Renormalisation

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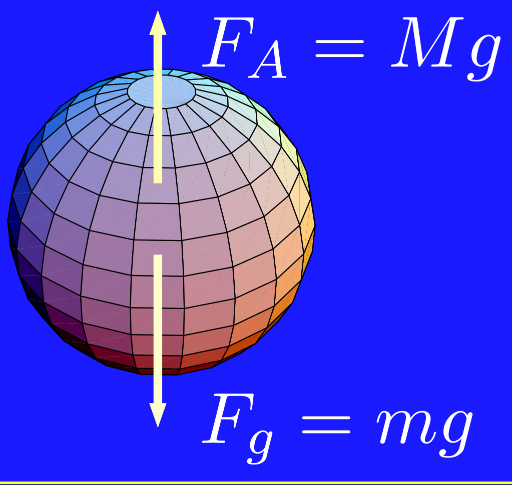
I. Introduction

II. Elements of Quantum Field Theory

III. Renormalisation

A simple example of renormalisation

Consider a ball (filled with air) submersed in water ...



m - mass of the ball

M - mass of a ball of water of the same size

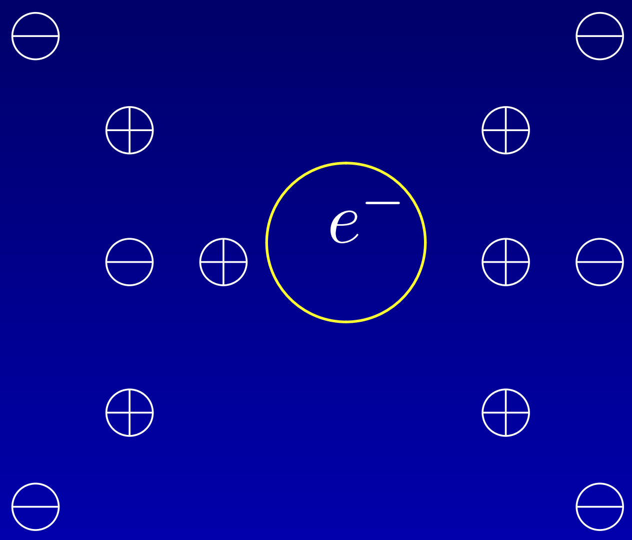
- resulting acceleration of the ball:

$$a = \frac{F_A - F_g}{m} = \frac{M - m}{m} g \rightarrow \frac{\text{density of water}}{\text{density of air}}\, g \approx 750\, g$$

- This is not observed!
- Not only the ball is accelerated upward, also a displaced quantity of surrounding water must be accelerated downward

$\Rightarrow$ bare mass m of the ball to be replaced by effective mass $m + \frac{1}{2} M$

- Consider an electron e^- submersed in its own electomagnetic field
- the field creates electron-positron pairs:

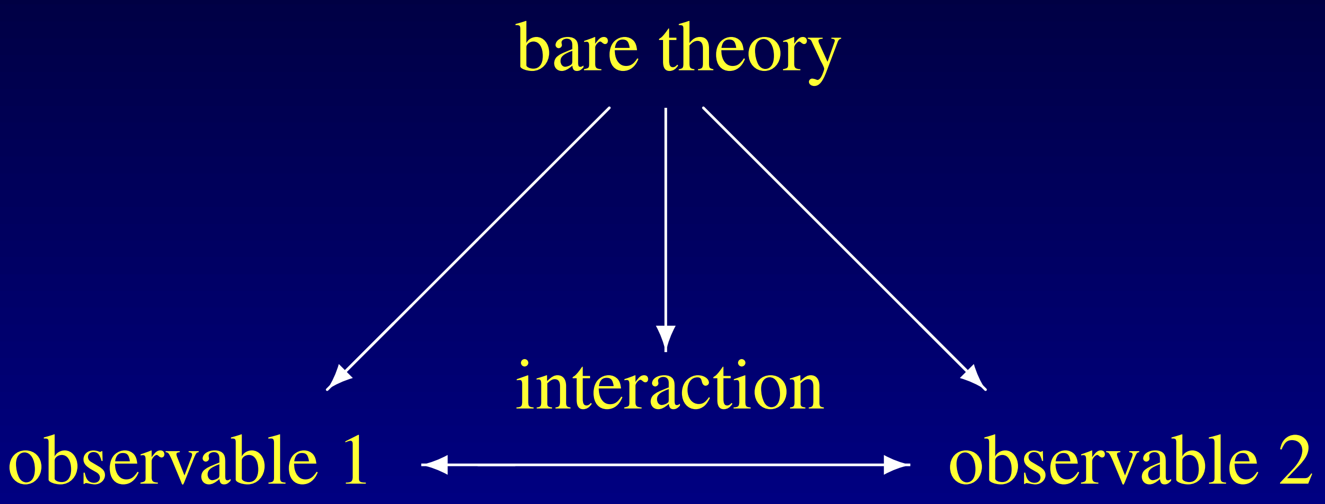


observer at distance r measures not the true (bare) charge of the electron, but an effective charge reduced by the excess of positrons over electrons inside a sphere of radius r

effective charge increases as $r \to 0$

- In contast to the ball in water, the electron cannot be taken out of its electromagnetic field $\Rightarrow$ the bare electron charge is not observable
- renormalisation is the transition from the (unobservable) bare quantities to the (observable) effective quantities

Quantum field theory



…experiments

How to find the bare theory?

- quantum field theory is quantum physics applied to systems with infinitely many degrees of freedom
- perturbation theory is good (at least for the electron)!

Idea: The bare theory …

… is classical field theory translated into operators on Hilbert space through the rules of quantum physics

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Quantisation

- starting point: action functional (for scalar field $\phi(t,x)$)

$$S = \int dt\, d^3\boldsymbol{x} \Big(\frac{1}{2} \Big(\frac{\partial \phi}{\partial t} \Big)^2 - \frac{1}{2} \sum_{i=1}^{3} \Big(\frac{\partial \phi}{\partial x^i} \Big)^2 + \frac{m^2}{2} \phi^2 + \frac{\lambda}{4} \phi^4 \Big)$$

- regard $\phi(t,\boldsymbol{x})$ as coordinates, $\pi(\boldsymbol{x},t) := \dfrac{\delta S}{\delta(\partial_t \phi(t,\boldsymbol{x}))}$ as momenta

- promote $\phi(t,\boldsymbol{x})$ and $\pi(t,\boldsymbol{x})$ to operators on Hilbert space and Poisson brackets to commutators

- leads to commutation relations

$$[\phi(t,\boldsymbol{x}), \phi(t,\boldsymbol{y})] = 0$$
$$[\pi(t,\boldsymbol{x}), \pi(t,\boldsymbol{y})] = 0$$
$$[\phi(t,\boldsymbol{x}), \pi(t,\boldsymbol{y})] = \mathrm{i}\delta(\boldsymbol{x}-\boldsymbol{y})$$

Free theory $\lambda = 0$

- consider global commutator $G(t,\boldsymbol{x};s,\boldsymbol{y}) = [\phi(t,\boldsymbol{x}),\phi(s,\boldsymbol{y})]$
- equal-time commutators provide initial conditions for solution of equation of motion $\left(\frac{\partial^2}{\partial t^2} + \Delta_{\boldsymbol{x}} + m^2\right)G(t,\boldsymbol{x};s,\boldsymbol{y}) = 0$
- causality: $G(t,\boldsymbol{x};s,\boldsymbol{y}) = 0$ if $(t-s)^2 - \|\boldsymbol{x}-\boldsymbol{y}\|^2 < 0$

  $\Rightarrow$ only expectation values of time-ordered products

  $\langle\psi|T\phi(t_1,\boldsymbol{x}_1)\ldots\phi(t_n,\boldsymbol{x}_n)|\psi\rangle$ are meaningful
- Problem: free theory corresponds to infinitely many harmonic oscillators with divergent vacuum energy

