Renormalisation of quantum field theories on noncommutative geometries

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Introduction

- Classical field theories for fundamental interactions (electroweak, strong, gravitational) of geometrical origin
- Quantum field theory for standard model (electroweak+strong) is renormalisable
- Quantisation of gravity is a tremendous challenge
- my favourite approach: noncommutative geometry

  . . . unifies standard model with gravity
  (at the level of classical field theories)

Can we make sense of renormalisation in NCG?

First step: construct quantum field theories on simple noncommutative geometries, e.g. the Moyal plane

The Moyal plane

Definition

The D-dimensional Moyal plane is the algebra of (rapidly decaying) functions over the D-dimensional Euclidean space, multiplied with the $\star$-product

$$(a \star b)(x) = \int d^D y \frac{d^D k}{(2\pi)^D} \, a(x + \tfrac{1}{2}\Theta \cdot k) b(x+y) \, \mathrm{e}^{iky}$$

with $\Theta = -\Theta^T \in M_D(\mathbb{R})$.

- $\star$-product is associative, but noncommutative
- $\star$-product is non-local

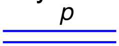
- construction of field theories with non-local interaction
- This non-locality has serious consequences for the renormalisation of the resulting quantum field theory

UV/IR-mixing

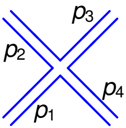
- naïve ϕ^4-action (ϕ-real, Euclidean space) on Moyal plane

$$S = \int d^4x \Big(\frac{1}{2}\partial_\mu\phi \star \partial^\mu\phi + \frac{m^2}{2}\phi\star\phi + \frac{\lambda}{4!}\phi\star\phi\star\phi\star\phi\Big)(x)$$

- Feynman rules:



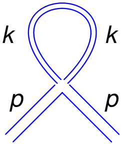
$$= \frac{1}{p^2+m^2}$$



$$= \frac{\lambda}{4!}\exp\Big(-\frac{\mathrm{i}}{2}\sum_{i<j} p_i^\mu p_j^\nu \theta_{\mu\nu}\Big)$$

- cyclic order of vertex momenta is essential
$\Rightarrow$ ribbon graphs

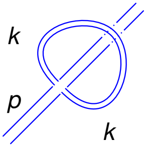
- one-loop two-point function, *planar contribution*:



$$= \frac{\lambda}{6}\int \frac{d^4k}{(2\pi)^4}\frac{1}{k^2+m^2}$$

to be treated by usual regularisation methods, can be put to 0

- non-planar contribution:



$$= \frac{\lambda}{12}\int \frac{d^4k}{(2\pi)^4}\frac{e^{ik\cdot\Theta\cdot p}}{k^2+m^2} = \frac{\lambda}{48\pi^2}\frac{m}{\|\Theta\cdot p\|}K_1\big(m\|\Theta\cdot p\|\big)$$

- non-planar graph finite (noncommutativity as a regulator), but $\sim p^{-2}$ for small momenta (renormalisation not possible)

$\Rightarrow$ leads to non-integrable integrals when inserted as subgraph into bigger graphs: UV/IR-mixing

Solution of the UV/IR-mixing problem

Theorem

The quantum field theory defined by the action

$$S = \int d^4x \Big(\frac{1}{2} \phi \star (\Delta + \Omega^2 \tilde{x}^2 + \mu^2) \phi + \frac{\lambda}{4!} \phi \star \phi \star \phi \star \phi \Big)(x)$$

with $\tilde{x} = 2\Theta^{-1} \cdot x$, ϕ *– real, Euclidean metric*
is perturbatively renormalisable to all orders in λ*.*

The additional oscillator potential $\Omega^2 \tilde{x}^2$

- implements mixing between large and small distance scales
- results from the renormalisation proof

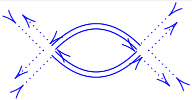
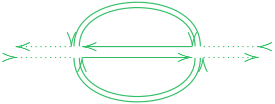
no other example of a renormalisable noncommutative quantum field theory with quadratic divergences is *known*

Intuitive renormalisation "proof"

Langmann-Szabo duality

$$\left.\begin{array}{rcl} \tilde{x} & \longmapsto & p \\ \phi(x) & \longmapsto & \frac{1}{\sqrt{|\det \pi\Theta|}}\hat{\phi}(p) \end{array}\right\} + \text{Fourier transformation}$$

- leaves $\int d^4x\,(\phi\star\phi\star\phi\star\phi)(x)$ and $\int d^4x\,(\phi\star\phi)(x)$ invariant
- transforms $\int d^4x\,(\phi\star\Delta\phi)(x)$ into $\int d^4x\,(\phi\star\tilde{x}^2\phi)(x)$

