# Renormalisation of quantum field theories on noncommutative geometries

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## Introduction

- Classical field theories for fundamental interactions (electroweak, strong, gravitational) are of geometrical origin
- Quantum field theory for standard model (electroweak+strong) is renormalisable
- Quantisation of gravity is a tremendous challenge
- Noncommutative geometry unifies standard model with gravity at the level of classical field theories
- Can we make sense of renormalisation in noncommutative geometry?
- First step: construct quantum field theories on simple noncommutative geometries, e.g. the Moyal plane

## The Moyal plane

**Definition:** The $D$-dimensional Moyal plane is the algebra of (rapidly decaying) functions over the $D$-dimensional Euclidean space, multiplied with the $\star$-product

$$(a\star b)(x) = \int d^D y \frac{d^D k}{(2\pi)^D}\, a(x+\tfrac{1}{2}\Theta\cdot k)b(x+y)\, \mathrm{e}^{\mathrm{i}ky}$$

with $\Theta=-\Theta^T \in M_D(\mathbb{R})$.

- $\star$-product is associative, but noncommutative
- $\star$-product is non-local

- construction of field theories with non-local interaction
- This non-locality has serious consequences for the renormalisation of the resulting quantum field theory

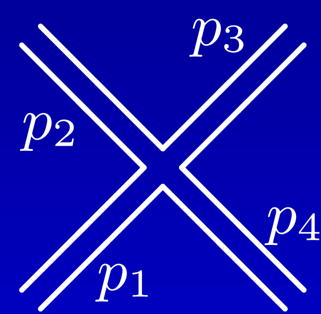
## UV/IR-mixing

- naïve $\phi^4$-action ($\phi$-real, Euclidean space) on Moyal plane

$$S = \int d^4x \Big( \frac{1}{2} \partial_\mu \phi \star \partial^\mu \phi + \frac{m^2}{2} \phi \star \phi + \frac{\lambda}{4!} \phi \star \phi \star \phi \star \phi \Big)(x)$$

- Feynman rules:

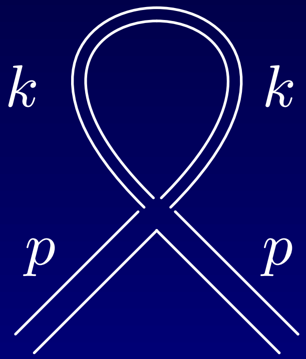

$$= \frac{1}{p^2 + m^2}$$




$$= \frac{\lambda}{4!} \exp \Big( -\frac{\mathrm{i}}{2} \sum_{i<j} p_i^\mu p_j^\nu \theta_{\mu\nu} \Big)$$

- cyclic order of vertex momenta is essential $\Rightarrow$ ribbon graphs

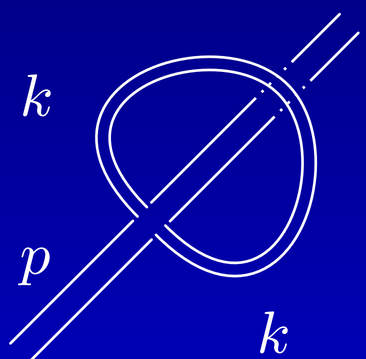
- one-loop two-point function, planar contribution:




$$= \frac{\lambda}{6} \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 + m^2}$$

to be treated by usual regularisation methods, can be put to 0

- non-planar contribution:




$$= \frac{\lambda}{12} \int \frac{d^4k}{(2\pi)^4} \frac{\mathrm{e}^{\mathrm{i}k\cdot\Theta\cdot p}}{k^2 + m^2} = \frac{\lambda}{48\pi^2} \frac{m}{\|\Theta\cdot p\|} K_1\big(m\|\Theta\cdot p\|\big)$$

- non-planar graph finite (noncommutativity as a regulator), but behaves $\sim p^{-2}$ for small momenta (renormalisation not possible)

$\Rightarrow$ leads to non-integrable integrals when inserted as subgraph into bigger graphs: UV/IR-mixing

# Solution of the UV/IR-mixing problem

**Theorem.** The quantum field theory defined by the action

$$S = \int d^4x \Big( \frac{1}{2}\phi \star (\Delta + \Omega^2 \tilde{x}^2 + m^2)\phi + \frac{\lambda}{4!}\phi \star \phi \star \phi \star \phi \Big)(x)$$

with

- $\tilde{x} = 2\Theta^{-1} \cdot x$
- $\phi$ – real
- Euclidean metric

is **perturbatively renormalisable to all orders** in $\lambda$.