  $\phi(t,\boldsymbol{x}) = \phi^-(t,\boldsymbol{x}) + \phi^+(t,\boldsymbol{x})$ (annihilation and creation)

  normal ordering = putting ϕ^+ left of ϕ^-

- Wick's theorem:
  connects operator product, normal product and commutators
- vacuum expectation values: use $\phi^{-}|0\rangle = 0$ and $\langle 0|\phi^{+} = 0$
  e.g.
  $$\begin{aligned}\langle 0|T\phi_1\phi_2\phi_3\phi_4|0\rangle &= \langle 0|T\phi_1\phi_2|0\rangle\langle 0|T\phi_3\phi_4|0\rangle \\ &+ \langle 0|T\phi_1\phi_3|0\rangle\langle 0|T\phi_2\phi_4|0\rangle \\ &+ \langle 0|T\phi_1\phi_4|0\rangle\langle 0|T\phi_2\phi_3|0\rangle\end{aligned}$$
- it suffices to compute two-point functions
  for the free scalar theory they are
  $$\langle 0_{\mathrm{f}}\big|T\phi_{\mathrm{f}}(t,\boldsymbol{x})\phi_{\mathrm{f}}(s,\boldsymbol{y})\big|0_{\mathrm{f}}\big\rangle = \lim_{\epsilon\to 0}\int dk_0 d^3\boldsymbol{k}\ \frac{\mathrm{e}^{-\mathrm{i}(k_0(t-s)-\boldsymbol{k}\cdot(\boldsymbol{x}-\boldsymbol{y}))}}{k_0^2-\|\boldsymbol{k}\|^2-m^2+\mathrm{i}\epsilon}$$
  (Feynman propagator)
- *S*-matrix (scattering matrix) $\psi_{\mathrm{out}} = S^{-1}\psi_{\mathrm{in}}$ from LSZ reduction formula
  axioms for S: Lorentz covariance, causality, unitarity

Interacting theory

- impossible (at least very difficult) to treat exactly
  - interaction picture (does not exist) $\rightarrow$ Gell-Mann-Low formula

$$\langle 0_\mathrm{i} | T \phi_\mathrm{i}(x_1) \ldots \phi_\mathrm{i}(x_n) | 0_\mathrm{i} \rangle = \frac{\langle 0_\mathrm{f} | T \phi_\mathrm{f}(x_1) \ldots \phi_\mathrm{f}(x_n)\, \mathrm{e}^{\mathrm{i} : S_\mathrm{int}[\phi_\mathrm{f}] :} | 0_\mathrm{f} \rangle}{\langle 0_\mathrm{f} | T \mathrm{e}^{\mathrm{i} : S_\mathrm{int}[\phi_\mathrm{f}] :} | 0_\mathrm{f} \rangle}$$

  - permits perturbative computation of time-ordered vacuum expectation values
- the expectation values diverge for all interesting models
  (no suprise because Gell-Mann-Low formula is illegally derived)
- perturbative renormalisation theory
  - computes (for renormalisable theories) the expectation values as formal power-series in λ
  - finite in each order, but sum of the series is divergent

- equivalent formulation: Feynman's path integral

$$\langle 0_{\rm i} | T\phi_{\rm i}(x_1)\ldots\phi_{\rm i}(x_n) | 0_{\rm i}\rangle = \frac{\int \mathcal{D}[\phi]\ \phi(x_1)\cdots\phi(x_n)\ {\rm e}^{{\rm i}S[\phi]}}{\int \mathcal{D}[\phi]\ {\rm e}^{{\rm i}S[\phi]}}$$

- interpretation:
  - sum over all histories of an event
  - amplitude of a history given by ${\rm e}^{{\rm i}S[\phi]}$
- path integral dominated by stationary points of the action
  - non-classical histories suppressed by interference
- convenient for computation (generating functionals, see below)
- mathematically not well-defined (in contrast to Wiener measure)
  - heuristic guide to remarkable developments in mathematics (e.g. Seiberg-Witten invariants)

