CONSTRUCTION and PROPERTIES of NONCOMMUTATIVE QUANTUM FIELDS

H. Grosse

Department of Physics, University of Vienna,
Vienna, Austria
[*] E-mail: harald.grosse@univie.ac.at

R. Wulkenhaar

Mathematisches Institut der Westfälischen Wilhelms-Universität
Münster, Germany
E-mail: raimar@math.uni-muenster.de

We review our recent successful attempt to construct the planar sector of a non-local scalar field model in four-dimensional Euclidean deformed space-time, which needs 4 (instead of 3) relevant/marginal operators in the defining Lagrangian. As we have shown earlier, this model is renormalizable up to all orders in perturbation theory. In addition a new fixed point appears, at which the beta function for the coupling constant vanishes. This way, we were able to tame the Landau ghost.

We next discuss Ward identities and Schwinger-Dyson equations and derive integral equations for the renormalized N-point functions. They are the starting point of an exact non-perturbative solution of the model.

Keywords: Noncommutative quantum fields, vanishing beta-function, non-perturbative construction, exactly solvable models

1. Introduction

Constructive methods led years ago to many beautiful ideas and results, but the main goal to construct a mathematical consistent model of a four-dimensional local quantum field theory has not been reached. We realized that certain models over deformed space-time behave much better. As a first step we report on the construction of a simple quantum field theory model on the Moyal space. Of course, this way we obtain models with non-local interactions.

The naïve application of this procedure to the ϕ^4-action leads on the four-dimensional Euclidean Moyal plane, defined by the $\star$-product: $(a\star b)(x) = \int \frac{d^4y\, d^4k}{(2\pi)^4}\; a(x{+}\frac{1}{2}\Theta{\cdot}k)b(x{+}y)\,\mathrm{e}^{iky}$ where $\Theta{=}-\Theta^T{\in}M_4(\mathbb{R})$ denotes a rank 4 matrix, to the action:

$$S = \int d^4x\Big(\frac{1}{2}\partial_\mu\phi\star\partial^\mu\phi + \frac{m^2}{2}\phi\star\phi + \frac{\lambda}{4}\phi\star\phi\star\phi\star\phi\Big)(x)\;. \tag{1}$$

The Feynman rules can be obtained easily. Since we obtain only cyclic invariance at the vertex, graphs are best drawn as Ribbon Graphs on Riemann surfaces with a certain genus and a certain number of boundary components. We obtain planar and non-planar graphs. The planar graphs still reveal UV divergences, the non-planar ones are finite for generic momenta. On the other hand for exceptional momenta (if sums of incoming or outgoing momenta vanish) the contributions develop an IR singularity, which spoils renormalizability! In our previous work[1] we realized that the UV/IR-mixing problem can be solved by adding a fourth relevant/marginal operator to the Lagrangian:

Theorem 1.1. *The quantum field theory defined by the action*

$$S = \int d^4x \Big(\frac{1}{2}\phi \star \big(\Delta + \Omega^2 \tilde{x}^2 + \mu^2\big)\phi + \frac{\lambda}{4!}\phi \star \phi \star \phi \star \phi \Big)(x) \;, \qquad (2)$$

where $\tilde{x} = 2\Theta^{-1} \cdot x$*, is perturbatively renormalizable to all orders in* λ*.*

The additional oscillator potential $\Omega^2\tilde{x}^2$ implements mixing between large and small distance scales and results from the renormalization proof. Maja Buric and Michael Wohlgenannt[2] found an interesting interpretation of this additional term: It results as the coupling of the scalar field to the scalar curvature within a truncation procedure.

The model is covariant under the Langmann-Szabo duality transformation[3] and becomes self-dual at $\Omega = 1$. Certain variants have also been treated, see[4] for a review. Evaluation of the β-functions for the coupling constants Ω, λ in first order of perturbation theory leads to a coupled dynamical system which indicates a fixed-point at $\Omega = 1$, while λ remains bounded.[5,6] The vanishing of the β-function at $\Omega = 1$ was next proved at three-loop order[7] and finally to all orders of perturbation theory.[8] It implies that there is no infinite renormalization of λ, which makes the non-perturbative construction simpler. The Landau ghost problem is solved.