- with  also its LS-dual is divergent
- also the LS-dual of  is divergent

renormalisation requires $\int d^4x\,(\phi\star\tilde{x}^2\phi)(x)$ in initial action

History of the renormalisation proof

- exact renormalisation group equation in matrix base
  [H. Grosse, R.W. (2004)]
  - simple interaction, complicated propagator
  - numerical determination of propagator asymptotics
  - power-counting from decay rate and ribbon graph topology
- multi-scale analysis in matrix base
  [V. Rivasseau, F. Vignes-Tourneret, R.W. (2005)]
  - rigorous bounds for the propagator (requires large Ω)
- multi-scale analysis in position space
  [R. Gurau, J. Magnen, V. Rivasseau, F. Vignes-Tourneret (2006)]
  - simple propagator (Mehler kernel), oscillating vertex
  - distinction between sum and difference of propagator ends

The matrix base of the Moyal plane

(non-vanishing components: $\theta=\Theta_{12}=-\Theta_{21}=\Theta_{34}=-\Theta_{43}$)

$$\phi(x)=\sum_{m_i,n_i\in\mathbb{N}} \phi_{\substack{m_1\; n_1\\ m_2\; n_2}} b_{\substack{m_1\; n_1\\ m_2\; n_2}}(x), \qquad b_{\substack{m_1\; n_1\\ m_2\; n_2}}(x)=f_{m_1n_1}(x^1,x^2)f_{m_2n_2}(x^3,x^4)$$

$$f_{m_1n_1}(x_1,x_2)=\frac{(x_1+\mathrm{i}x_2)^{\star m_1}}{\sqrt{m_1!(2\theta)^{m_1}}}\star\left(2\mathrm{e}^{-\frac{1}{\theta}(x_1^2+x_2^2)}\right)\star\frac{(x_1-\mathrm{i}x_2)^{\star n_1}}{\sqrt{n_1!(2\theta)^{n_1}}}$$
$$=2(-1)^{m_1}\sqrt{\frac{m_1!}{n_1!}}\mathrm{e}^{\mathrm{i}\varphi(n_1-m_1)}\left(\sqrt{\tfrac{2}{\theta}}\rho\right)^{n_1-m_1}\mathrm{e}^{-\frac{\rho^2}{\theta}}L_{m_1}^{n_1-m_1}\left(\tfrac{2}{\theta}\rho^2\right)$$

satisfies:
$$(b_{mn}\star b_{kl})(x)=\delta_{nk}b_{ml}(x)$$
$$\int d^4x\, b_{mn}(x)=\sqrt{\det(2\pi\Theta)}\,\delta_{mn}$$

non-local $\star$-product becomes simple *matrix product*

$$S[\phi]=\sqrt{\det(2\pi\Theta)}\sum_{m,n,k,l\in\mathbb{N}^2}\left(\frac{1}{2}\phi_{mn}\Delta_{mn;kl}\phi_{kl}+\frac{\lambda}{4!}\phi_{mn}\phi_{nk}\phi_{kl}\phi_{lm}\right)$$

calculation of propagator $G = \Delta^{-1}$ via Meixner polynomials

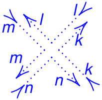
- $G_{mn;kl} \neq 0$ only for $m - l = n - k$ due to angular momentum conservation from $SO(2) \times SO(2)$-symmetry

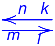
$$G_{\substack{m_1 \\ m_2}\substack{m_1+h_1 \\ m_2+h_2};\substack{l_1+h_1 \\ l_2+h_2}\substack{l_1 \\ l_2}} = \frac{\theta}{8\Omega} \sum_{u_1=0}^{\min(m_1,l_1)} \sum_{u_2=0}^{\min(m_2,l_2)} \int_0^1 dt\, \frac{t^{\frac{\mu^2\theta}{8\Omega}+\alpha}(1-t)^\beta}{\big(1-\frac{(1-\Omega)^2}{(1+\Omega)^2}t\big)^{2+2\alpha+\beta}}$$
$$\times \Big(\frac{1-\Omega}{1+\Omega}\Big)^\beta \Big(\frac{4\Omega}{(1+\Omega)^2}\Big)^{2+2\alpha} \prod_{i=1}^2 \frac{\sqrt{m_i!\,l_i!\,(m_i+h_i)!\,(l_i+h_i)!}}{(m_i-u_i)!\,(l_i-u_i)!\,(h_i+u_i)!\,u!}$$
$$= \frac{\theta}{2(1+\Omega)^2} \sum_{u_1=0}^{\min(m_1,l_1)} \sum_{u_2=0}^{\min(m_2,l_2)} {}_2F_1\Bigg(\begin{matrix} 1+\beta\,,\ \frac{\mu^2\theta}{8\Omega}-\alpha \\ 2+\frac{\mu^2\theta}{8\Omega}+\alpha+\beta \end{matrix}\Bigg|\frac{(1-\Omega)^2}{(1+\Omega)^2}\Bigg)$$
$$\times \Big(\frac{1-\Omega}{1+\Omega}\Big)^\beta B\big(1+\tfrac{\mu^2\theta}{8\Omega}+\alpha,\, 1+\beta\big) \prod_{i=1}^2 \frac{\sqrt{m_i!\,l_i!\,(m_i+h_i)!\,(l_i+h_i)!}}{(m_i-u_i)!\,(l_i-u_i)!\,(h_i+u_i)!\,u!}$$