Euclidean quantum field theory

- partition function

$$\langle \phi(x_1) \dots \phi(x_n) \rangle = \frac{\int \mathcal{D}[\phi]\ \phi(x_1) \cdots \phi(x_n)\ \mathrm{e}^{-S[\phi]}}{\int \mathcal{D}[\phi]\ \mathrm{e}^{-S[\phi]}}$$

  expectation value in quantum statistical equilibrium

  - formally the analytic continuation to $t \in \mathbb{C}$
  - justified under Osterwalder-Schrader axioms
  - describes generalised spin system in statistical physics

- Euclidean framework permits

  - rigorous construction of QFT's in lower dimension
  - simulation of critical behaviour in lattice models
  - renormalisation group methods

Generating functionals

powerful method to generate formal expressions for expectation values of time-ordered products (Green's functions)

- general Green's functions

$$Z[J] = \int \mathcal{D}[\phi]\; \mathrm{e}^{\mathrm{i}(S[\phi] - \int d^4x\, \phi(x) J(x))}$$

$$\langle 0_\mathrm{i} | T\phi_\mathrm{i}(x_1)\dots\phi_\mathrm{i}(x_n) | 0_\mathrm{i}\rangle = \frac{1}{Z[0]}\left(\frac{\mathrm{i}^n \delta^n Z[J]}{\delta J(x_1)\dots\delta J(x_n)}\right)_{J=0}$$

- perturbation theory in λ: $S[\phi] = S_\mathrm{bil}[\phi] + S_\mathrm{int}[\phi]$

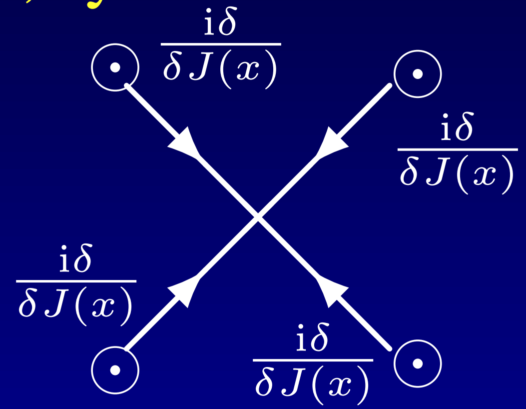
$$Z[J] = \sum_{k=0}^{\infty} \frac{1}{k!}\left(\mathrm{i}S_\mathrm{int}\left[\frac{\mathrm{i}\delta}{\delta J}\right]\right)^k Z_\mathrm{f}$$

$$Z_\mathrm{f} = Z_\mathrm{f}[0]\;\exp\left(-\frac{1}{2}\int d^4x\, d^4y\; J(x)\langle 0_\mathrm{f}|T\phi(x)\phi(y)|0_\mathrm{f}\rangle J(y)\right)$$

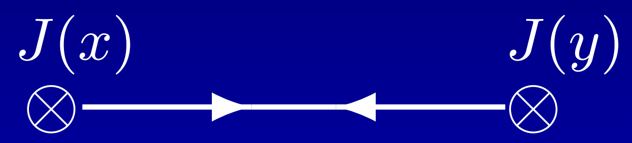
Feynman graphs

- for $S_{\text{int}} = \frac{\lambda}{4!}\int d^4x\,(\phi(x))^4$, symbolise J-derivatives by vertex

$$S_{\text{int}}\left[\mathrm{i}\frac{\delta}{\delta J}\right] = \frac{\lambda}{4!}\int d^4x \left(\quad \right)$$



acting on $\quad \exp\left(-\frac{1}{2}\int d^4x\, d^4y \quad \right)$



  - formal expressions (integrals diverge), ensures combinatorics
- generating functional for connected graphs $W[J] = \ln Z[J]$
- generating functional for one-particle irreducible (1PI) graphs
  - define classical fields $\phi_{c\ell}(x) = \frac{\delta W[J]}{\delta J(x)}$
  - Legendre transformation

$$\Gamma[\phi_{c\ell}] = \int d^4x\; \phi_{c\ell}(x) J(x) - W[J]\Big|_{J=J[\phi_{c\ell}]}$$

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Treatment of divergences

- the space-time integrals encoded by Feynman graphs diverge
  - interaction picture illegal (or: not yet understood)
  - product of distributions not globally defined
  - bare parameters not observable; not necessarily bounded
  - space-time at short distances could be different