The additional oscillator potential $\Omega^2 \tilde{x}^2$

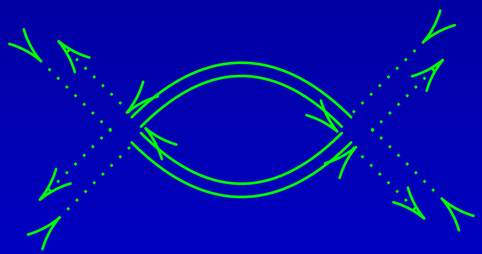
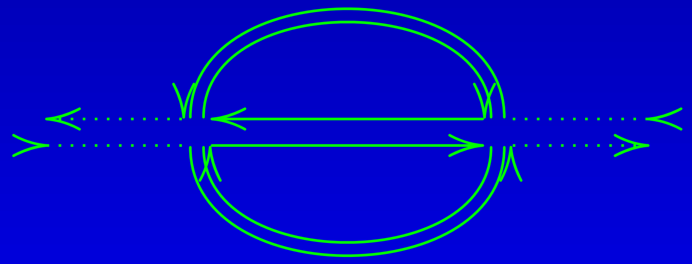
- implements the mixing between large and small distance scales
- results from the renormalisation proof

For the time being, this is the only renormalisable noncommutative quantum field theory with quadratic divergences

# Intuitive renormalisation "proof"

**Langmann-Szabo duality** between position and momentum space:

$$\left.\begin{array}{rcl} \tilde{x} & \longmapsto & p \\ \phi(x) & \longmapsto & \dfrac{1}{\sqrt{|\det \pi\Theta|}}\hat{\phi}(p) \end{array}\right\} + \text{Fourier transformation}$$

- leaves $\int d^4x\,(\phi\star\phi\star\phi\star\phi)(x)$ and $\int d^4x\,(\phi\star\phi)(x)$ invariant
- transforms $\int d^4x\,(\phi\star\Delta\phi)(x)$ into $\int d^4x\,(\phi\star\tilde{x}^2\phi)(x)$
- with  also its Langmann-Szabo dual is divergent
- thus, also the LS-dual of  is divergent

$\Rightarrow$ renormalisation requires $\int d^4x\,(\phi\star\tilde{x}^2\phi)(x)$ in initial action

## History of the renormalisation proof

- exact renormalisation group equation in matrix base
  [H. Grosse, R.W. (2004)]
  - simple interaction, complicated propagator
  - numerical determination of propagator asymptotics
  - power-counting from decay rate and ribbon graph topology
- multi-scale analysis in matrix base
  [V. Rivasseau, F. Vignes-Tourneret, R.W. (2005)]
  - rigorous bounds for the propagator (requires large $\Omega$)
- multi-scale analysis in position space
  [R. Gurau, J. Magnen, V. Rivasseau, F. Vignes-Tourneret (2006)]
  - simple propagator (Mehler kernel), oscillating vertex
  - distinction between sum and difference of propagator ends

# The matrix base

representation $\phi(x) = \sum_{m,n\in\mathbb{N}^2} \phi_{mn} b_{mn}(x)$ in basis $\{b_{mn}(x)\}_{m,n\in\mathbb{N}^2}$:

$$(b_{mn} \star b_{kl})(x) = \delta_{nk} b_{ml}(x) \qquad \int d^4x \, b_{mn}(x) = \sqrt{\det(2\pi\Theta)}\, \delta_{mn}$$

- non-local $\star$-product becomes simple matrix product:

$$S[\phi] = \sqrt{\det(2\pi\Theta)} \sum_{m,n,k,l\in\mathbb{N}^2} \Big( \frac{1}{2} \phi_{mn} \Delta_{mn;kl} \phi_{kl} + \frac{\lambda}{4!} \phi_{mn}\phi_{nk}\phi_{kl}\phi_{lm} \Big)$$

- kinetic term $\Delta_{mn;kl}$ and propagator $G = \Delta^{-1}$ complicated!

