Renormalisation of scalar quantum field theory on 4D-Moyal plane

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The Moyal plane

Definition: The four-dimensional Moyal plane is the algebra of (rapidly decaying) functions over the four-dimensional euclidean space, multiplied with the $\star$-product

$$(a \star b)(x) = \int d^4y \frac{d^4k}{(2\pi)^4}\, a(x+\tfrac{1}{2}\theta\cdot k)b(x+y)\,\mathrm{e}^{\mathrm{i}ky}$$

with $\theta=-\theta^T \in M_4(\mathbb{R})$.

- $\star$-product is associative, but noncommutative
- $\star$-product is non-local

- construction of field theories with non-local interaction
- This non-locality has serious consequences for the renormalisation of the resulting quantum field theory

The main result

Theorem. The quantum field theory defined by the action

$$S = \int d^4x \Big(\frac{1}{2}\phi \star (\Delta + \Omega^2 \tilde{x}^2 + \mu^2)\phi + \frac{\lambda}{4!}\phi \star \phi \star \phi \star \phi \Big)(x)$$

with

- $\tilde{x} = 2\theta^{-1} \cdot x$
- ϕ – real
- euclidean metric

is **perturbatively renormalisable to all orders** in λ.

The additional oscillator potential $\Omega^2 \tilde{x}^2$

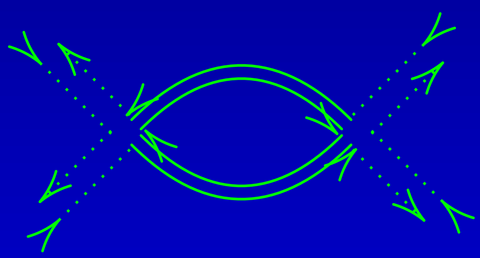
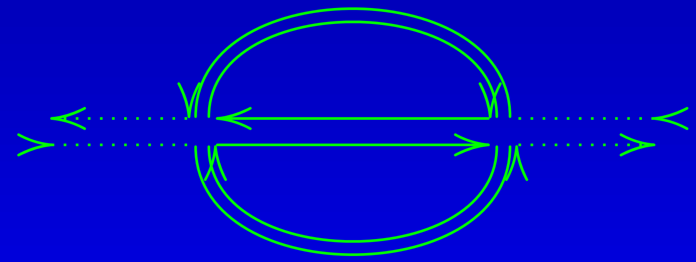
- implements the mixing between large and small distance scales
- results from the renormalisation proof

For the time being, this is the only renormalisable noncommutative quantum field theory with quadratic divergences

Intuitive renormalisation "proof"

Langmann-Szabo duality between position and momentum space:

$$\left.\begin{aligned} \tilde{x} &\longmapsto p \\ \phi(x) &\longmapsto \frac{1}{\sqrt{|\det \pi\theta|}}\hat{\phi}(p) \end{aligned}\right\} + \text{Fourier transformation}$$

- leaves $\int d^4x\,(\phi\star\phi\star\phi\star\phi)(x)$ and $\int d^4x\,(\phi\star\phi)(x)$ invariant
- transforms $\int d^4x\,(\phi\star\Delta\phi)(x)$ into $\int d^4x\,(\phi\star\tilde{x}^2\phi)(x)$
- with  also its Langmann-Szabo dual is divergent
- thus, also the LS-dual of  is divergent

$\Rightarrow$ renormalisation requires $\int d^4x\,(\phi\star\tilde{x}^2\phi)(x)$ in initial action

History of the renormalisation proof

- exact renormalisation group equation in matrix base
  [H. Grosse, R.W. (2004)]
  - simple interaction, complicated propagator
  - numerical determination of propagator asymptotics
  - power-counting from decay rate and ribbon graph topology
- multi-scale analysis in matrix base
  [V. Rivasseau, F. Vignes-Tourneret, R.W. (2005)]
  - rigorous bounds for the propagator (requires large Ω)
- multi-scale analysis in position space
  [R. Gurau, J. Magnen, V. Rivasseau, F. Vignes-Tourneret (2006)]
  - simple propagator (Mehler kernel), oscillating vertex
  - distinction between sum and difference of propagator ends

The matrix base

representation $\phi(x) = \sum_{m,n\in\mathbb{N}^2} \phi_{mn} b_{mn}(x)$ in basis $\{b_{mn}(x)\}_{m,n\in\mathbb{N}^2}$:

$$(b_{mn} \star b_{kl})(x) = \delta_{nk} b_{ml}(x) \qquad \int d^4x\, b_{mn}(x) = \sqrt{\det(2\pi\theta)}\, \delta_{mn}$$

