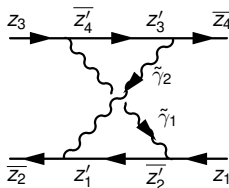
- $\theta \in \mathbb{Z} \Rightarrow$ divergence for **all** $\tilde{\gamma}_1$
local counterterm
- $\theta = \frac{p}{q} \in \mathbb{Q} \Rightarrow$ divergence for **some** $\tilde{\gamma}_1$
translation to matrices:
counterterm $q^2 \int_{[0, \frac{1}{q}]^4} d^2y d^2x \left(\text{Tr}(\mathcal{M}^\dagger \mathcal{M})(x, y) \right)^2$
unfamiliar, but **local on 4-torus**
- $\theta \notin \mathbb{Q} \Rightarrow$ **only solution is** $\tilde{\gamma}_1 = \tilde{\gamma}_2^* = 0$
additionally: finiteness of integrals for $\tilde{\gamma}_1 + i\tilde{\gamma}_2^* \neq 0$ requires

Diophantine condition on θ :

$$\forall n \in \mathbb{Z} \setminus \{0\} \exists C, \delta > 0 \text{ s.t. } \inf_{m \in \mathbb{Z}} |n\theta - m| \geq C|n|^{-(1+\delta)}$$

non-local counterterm $\left(\text{Tr}_{\mathcal{A}_\theta^4} [(\varphi, \varphi)_{\mathcal{A}_\theta^4}] \right)^2$

similar discussion for second non-planar graph



(proof of finiteness under Diophantine condition much more complicated)

Result

The scalar field theory on the Schwartz module $\mathcal{S}(\mathbb{R}^2) \ni \phi$ over the 4-dimensional noncommutative torus $\mathcal{A} = \mathcal{A}_\theta^4$ (θ irrational and Diophantine), defined by the action

$$\mathcal{S}[\phi, \phi^\dagger]$$

$$= \langle \nabla \phi, \nabla \phi \rangle_{\mathcal{A}, 1} + \text{Tr}_{\mathcal{A}} \left(\mu^2 (\phi, \phi)_{\mathcal{A}} + \frac{\lambda}{2} (\phi, \phi)_{\mathcal{A}}^2 \right) + \frac{\lambda'}{2} \left(\text{Tr}_{\mathcal{A}} ((\phi, \phi)_{\mathcal{A}}) \right)^2$$

is one-loop renormalisable

Outlook: What about higher loop order?

Bargmann module inappropriate (**no positivity**)

work directly with Schwartz module $\mathcal{S}(\mathbb{R}^2)$

- propagator becomes **2-dim. Mehler kernel**

$$H^{-1}(x, y) = \frac{\omega^2}{4\pi^2} \int_0^\infty \frac{d\beta}{\sinh(2\omega\beta)} e^{-\frac{\omega}{4} (\coth(\omega\beta)|x-y|^2 + \tanh(\omega\beta)|x+y|^2)}$$

(cf. **4-dim.** Mehler kernel for renormalisable Moyal model)

- vertex reads after Poisson resummation $(x, y, z, t \in \mathbb{R}^2)$

$$V(x, y, z, t) = \frac{\lambda}{(2\theta)^4} \sum_{m, n \in \mathbb{Z}^2} \delta(x-y+z-t) e^{\frac{2i\pi}{\theta} m(x-y) + 2i\pi n(x-t)}$$

- compare with Moyal vertex $(x, y, z, t \in \mathbb{R}^4)$

$$V_*(x, y, z, t) = \frac{\lambda}{(\pi\theta)^4} \delta(x-y+z-t) e^{2i\theta^{-1}(x,y) + 2i\theta^{-1}(z,t)}$$

Mehler + Moyal in x -space well understood [Orsay group]

→ extend this analysis by the new γ - x -interaction