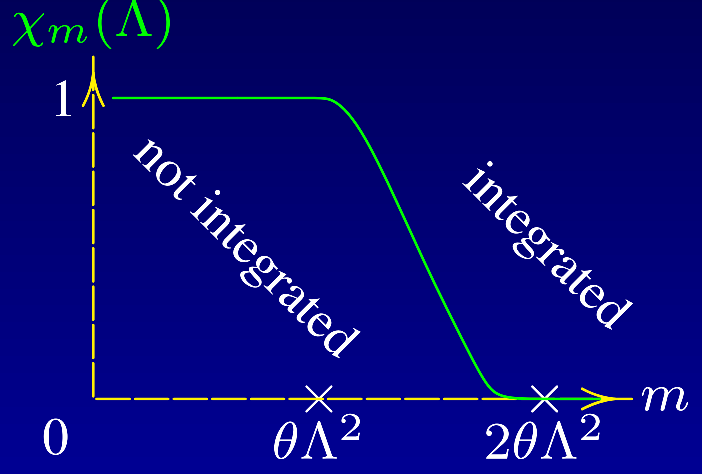
calculation of $G_{mn;kl}$ leads to Meixner polynomials:

- $G_{mn;kl}$ is finite sum over hypergeometric functions
- all matrix elements $G_{mn;kl}$ are non-negative
- $G_{mn;kl} \neq 0$ only for $m - l = n - k$ due to angular momentum conservation from $SO(2) \times SO(2)$-symmetry

# First proof: exact renormalisation group equations

QFT defined via partition function $Z[J] = \int \mathcal{D}[\phi]\, e^{-S[\phi]-\mathrm{tr}(\phi J)}$

- Wilson's strategy: integration of field modes $\phi_{mn}$ with indices $\geq \theta\Lambda^2$ yields effective action $L[\phi,\Lambda]$




- variation of cut-off function $\chi(\Lambda)$ with $\Lambda$ modifies the effective action:

$\Rightarrow$ exact renormalisation group equation [Polchinski equation]

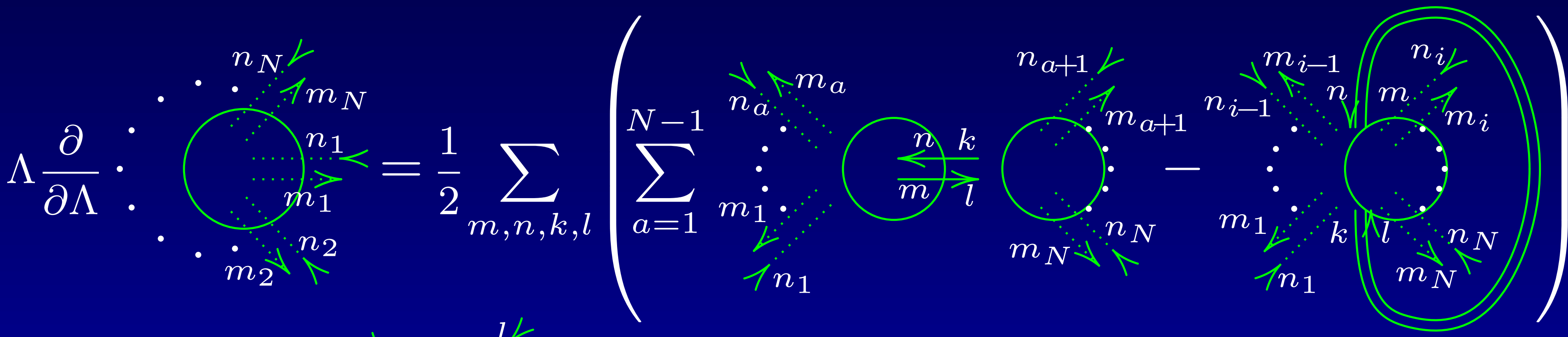
$$\Lambda \frac{\partial L[\phi,\Lambda]}{\partial \Lambda} = \sum_{m,n,k,l} \frac{1}{2} Q_{mn;kl}(\Lambda) \left( \frac{\partial L[\phi,\Lambda]}{\partial \phi_{mn}} \frac{\partial L[\phi,\Lambda]}{\partial \phi_{kl}} - \frac{1}{V_\Theta} \frac{\partial^2 L[\phi,\Lambda]}{\partial \phi_{mn}\, \partial \phi_{kl}} \right)$$