- non-local $\star$-product becomes simple matrix product:

$$S[\phi] = \sqrt{\det(2\pi\theta)} \sum_{m,n,k,l\in\mathbb{N}^2} \Big(\frac{1}{2}\phi_{mn}\Delta_{mn;kl}\phi_{kl} + \frac{\lambda}{4!}\phi_{mn}\phi_{nk}\phi_{kl}\phi_{lm}\Big)$$

- kinetic term $\Delta_{mn;kl}$ and propagator $G = \Delta^{-1}$ complicated!

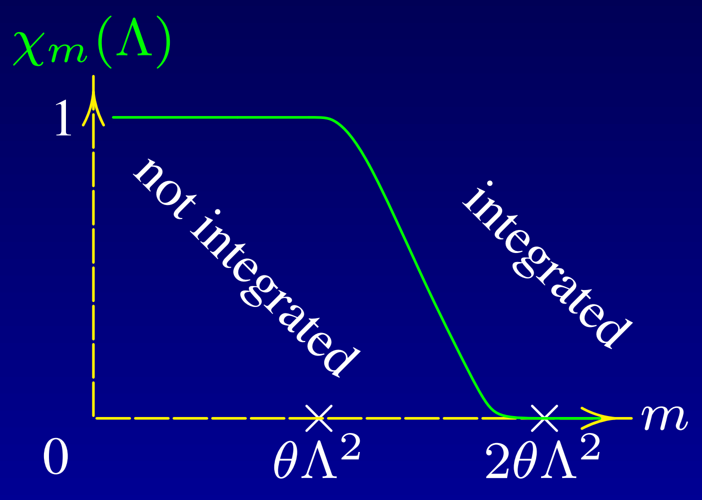
calculation of $G_{mn;kl}$ leads to Meixner polynomials:

- $G_{mn;kl}$ is finite sum over hypergeometric functions
- all matrix elements $G_{mn;kl}$ are non-negative
- $G_{mn;kl} \neq 0$ only for $m - l = n - k$ due to angular momentum conservation from $SO(2) \times SO(2)$-symmetry

First proof: exact renormalisation group equations

QFT defined via partition function $Z[J] = \int \mathcal{D}[\phi]\, \mathrm{e}^{-S[\phi]-\mathrm{tr}(\phi J)}$

- Wilson's strategy: integration of field modes ϕ_{mn} with indices $\geq \theta\Lambda^2$ yields effective action $L[\phi, \Lambda]$



- variation of cut-off function $\chi(\Lambda)$ with Λ modifies the effective action:

$\Rightarrow$ exact renormalisation group equation [Polchinski equation]

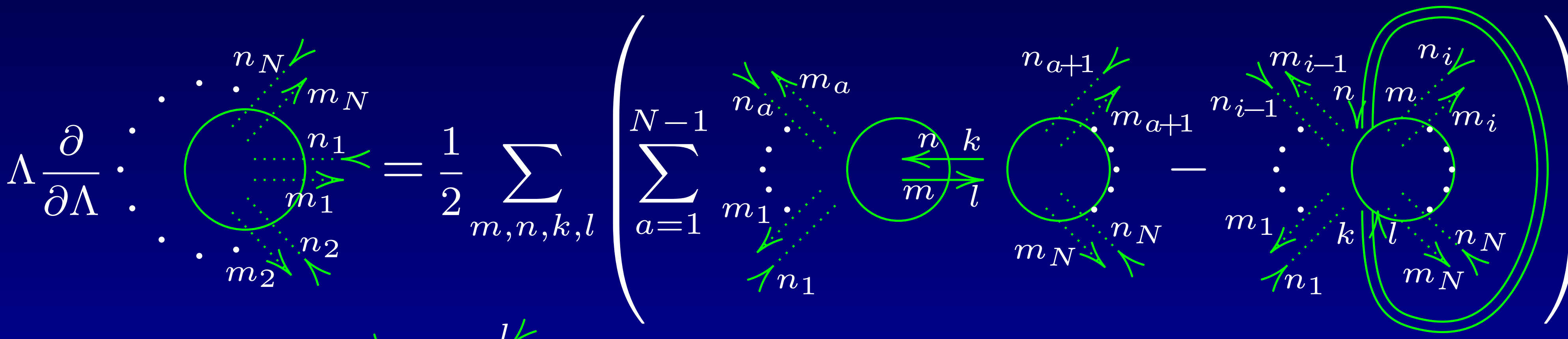
$$\Lambda \frac{\partial L[\phi,\Lambda]}{\partial \Lambda} = \sum_{m,n,k,l} \frac{1}{2} Q_{mn;kl}(\Lambda) \left(\frac{\partial L[\phi,\Lambda]}{\partial \phi_{mn}} \frac{\partial L[\phi,\Lambda]}{\partial \phi_{kl}} - \frac{1}{V_\theta} \frac{\partial^2 L[\phi,\Lambda]}{\partial \phi_{mn}\, \partial \phi_{kl}} \right)$$