with $\alpha = \frac{1}{2}\sum_{i=1}^2 (h_i + 2u_i) \geq 0 \qquad \beta = \sum_{i=1}^2 (m_i + l_i - 2u_i) \geq 0$

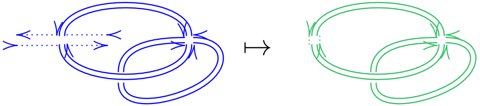
- all matrix elements $G_{mn;kl}$ are non-negative
- $G_{mn;kl}$ is finite sum over hypergeometric functions

Ribbon graphs

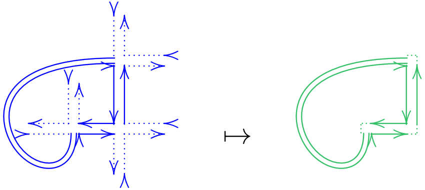
Feynman graphs are ribbon graphs with V vertices  and

I edges  $= G_{mn;kl}$ and N external legs

- leads to F faces, B of them with external legs
- ribbon graph can be drawn on Riemann surface of genus $g = 1 - \frac{1}{2}(F - I + V)$ with B holes



$F = 1$ $\quad g = 1$
$I = 3$ $\quad B = 1$
$V = 2$ $\quad N = 2$

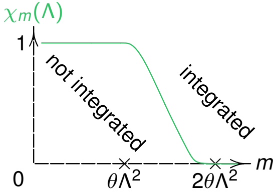


$L = 2$ $\quad g = 0$
$I = 3$ $\quad B = 2$
$V = 3$ $\quad N = 6$

First proof: exact renormalisation group equations

QFT defined via partition function $Z[J] = \int \mathcal{D}[\phi]\, e^{-S[\phi]-\mathrm{tr}(\phi J)}$

- Wilson's strategy: integration of field modes ϕ_{mn} with indices $\geq \theta\Lambda^2$ yields effective action $L[\phi,\Lambda]$
- variation of cut-off function $\chi(\Lambda)$ with Λ modifies effective action:



exact renormalisation group equation [Polchinski equation]

$$\Lambda \frac{\partial L[\phi,\Lambda]}{\partial \Lambda} = \sum_{m,n,k,l} \frac{1}{2} Q_{mn;kl}(\Lambda) \left(\frac{\partial L[\phi,\Lambda]}{\partial \phi_{mn}} \frac{\partial L[\phi,\Lambda]}{\partial \phi_{kl}} - \frac{1}{V_\Theta} \frac{\partial^2 L[\phi,\Lambda]}{\partial \phi_{mn}\,\partial \phi_{kl}} \right)$$

with $Q_{mn;kl}(\Lambda) = \Lambda \frac{\partial (G_{mn;kl}\, \chi_{mn;kl}(\Lambda))}{\partial \Lambda}$ $\qquad V_\Theta = \sqrt{\det(2\pi\Theta)}$

- renormalisation = proof that there exists a regular solution which depends on only a finite number of initial data

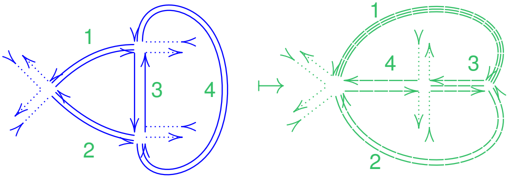
Second proof: multi-scale analysis

- propagator cut into slices: $G_{mn;kl} = \sum_{i=1}^{\infty} G^i_{mn;kl}$
  estimations:

$$0 \leq G^i_{mn;kl} \leq K_1 M^{-i} \mathrm{e}^{-c_1 M^{-i}(\|m\|+\|n\|+\|k\|+\|l\|)} \delta_{m-l,-(k-n)}$$

$$\sum_l \Big(\max_{n(l),k(l)} G^i_{mn;kl} \Big) \leq K_2 M^{-i} \mathrm{e}^{-c_2 M^{-i}\|m\|}$$