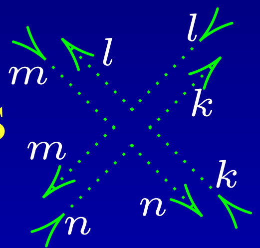
with $Q_{mn;kl}(\Lambda) = \Lambda \frac{\partial (G_{mn;kl}\, \chi_{mn;kl}(\Lambda))}{\partial \Lambda}$ $\qquad V_\Theta = \sqrt{\det(2\pi\Theta)}$

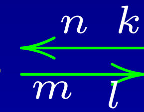
- renormalisation = proof that there exists a regular solution which depends on only a finite number of initial data

# Ribbon graphs

we solve the Polchinski equation iteratively by ribbon graphs




with $V$ vertices 

(iteration start), $I$ edges 

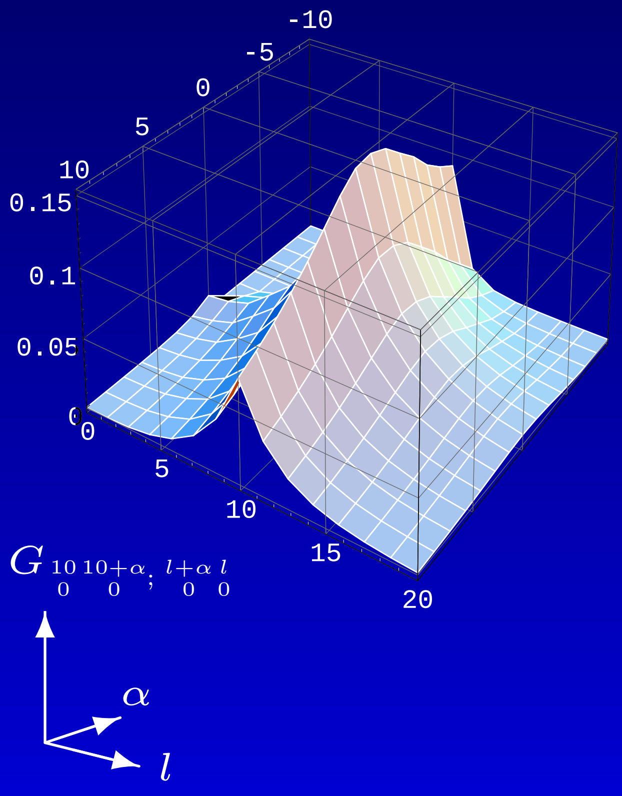
$= Q_{mn;kl}(\Lambda)$

- leads to $F$ faces, $B$ of them with external legs
- ribbon graph can be drawn on Riemann surface of genus $g = 1 - \frac{1}{2}(F - I + V)$ with $B$ holes
- amplitudes of the graphs $L^{(V,B,g)}_{m_1 n_1;\dots;m_N n_N}(\Lambda)$

integration of the Polchinski equation from $\infty$ to $\Lambda$, if bounded, otherwise from renormalisation scale $\Lambda_R$ to $\Lambda$ (requires initial value)

# Scaling behaviour

$$\Big|L^{(V,B,g)}_{m_1n_1;\dots;m_Nn_N}[\Lambda]\Big| \leq \big(\sqrt{\theta}\Lambda\big)^{(4-N)+4(1-B-2g)}\;\mathrm{pol}^{4V-N}\Big[\frac{\{m_i^r,n_i^r\}}{\theta\Lambda^2}\Big]\,\mathrm{pol}^{2V-\frac{N}{2}}\Big[\ln\frac{\Lambda}{\Lambda_R}\Big]$$