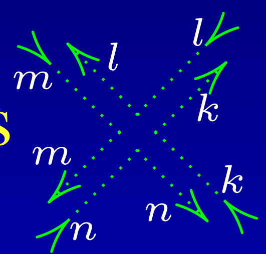
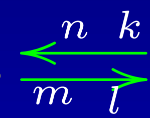
with $Q_{mn;kl}(\Lambda) = \Lambda \frac{\partial (G_{mn;kl}\, \chi_{mn;kl}(\Lambda))}{\partial \Lambda}$ $\qquad V_\theta = \sqrt{\det(2\pi\theta)}$

- renormalisation = proof that there exists a regular solution which depends on only a finite number of initial data

Ribbon graphs

we solve the Polchinski equation iteratively by ribbon graphs



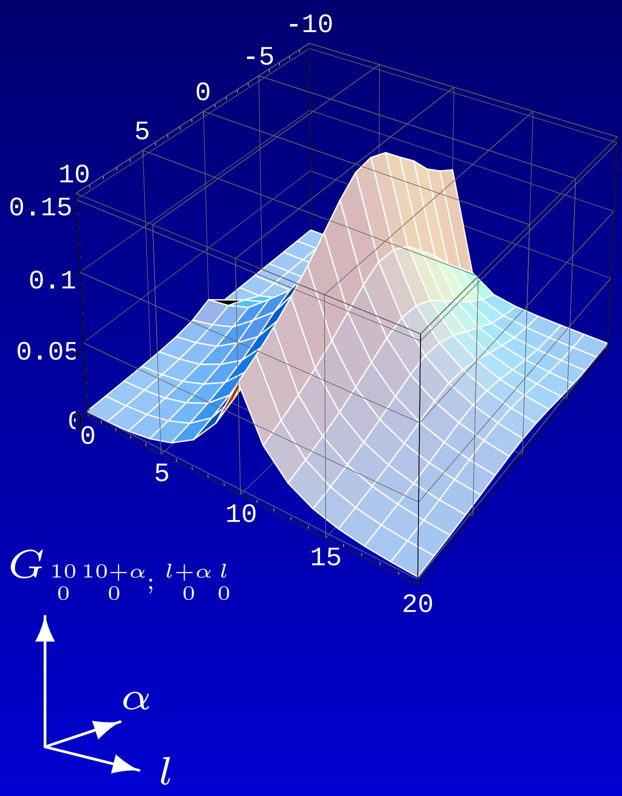
with V vertices  (iteration start), I edges  $= Q_{mn;kl}(\Lambda)$

- leads to F faces, B of them with external legs
- ribbon graph can be drawn on Riemann surface of genus $g = 1 - \frac{1}{2}(F - I + V)$ with B holes
- amplitudes of the graphs $L^{(V,B,g)}_{m_1 n_1;\dots;m_N n_N}(\Lambda)$

integration of the Polchinski equation from ∞ to Λ, if bounded, otherwise from renormalisation scale Λ_R to Λ (requires initial value)

Scaling behaviour

$$\left|L^{(V,B,g)}_{m_1n_1;\dots;m_Nn_N}[\Lambda]\right| \leq \left(\sqrt{\theta}\Lambda\right)^{(4-N)+4(1-B-2g)} \;\mathrm{pol}^{4V-N}\left[\frac{\{m^r_i,n^r_i\}}{\theta\Lambda^2}\right]\mathrm{pol}^{2V-\frac{N}{2}}\left[\ln\frac{\Lambda}{\Lambda_R}\right]$$



Proof: behaviour of $\overset{n\quad k}{\underset{m\quad l}{\rightleftarrows}} = Q_{mn;kl}(\Lambda)$