- solution: renormalisation = (infinite) recursive adjustment (order by order in perturbation theory) of the bare parameters so that
  - a finite number of observables is tuned to “experiment”
  - all other (infinitely many) observables are bounded
  - the axioms of the S-matrix are satisfied
  - possible symmetries are preserved

Regularisation

- crucial point: observables are measured at some scale μ
- introduce a regulator ϵ, e.g.
  - lattice of spacing $\frac{\epsilon}{\mu}$ (and possibly finite volume)
  - frequency cut-off $\|p\| < \frac{\mu}{\epsilon}$ in momentum space
  - dimensional regularisation $\int d^4k \mapsto \mu^{2\epsilon} \int d^{4-2\epsilon}k$
  - analytical regularisation $\frac{1}{p^2+m^2} \mapsto \frac{\mu^{2\epsilon}}{(p^2+m^2)^{1+\epsilon}}$
- then:

$$\mathcal{RO}_i = \mathcal{RO}_i[\mathcal{B}_j, \epsilon, \mu] \;, \quad i = 0, \ldots, n$$
$$\mathcal{IO}_k = \mathcal{IO}_k[\mathcal{B}_j, \epsilon, \mu]$$

where

$\mathcal{RO}_i, \mathcal{IO}_k$ – relevant/irrelevant observables, $\mathcal{B}_j$ – bare parameters

- invert the first relation to $\mathcal{B}_j = \mathcal{B}_j[\mathcal{RO}_i, \epsilon, \mu]$

  prove that $\mathcal{IO}_k[\mathcal{B}_j[\mathcal{RO}_i, \epsilon, \mu], \epsilon, \mu]$ converges for $\epsilon \to 0$, for all k

Perturbation theory

- In practise, the solution for the bare parameters is only possible in perturbation theory: $\{\mathcal{RO}_i\} = \{\lambda, \mathcal{RO}'_i\}$
  then: $\mathcal{B}_j = \sum_{l=0}^{\infty} \lambda^l \mathcal{B}_j^{(l)}[\mathcal{RO}'_i, \epsilon, \mu]$
  leads to
  $$\mathcal{IO}_k\Big[\sum_{l=0}^{\infty} \lambda^l \mathcal{B}_j^{(l)}[\mathcal{RO}'_i, \epsilon, \mu], \epsilon, \mu\Big] = \sum_{\ell=1}^{\infty} \lambda^\ell \mathcal{IO}_k^{(\ell)}$$
- This is a recursive procedure.
  D. Kreimer, soon joint by A. Connes, discovered that this procedure corresponds to the computation of the antipode of a Hopf algebra of Feynman graphs.
- There is a global solution for the antipode, Zimmermann's forest formula. It deals with the disentanglement of overlapping divergences.

Renormalisation group

- so far: $\mu = \text{const}$ – normalisation scale

- different viewpoint [Wilson]:
  - bare observables defined at fundamental cut-off scale $\frac{\mu}{\epsilon_0}$
  - QFT computes observables at smaller scale $\frac{\mu}{\epsilon_1}$, $\epsilon < \epsilon_1 \leq 1$
  - regard observables at $\frac{\mu}{\epsilon_1}$ as bare observables for another QFT-computation of observables at scale $\frac{\mu}{\epsilon_2}$, and so on

- consequence: observables together with a scale form a semigroup, the renormalisation group

- dimensional analysis: $\mathcal{O}_i[\frac{\mu}{\epsilon}] \sim \left(\frac{\epsilon}{\epsilon_0}\right)^{\omega_i} \mathcal{O}_i[\frac{\mu}{\epsilon_0}]$

  $\omega_i < 0$ for irrelevant observables, $\lim_{\epsilon_0 \to 0}$ exists

  $\omega_i \geq 0$ for relevant observables, only finitely many are OK

- other point: if $\frac{\mu}{\epsilon_0}$ fundamantal scale, then irrelevant observables are small at scale $\frac{\mu}{\epsilon} \ll \frac{\mu}{\epsilon_0}$

  explains why experiments can only see renormalisable models: renormalisable (relevant) operators survive to laboratory energies, non-renormalisable (irrelevant) ones are scaled away