Proof: behaviour of $\underset{m\;\;\;l}{\overset{n\;\;\;k}{\rightleftarrows}} = Q_{mn;kl}(\Lambda)$

- $\displaystyle\max_{m,n,k,l}|Q_{mn;kl}(\Lambda)| \leq \frac{C}{\theta\Lambda^2}$

  index volume $\sim \theta^2\Lambda^4$

  $\Big\}\rightarrow$ scaling exponent $4-N$

- $G_{mn;kl}$ has for given $m$ sharp maximum at $l{=}m$:

  $\displaystyle\sum_l\Big(\max_{n,k}|Q_{\underline{m}n;kl}(\Lambda)|\Big) \leq \frac{C'}{\theta\Lambda^2}$

  $\Big\}\rightarrow$ volume factor not necessary

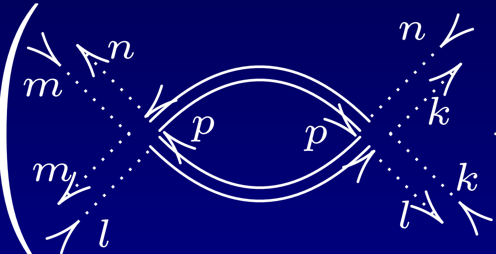
All non-planar graphs ($B \geq 1$ and/or $g \geq 0$) and all planar graphs with $N > 4$ are irrelevant!

## Renormalisation

Problem: infinitely many planar 2- and 4-leg graphs are divergent

Solution: discrete Taylor expansion about reference graphs:

$$L^{(2,1,0)}_{mn;nk;kl;lm}[\Lambda] = -\int_\Lambda^\infty \frac{d\Lambda'}{\Lambda'} \sum_{p\in\mathbb{N}^2} \Bigg($$

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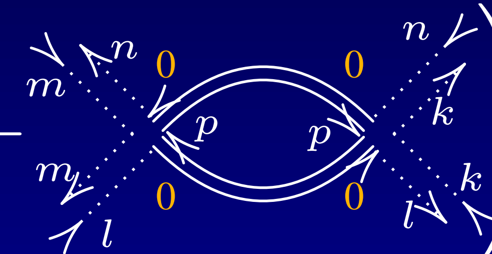
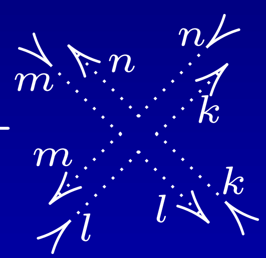
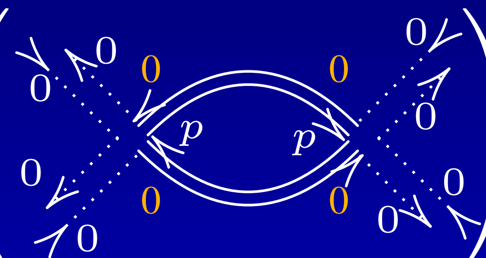



$$\Bigg)(\Lambda')$$

$$+$$




$$\Bigg[\int_{\Lambda_R}^{\Lambda} \frac{d\Lambda'}{\Lambda'} \sum_{p\in\mathbb{N}^2} \Bigg($$




$$\Bigg)(\Lambda') + L^{(2,1,0)}_{00;00;00;00}[\Lambda_R]\Bigg]$$

- difference expressed in terms of $Q_{mp;pm}(\Lambda) - Q_{0p;p0}(\Lambda) \sim \Lambda^{-4}$
- $1^{\text{st}}$ integral converges, $2^{\text{nd}}$ integral needs a single initial condition
- accordingly for $L^{(V,1,0)}_{{m^1 \atop m^2}{n^1 \atop n^2};{n^1 \atop n^2};{m^1 \atop m^2}}$ and $L^{(V,1,0)}_{{m^1+1 \atop m^2}{n^1+1 \atop n^2};{n^1 \atop n^2}{m^1 \atop m^2}}$

> The model is renormalisable by normalisation conditions for mass, field amplitude, coupling constant and oscillator frequency