- $\max\limits_{m,n,k,l} |Q_{mn;kl}(\Lambda)| \leq \dfrac{C}{\theta\Lambda^2}$
  index volume $\sim \theta^2\Lambda^4$
  $\Big\}\rightarrow$ scaling exponent $4-N$
- $G_{mn;kl}$ has for given m sharp maximum at $l{=}m$:
  $\displaystyle\sum_l\Big(\max_{n,k}|Q_{\underline{m}n;kl}(\Lambda)|\Big) \leq \frac{C'}{\theta\Lambda^2}$
  $\Big\}\rightarrow$ volume factor not necessary

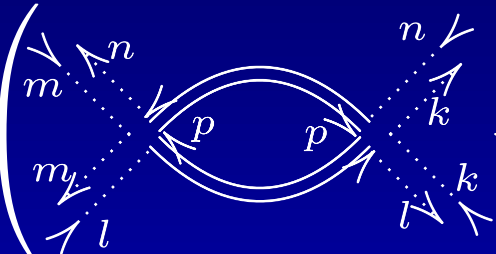
All non-planar graphs ($B \geq 1$ and / or $g \geq 0$) and all planar graphs with $N > 4$ are irrelevant!

Integration procedure

Problem: as planar 2- and 4-leg graphs are integrated from Λ_R to Λ, we seem to need infinitely many initial data

Solution: discrete Taylor expansion about reference graphs with vanishing indices:

$$L^{(2,1,0)}_{mn;nk;kl;lm}[\Lambda] = -\int_{\Lambda}^{\infty} \frac{d\Lambda'}{\Lambda'} \sum_{p\in\mathbb{N}^2} \Bigg($$



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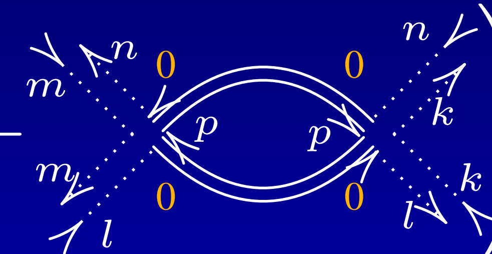
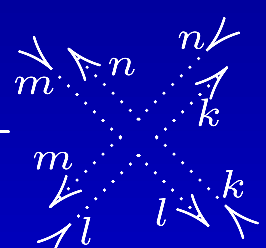
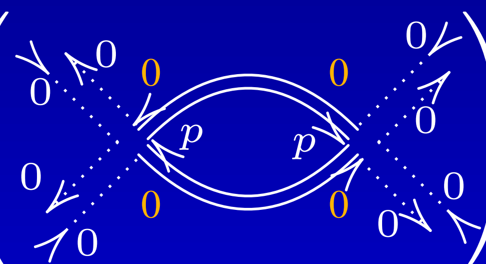


$$\Bigg)(\Lambda')$$

$$+$$



$$\Bigg[\int_{\Lambda_R}^{\Lambda} \frac{d\Lambda'}{\Lambda'} \sum_{p\in\mathbb{N}^2} \Bigg($$



$$\Bigg)(\Lambda') + L^{(2,1,0)}_{00;00;00;00}[\Lambda_R]\Bigg]$$

- difference of graphs can be expressed in terms of difference of propagators $Q_{mp;pm}(\Lambda) - Q_{0p;p0}(\Lambda)$
- this difference decays with Λ^{-4} $\Rightarrow$ first integral converges
- second integral requires a single initial condition

accordingly: mixed boundary conditions for

- $L^{(V,1,0)}_{\substack{m^1\\m^2}\substack{n^1\\n^2};\substack{n^1\\n^2};\substack{m^1\\m^2}}$ (2 conditions: quadratic+logarithmic divergence)
- $L^{(V,1,0)}_{\substack{m^2\\m^1}\substack{n^2\\n^1};\substack{n^2\\n^1}\substack{k^2\\k^1};\substack{k^2\\k^1}\substack{l^2\\l^1};\substack{l^2\\l^1}\substack{m^2\\m^1}}$
- $L^{(V,1,0)}_{\substack{m^1+1\\m^2}\substack{n^1+1\\n^2};\substack{n^1\\n^2}\substack{m^1\\m^2}} = L^{(V,1,0)}_{\substack{m^2\\m^1+1}\substack{n^2\\n^1+1};\substack{n^2\\n^1}\substack{m^2\\m^1}}$ (symmetry in θ)