- induces scale attribution $i_\delta \in \mathbb{N}^+$ for each edge δ of the graph
- $SO(2) \times SO(2)$ symmetry implemented by dual graphs (vertices $\Leftrightarrow$ faces)



- index-difference (= angular momentum) conserved at propagators and vertices

index assignment in dual graphs

- given external indices
- reference indices at each internal vertex
- index differences between opposite sides of propagators in the complement of a maximal tree

$\Rightarrow$ $\sum_{\text{index differences}} \to$ factor M^{-i} preserved

$\sum_{\text{reference indices}} \to$ factor M^{2i} from $\sum_{m\in\mathbb{N}^2} \mathrm{e}^{-M^{-i}\|m\|}$

- power-counting degree of divergence for dual subgraphs
  $2\,\#(\text{inner vertices}) - \#(\text{edges})$
  $= 2(F-B)-I = 4-4g-2V+I-2B = (2-\frac{N}{2}) - 2(2g+B-1)$

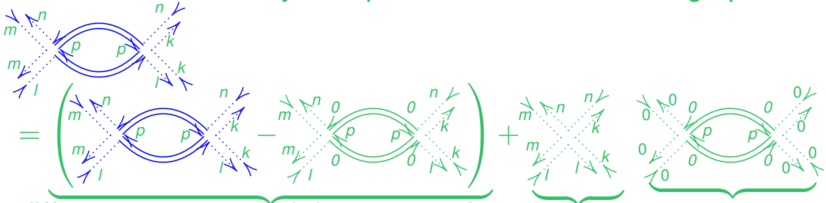
Conclusion

All non-planar graphs and all planar graphs with ≥ 4 external legs are convergent

Renormalisation

Problem: infinitely many planar 2- and 4-leg graphs diverge

Solution: discrete Taylor expansion about reference graphs:



difference expressed in terms of

$|G_{np;pn} - G_{0p;p0}| \le K_3 M^{-i} \frac{\|n\|}{M^i} e^{-c_3\|p\|}$

$=1$

put to renormalised value

- similar for all $A^{planar}_{mn;nk;kl;lm}$ $A^{planar}_{mn;nm}$ and $A^{planar}_{\substack{m^1+1\\m^2}\,\substack{n^1+1\\n^2};\substack{n^1\\n^2}\,\substack{m^1\\m^2}}$

Renormalisation of noncommutative ϕ^4_4-model to all orders

by normalisation conditions for mass, field amplitude, coupling constant and oscillator frequency

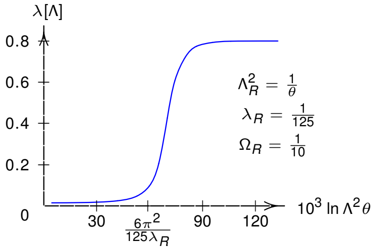
The β-function

one-loop calculation

$$\frac{\lambda[\Lambda]}{\Omega^2[\Lambda]} = \text{const}$$

$\Omega^2[\Lambda] \leq 1$

($\lambda[\Lambda]$ diverges in commutative case)



- perturbation theory remains valid at all scales!
- non-perturbative construction of the model seems possible!

New point of view

The duality-covariant ϕ^4-model on 4D-Moyal plane is an example where noncommutative quantum field theories are nicer than commutative ones (in contrast to the public opinion)!

Third proof: position space

- propagator given by Mehler kernel:

$$G(x,y) = \int_0^\infty \frac{\Omega^2\, dt}{2\theta\pi^2 \sinh^2(2\Omega t)}\, e^{-\frac{\Omega}{4\theta}\coth(\Omega t)\|x-y\|^2 - \frac{\Omega}{2\theta}\tanh(\Omega t)\|x+y\|^2 - \frac{m^2\theta}{2}t}$$

  - multi-scale approach:
    divide integral into slices $M^{-2i} \leq t \leq M^{-2(i-1)}$ $\qquad M > 1$
  - $0 \leq G^i(x,y) \leq KM^{2i} e^{-c(M^i\|x-y\| + M^{-i}\|x+y\|)}$

- vertex

$$V(x_1,\dots,x_4) = \frac{\lambda}{4!\pi^4\theta^4}\delta(x_1-x_2+x_3-x_4)\, e^{2\mathrm{i}\sum_{1\leq i<j\leq 4} x_i^\mu (\Theta^{-1})_{\mu\nu} x_j^\nu}$$

- integration over short $(x-y)$ and long $(x+y)$ distances
  possible divergence for $i \to \infty$, i.e. $t \to 0$