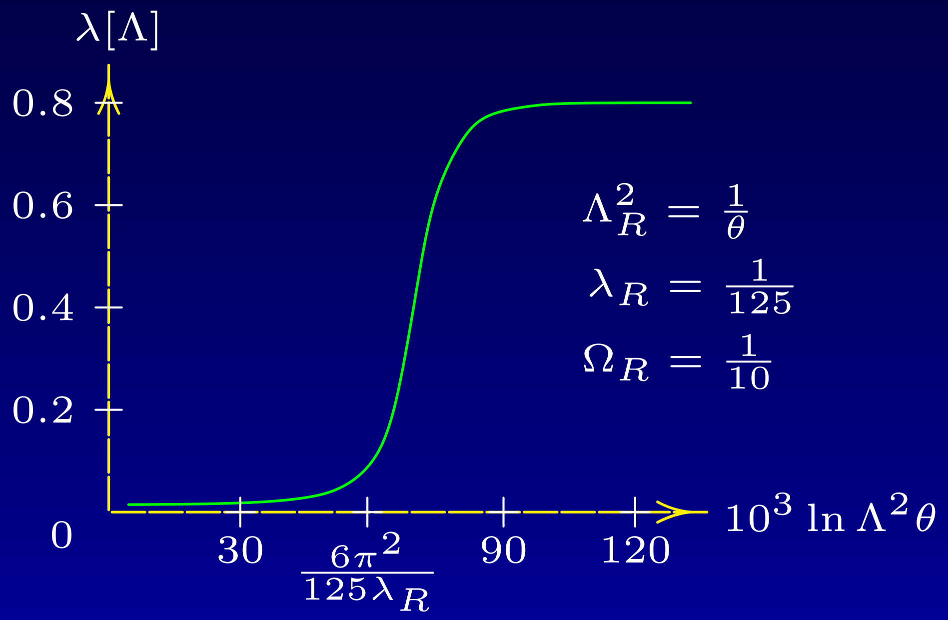
## The $\beta$-function

one-loop calculation:

$$\frac{\lambda[\Lambda]}{\Omega^2[\Lambda]} = \text{const}$$

$$\Omega^2[\Lambda] \leq 1$$

($\lambda[\Lambda]$ diverges in commutative case)




- perturbation theory remains valid at all scales!
- non-perturbative construction of the model seems possible!

The presented model is an example where noncommutative quantum field theories are better behaved than commutative ones (in contrast to the public opinion)!

# Second proof: multi-scale analysis

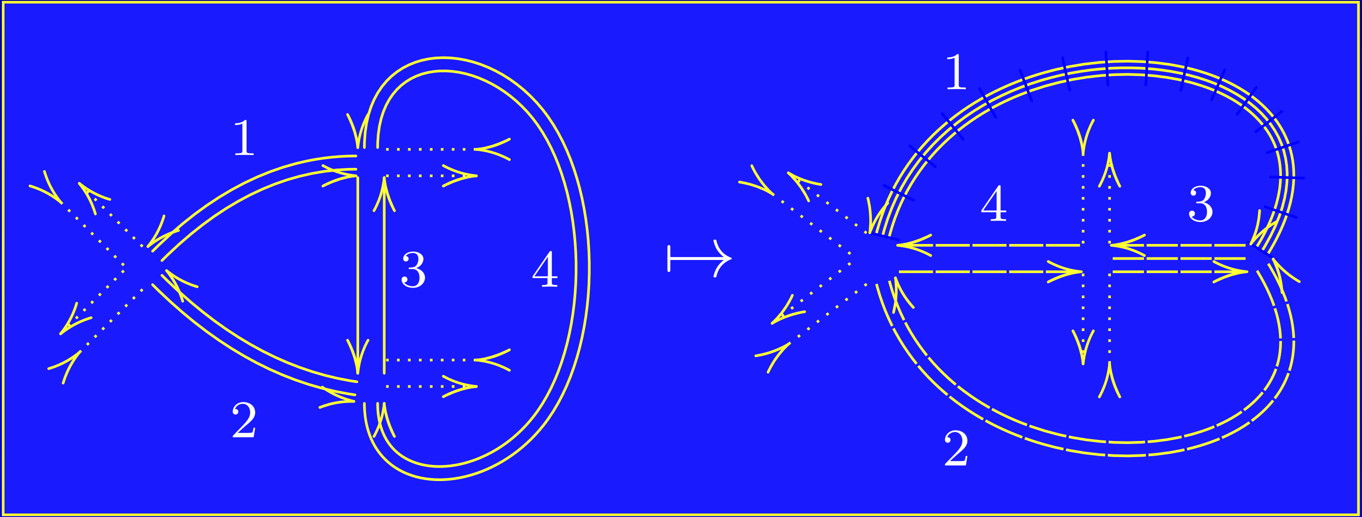
- propagator cut into slices: $G_{mn;kl} = \sum_{i=1}^{\infty} G^i_{mn;kl}$; estimations:

$$0 \leq G^i_{mn;kl} \leq K_1 M^{-i} \,\mathrm{e}^{-c_1 M^{-i}(\|m\|+\|n\|+\|k\|+\|l\|)}\, \delta_{m-l,-(k-n)}$$

$$\sum_l \Big( \max_{n(l),k(l)} G^i_{mn;kl} \Big) \leq K_2 M^{-i} \,\mathrm{e}^{-c_2 M^{-i}\|m\|}$$

- induces scale attribution $i_\delta \in \mathbb{N}^+$ for each edge $\delta$ of the graph

- $SO(2) \times SO(2)$ symmetry implemented by dual graphs (vertices $\Leftrightarrow$ faces)




- conserved angular momentum = index difference index assignment in dual graphs:

– given external indices
– reference indices at each internal vertex
– index differences between opposite sides of propagators in the complement of a maximal tree

$\Rightarrow$ $\sum_{\text{index differences}} \to$ factor $M^{-i}$ preserved

$\sum_{\text{reference indices}} \to$ factor $M^{2i}$ from $\sum_{m\in\mathbb{N}^2} \mathrm{e}^{-M^{-i}\|m\|}$

- estimation of subgraphs of the dual graph by

$2\,\#(\text{inner vertices}) - \#(\text{edges})$
$= 2(F{-}B) - I = 4{-}4g{-}2V{+}I - 2B = (2{-}\frac{N}{2}) - 2(2g{+}B{-}1)$

for appropriate choice of reference points according to scale attribution ($\exists$ algorithm)

## Third proof: position space

- propagator given by Mehler kernel:

  $G(x,y) =$
  $$\int_0^\infty \frac{\Omega^2\, dt}{2\theta\pi^2 \sinh^2(2\Omega t)}\, \mathrm{e}^{-\frac{\Omega}{4\theta}\coth(\Omega t)\|x-y\|^2-\frac{\Omega}{2\theta}\tanh(\Omega t)\|x+y\|^2-\frac{m^2\theta}{2}t}$$

  - multi-scale approach:
    divide integral into slices $M^{-2i} \leq t \leq M^{-2(i-1)}$ $\qquad M>1$
  - $0 \leq G^i(x,y) \leq K M^{2i} \mathrm{e}^{-c(M^i\|x-y\|+M^{-i}\|x+y\|)}$

- vertex $V(x_1,\dots,x_4) =$
  $\frac{\lambda}{4!\pi^4\theta^4}\delta(x_1-x_2+x_3-x_4)\,\mathrm{e}^{2\mathrm{i}\sum_{1\leq i<j\leq 4} x_i^\mu(\Theta^{-1})_{\mu\nu}x_j^\nu}$

- integration over short $(x-y)$ and long $(x+y)$ distance variables
  possible divergence for $i\to\infty$, i.e. $t\to 0$

- first approximation: ignore vertex phases
  - short variables bring $M^{-4i}$, long distances cost $M^{4i}$
  - eliminate most dangerous (Gallavotti-Nicolò algorithm) long distances using vertex $\delta$'s
  - orientable graphs: $(V-1)$ $\delta$-functions
    $\prod M^{-\omega} \quad \omega = 4(V-1)-(4V-N)$ classical power-counting
  - non-orientable graphs: $V$ $\delta$-functions
    $\prod M^{-\omega} \quad \omega = 4V-(4V-N) > 0$ always convergent!
- consideration of vertex phases only for orientable graphs
  - total phase from contraction to rosette
  - intersecting lines (non-planarity) yield phase $\mathrm{i}y_i^\mu \Theta^{-1}_{\mu\nu} y_j^\nu$ in long variables which overcompensates the cost $M^{4i}$
- renormalisation of planar graphs by Taylor expansion in external variables connected by short variables