- relevant operators, normalised at $\frac{\mu}{\epsilon}$, become huge at $\frac{\mu}{\epsilon_0}$ unless $\omega = 0$ (marginal operators)

  – for $\omega = 0$ the subleading logarithmic corrections are important

  – the relevant operator for the coupling constant, in which we expand perturbatively, must be small for all scales

Non-perturbative renormalisation

- Models in which the coupling constant remains small for all scales are believed to exist non-perturbatively
- this includes non-Abelian Yang-Mills theories (which are asymptotically free), but not QED and ϕ^4
- non-perturbative investigations of Yang-Mills theory (mostly on lattices) are very difficult

Noncommutative ϕ^4-model

Moyal plane: algebra of Schwartz class functions with product

$$(a \star b)(x) = \int d^4y \frac{d^4k}{(2\pi)^4}\, a(x+\tfrac{1}{2}\theta\cdot k)b(x+y)\,\mathrm{e}^{\mathrm{i}ky}$$

Theorem. (H. Grosse, R.W.)

The quantum field theory defined by the action

$$S = \int d^4x \Big(\frac{1}{2}\phi \star (\Delta + \Omega^2\tilde{x}^2 + \mu^2)\phi + \frac{\lambda}{4!}\phi\star\phi\star\phi\star\phi\Big)(x)$$

with $\tilde{x} = 2\theta^{-1}\cdot x$, euclidean metric, ϕ a real field, is perturbatively renormalisable to all orders in λ.

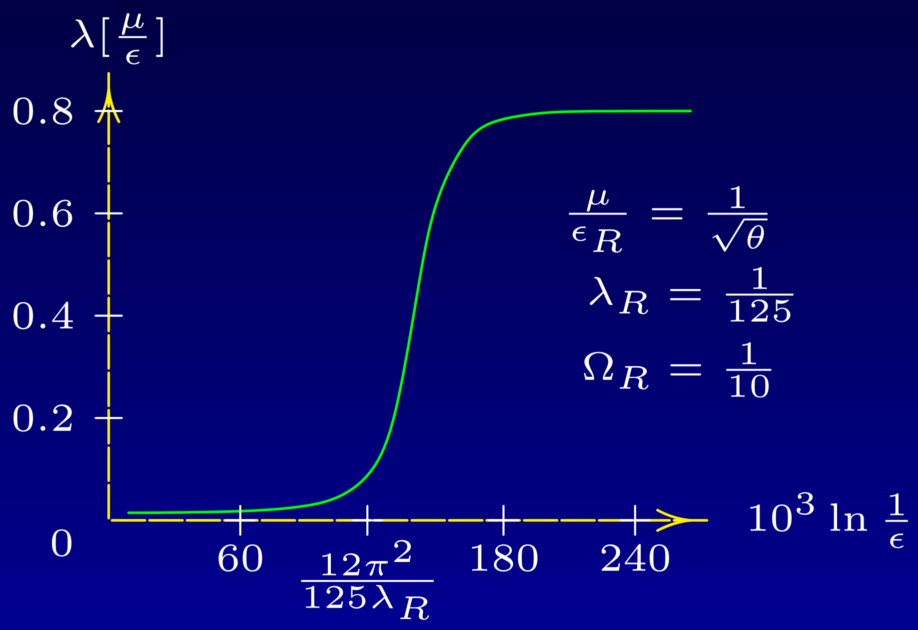
The additional oscillator potential $\Omega^2\tilde{x}^2$

- implements the mixing between large and small distance scales
- results from the renormalisation proof

One-loop calculation of the model

$$\frac{\lambda[\frac{\mu}{\epsilon}]}{\Omega^2[\frac{\mu}{\epsilon}]} = \text{const}$$

$$\Omega^2[\tfrac{\mu}{\epsilon}] \leq 1$$



- perturbation theory remains valid at all scales!
- non-perturbative construction of the model seems possible!

This is a joint project with V. Rivasseau and F. Vignes-Tourneret. The first step, a new perturbative renormalisation proof of the model, is already accomplished.