- first approximation: ignore vertex phases
  - short variables bring M^{-4i}, long distances cost M^{4i}
  - eliminate most dangerous (Gallavotti-Nicolò algorithm) long distances using vertex δ's
  - orientable graphs: $(V-1)$ δ-functions
    $\prod M^{-\omega} \quad \omega = 4(V-1) - (4V-N)$ classical power-counting
  - non-orientable graphs: V δ-functions
    $\prod M^{-\omega} \quad \omega = 4V - (4V-N) > 0$ always convergent!
- consideration of vertex phases only for orientable graphs
  - total phase from contraction to rosette
  - intersecting lines (non-planarity) yield phase $\mathrm{i} w_i^{\mu} \Theta^{-1}_{\mu\nu} w_j^{\nu}$ in long variables w which overcompensates the cost M^{4i}
- renormalisation of planar graphs by Taylor expansion in external variables connected by short variables

Further developments

explicit computation of the Schwinger parametric representation in position space [R. Gurau, V. Rivasseau (2006)]

$$\mathcal{A}_G(x_1,\dots,x_N)=K\int_0^\infty \frac{\prod_{l=1}^{I(G)}\big(d\alpha_l(1-t_l^2)^{\frac{D}{2}}\big)}{\big(U_G[t_1,\dots,t_l]\big)^{\frac{D}{2}}}\,\mathrm{e}^{-\frac{V_G[t_1,\dots,t_l,x_1,\dots,x_N]}{U_G[t_1,\dots,t_l]}}$$

with $t_l=\tanh\frac{\alpha_l}{2}$

- U_G, V_G – polynomials analogous to Symanzik polynomials in commutative case
- explicit formulae for orientable graphs G
- leading terms correctly detect topology g, B
- starting point for dimensional regularisation in D dimensions

NC Gross-Neveu model [F. Vignes-Tourneret, 2006]

- action functional (simplest case)

$$S[\bar\psi,\psi] = \int d^2x\Big(\bar\psi\big(-\mathrm{i}\,\partial\!\!\!/ + \Omega\,\tilde{x}\!\!\!/ + m + \mathrm{i}\delta m\gamma\Theta^{-1}\gamma\big)\psi + \tfrac{\lambda}{4}\bar\psi\star\psi\star\bar\psi\star\psi\Big)(x)$$

- $\Omega\,\tilde{x}\!\!\!/$ is external magnetic field, not oscillator potential

- $$G(x,y) = -\frac{\Omega}{\theta\pi}\int_0^\infty dt\,\frac{\mathrm{e}^{-(m^2+\mathrm{i}\Omega\gamma\Theta\gamma)t}}{\sinh(\frac{4\Omega t}{\theta})}\mathrm{e}^{-\frac{\Omega}{\theta}(x-y)^2\coth(\frac{4\Omega t}{\theta})+\mathrm{i}\Omega(x-y)\Theta^{-1}(x+y)} \times\Big(\frac{2\mathrm{i}\Omega}{\theta}\coth(\tfrac{4\Omega t}{\theta})(x\!\!\!/ - y\!\!\!/) + \Omega(\tilde{x}\!\!\!/ - \tilde{y}\!\!\!/) - m\Big)$$

- no Gaussian decay with $\|x+y\|$, proof more complicated

- renormalisable to all orders, limit $\Omega\to 0$ exists
  one new counterterm $\mathrm{i}\delta m\gamma\Theta^{-1}\gamma$ required

- model with spin, matrix base proof hardly possible

Summary

- Renormalisation is compatible with noncommutative geometry
- We can renormalise models with new types of degrees of freedom, such as dynamical matrix models
- Equivalence of renormalisation schemes is confirmed
- Rigorous construction of noncommutative quantum field theories is promising
- Important tools (multi-scale analysis) are worked out

Outlook

Renormalisation of *Yang-Mills theory* on the Moyal plane

- action functional
$$\int d^4x \Big(\lambda_1 [X_\mu, X_\nu]_\star [X^\mu, X^\nu]_\star + \lambda_2 (X_\mu X^\mu)^2\Big)(x)$$
with covariant coordinates $X_\mu = (\Theta^{-1})_{\mu\nu} x^\nu + A_\mu$
- gauge and Lorentz invariant, oscillator potential $\tilde{x}^2 A^2$
- difficulties: find solution of classical field equation (A-linear term!) to quantise about; gauge fixing unclear

Rigorous construction of nc Gross-Neveu model

- realisation by determinants due to Pauli's principle

Rigorous construction of nc ϕ^4 model

- probably very different from fermionic case