## Further development

- explicit computation of the Schwinger parametric representation in position space [R. Gurau, V. Rivasseau (2006)]

$$\mathcal{A}_G(x_1,\dots,x_N)=K\int_0^\infty \frac{\prod_{l=1}^{I(G)}\Big(d\alpha_l(1-t_l^2)^{\frac{D}{2}}\Big)}{\big(U_G[t_1,\dots,t_I]\big)^{\frac{D}{2}}}\,\mathrm{e}^{-\frac{V_G[t_1,\dots,t_I,x_1,\dots,x_N]}{U_G[t_1,\dots,t_I]}}$$

  with $t_l=\tanh\frac{\alpha_l}{2}$

  – $U_G$, $V_G$ – polynomials analogous to Symanzik polynomials in commutative case
  – explicit formulae for orientable graphs $G$
  – leading terms correctly detect topology $g$, $B$
  – starting point for dimensional regularisation in $D$ dimensions

# Noncommutative Gross-Neveu model [F. Vignes-Tourneret, 2006]

- action functional (simplest case)

$$S[\bar\psi,\psi] = \int d^2x \Big(\bar\psi\big(-\mathrm{i}\,\not\partial + \Omega\,\tilde{\not x} + m + \mathrm{i}\delta m\gamma\Theta^{-1}\gamma\big)\psi + \frac{\lambda}{4}\bar\psi\star\psi\star\bar\psi\star\psi\Big)(x)$$

- $\Omega\,\tilde{\not x}$ is external magnetic field, not oscillator potential

- $$G(x,y) = -\frac{\Omega}{\theta\pi}\int_0^\infty dt\,\frac{\mathrm{e}^{-(m^2+\mathrm{i}\Omega\gamma\Theta\gamma)t}}{\sinh(\frac{4\Omega t}{\theta})}\mathrm{e}^{-\frac{\Omega}{\theta}(x-y)^2\coth(\frac{4\Omega t}{\theta})+2\mathrm{i}\Omega x\Theta^{-1}y} \times\Big(\frac{2\mathrm{i}\Omega}{\theta}\coth(\frac{4\Omega t}{\theta})(\not x-\not y) + \Omega(\tilde{\not x}-\tilde{\not y}) - m\Big)$$

- no Gaussian decay with $\|x+y\|$, makes proof more complicated

- renormalisable to all orders, limit $\Omega\to 0$ exists
  one new counterterm $\mathrm{i}\delta m\gamma\Theta^{-1}\gamma$ required

- model with spin, matrix base proof hardly possible

## Summary

- Renormalisation is compatible with noncommutative geometry
- We can renormalise models with new types of degrees of freedom, such as dynamical matrix models
- Equivalence of renormalisation schemes is confirmed
- Rigorous construction of noncommutative quantum field theories is promising
- Important tools (multi-scale analysis) are worked out

# Outlook

- Renormalisation of Yang-Mills theory on the Moyal plane
  - action functional
    $$\int d^4x \Big(\lambda_1 [X_\mu, X_\nu]_\star [X^\mu, X^\nu]_\star + \lambda_2 (X_\mu X^\mu)^2\Big)(x)$$
    with covariant coordinates $X_\mu = (\Theta^{-1})_{\mu\nu} x^\nu + A_\mu$
  - gauge and Lorentz invariant, contains oscillator potential $\tilde{x}^2 A^2$
  - difficulty: find solution of classical field equation ($A$-linear term!) to quantise about
  - gauge fixing unclear
- rigorous construction of nc Gross-Neveu model
  - make use of Pauli's principle and realisation by determinants
- rigorous construction of nc $\phi^4$ model
  - probably very different from fermionic case