with same index dependence as in initial action

The model is renormalisable by normalisation conditions for

- mass
- field amplitude
- coupling constant
- oscillator frequency

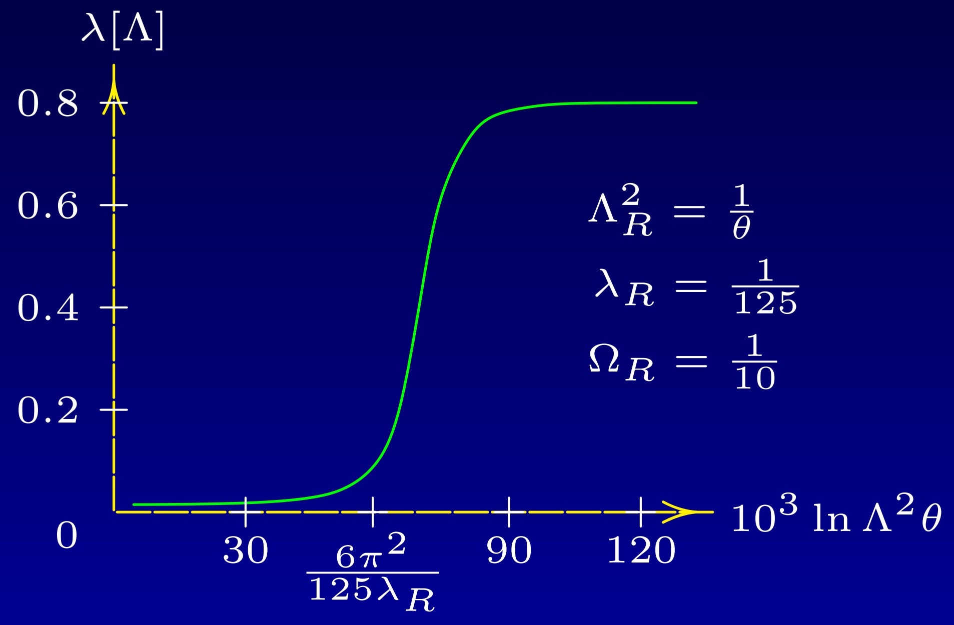
The β-function

one-loop calculation:

$$\frac{\lambda[\Lambda]}{\Omega^2[\Lambda]} = \text{const}$$

$$\Omega^2[\Lambda] \leq 1$$

($\lambda[\Lambda]$ diverges in commutative case)



- perturbation theory remains valid at all scales!
- non-perturbative construction of the model seems possible!

The presented model is an example where noncommutative quantum field theories are better behaved than commutative ones (in contrast to the public opinion)!

Second proof: multi-scale analysis

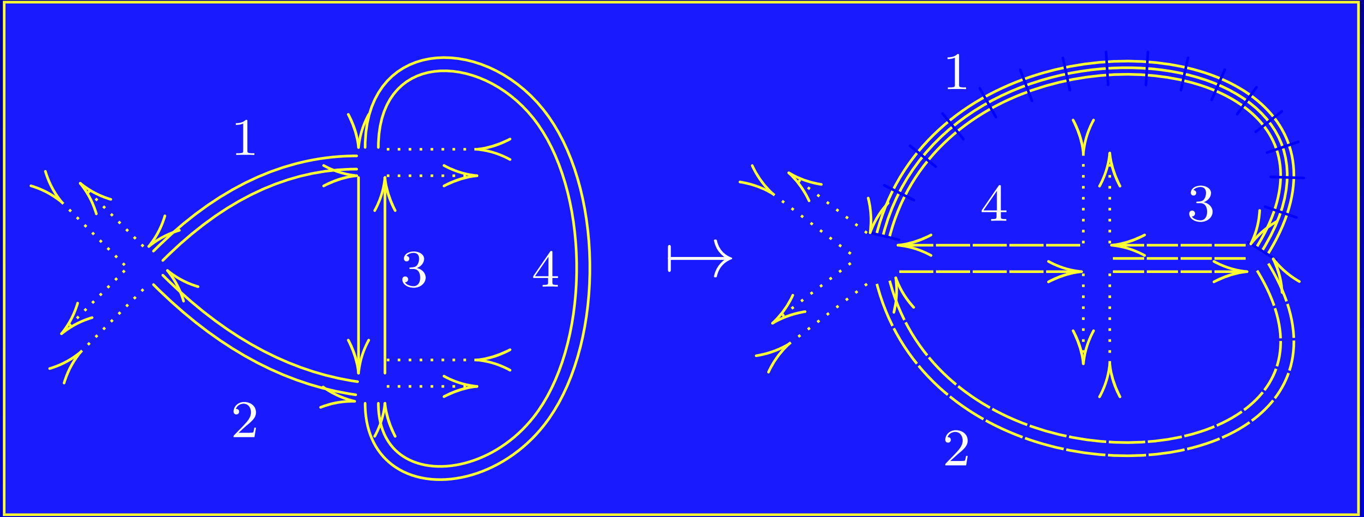
- propagator cut into slices: $G_{mn;kl} = \sum_{i=1}^{\infty} G^i_{mn;kl}$; estimations:

$$0 \le G^i_{mn;kl} \le K_1 M^{-i}\, \mathrm{e}^{-c_1 M^{-i}(\|m\|+\|n\|+\|k\|+\|l\|)}\, \delta_{m-l,-(k-n)}$$

$$\sum_l \Big(\max_{n(l),k(l)} G^i_{mn;kl} \Big) \le K_2 M^{-i}\, \mathrm{e}^{-c_2 M^{-i}\|m\|}$$

- induces scale attribution $i_\delta \in \mathbb{N}^+$ for each edge δ of the graph

- $SO(2) \times SO(2)$ symmetry implemented by dual graphs (vertices ⇔ faces)



- conserved angular momentum = index difference index assignment in dual graphs:

- given external indices
- reference indices at each internal vertex
- index differences between opposite sides of propagators in the complement of a maximal tree

$\Rightarrow \quad \sum_{\text{index differences}} \to$ factor M^{-i} preserved

$\sum_{\text{reference indices}} \to$ factor M^{2i} from $\sum_{m\in\mathbb{N}^2} \mathrm{e}^{-M^{-i}\|m\|}$

- estimation of (sub)graphs by $\#(\text{edges}) - 2\,\#(\text{inner vertices})$ for appropriate choice of reference points
- amounts to choice of maximal tree according to scale attribution

$\rightarrow$ plan is to continue this approach to a constructive renormalisation of the noncommutative ϕ^4_4-model

Third proof: position space

- propagator given by Mehler kernel:

  $G(x,y) =$

  $$\int_0^\infty \frac{\Omega^2\,dt}{2\theta\pi^2\sinh^2(2\Omega t)}\;\mathrm{e}^{-\frac{\Omega}{4\theta}\coth(\Omega t)\|x-y\|^2-\frac{\Omega}{2\theta}\tanh(\Omega t)\|x+y\|^2-\frac{m^2\theta}{2}t}$$

  - multi-scale approach:
    devide integral into slices $M^{-2i} \leq t \leq M^{-2(i-1)}$ $\quad M > 1$
  - $0 \leq G^i(x,y) \leq KM^{2i}\mathrm{e}^{-c(M^i\|x-y\|+M^{-i}\|x+y\|)}$

- vertex

  $V(x_1,\dots,x_4) = \frac{\lambda}{4!\pi^4\theta^4}\delta(x_1-x_2+x_3-x_4)\,\mathrm{e}^{2\mathrm{i}\sum_{1\leq i<j\leq 4} x_i^\mu(\theta^{-1})_{\mu\nu}x_j^\nu}$

- integration over short $(x-y)$ and long $(x+y)$ distance variables
  possible divergence for $i\to\infty$, i.e. $t\to 0$

- first approximation: ignore vertex phases
  - short variables bring M^{-4i}, long distances cost M^{4i}
  - eliminate most dangerous (Gallavotti-Nicolò algorithm) long distances using vertex δ's
  - orientable graphs: $(V-1)$ δ-functions
    $\prod M^{-\omega} \qquad \omega = 4(V-1) - (4V-N)$ classical power-counting
  - non-orientable graphs: V δ-functions
    $\prod M^{-\omega} \qquad \omega = 4V - (4V-N) > 0$ always convergent!
- consideration of vertex phases only for orientable graphs
  - total phase from contraction to rosette
  - intersecting lines (non-planarity) yield phase $\mathrm{i}y_i^\mu \theta^{-1}_{\mu\nu} y_j^\nu$ in long variables which overcompensates the cost M^{4i}
- renormalisation of planar graphs by Taylor expansion in external variables connected by short variables