The vanishing of the β-function to all orders has been obtained using a Ward identity.[8] We extend this work and derive an integral equation for the two-point function alone by using the Ward identity and Schwinger-Dyson equations. Usually, Schwinger-Dyson equations couple the two-point function to the four-point function. In our model, we show that the Ward identity allows to express the four-point function in terms of the two-point function, resulting in an equation for the planar two-point function alone. This is achieved in the first step for the bare two-point function. We are able to perform the mass and wavefunction renormalization directly in the integral equation, giving a *self-consistent non-linear equation for the renormalized two-point function alone*.[9] Higher n-point functions fulfill a *linear* (inhomogeneous) Schwinger-Dyson equation, with the inhomogeneity given by m-point functions with $m < n$. This means that solving our equation for the two-point function leads to a full non-perturbative construction of this interacting quantum field theory in four dimensions. Recently we reduced the question of solving this model to solving one non-linear integral equation in one variable.[10]

Of course, the next question concerns the non-planar sector of this model. We know the appropriate Ward identities, an extension of the reviewed ideas to this sector is under discussion.

In the case of the Φ^4 model with negative coupling constant, it was possible to sum up the planar graphs, but the non-planar graphs cannot be summed up, due to lack of stability.[11,12] In the present model, we have a positive coupling constant and stability is not a problem. Nevertheless the construction of the full model is still a hard task. A new summation technique has been invented recently for such a situation.[13] It has been applied to the two-dimensional model already.[14]

We hope that a detailed analysis of our model will help for a non-perturbative treatment of more realistic quantum field theories.

2. Matrix Model

It is convenient to write the action (2) in the matrix base of the Moyal space.[1] It simplifies enormously at the self-duality point $\Omega = 1$. We write down the resulting action functionals for the *bare* quantities, which involves the bare mass μ_{bare} and the wave function renormalization $\phi \mapsto Z^{\frac{1}{2}}\phi$. For simplicity we fix the length scale to $\theta = 4$. This gives

$$S = \sum_{m,n\in\mathbb{N}^2_\Lambda} \frac{1}{2}\phi_{mn}H_{mn}\phi_{nm} + V(\phi)\ , \tag{3}$$

$$H_{mn} = Z\big(\mu^2_{bare} + |m| + |n|\big)\ , \qquad V(\phi) = \frac{Z^2\lambda}{4}\sum_{m,n,k,l\in\mathbb{N}^2_\Lambda} \phi_{mn}\phi_{nk}\phi_{kl}\phi_{lm}\ , \tag{4}$$

It is already used that this model has no renormalization of the coupling constant.[8] All summation indices $m, n, \ldots$ belong to $\mathbb{N}^2$, with $|m| := m_1 + m_2$. The symbol $\mathbb{N}^2_\Lambda$ refers to a cut-off in the matrix size. The scalar field is real, $\phi_{mn} = \overline{\phi_{nm}}$.

3. Ward Identity

The key step in the proof[8] that the β-function vanishes is the discovery of a Ward identity induced by inner automorphisms $\phi \mapsto U\phi U^\dagger$. Inserting into the connected graphs the special insertion vertex

$$V^{ins}_{ab} := \sum_n (H_{an} - H_{nb})\phi_{bn}\phi_{na} \tag{5}$$

is the same as the difference of graphs with external indices b and a, respectively, $Z(|a|-|b|)G^{ins}_{[ab]\ldots} = G_{b\ldots} - G_{a\ldots}$. Here the dots stand for the remaining face indices. We have used $H_{an} - H_{nb} = Z(|a| - |b|)$.

We write Feynman graphs in the self-dual ϕ^4_4-model as ribbon graphs on a genus-g Riemann surface with B external faces. Adding for each external face an external vertex to get a closed surface, the matrix index is constant at every face. Inserting the special vertex V_{ab}^{ins} leads, however, to an index jump from a to b in an external face which meets an external vertex. The corresponding external sources at the jumped face are thus J_{na} and J_{bm} for some other indices m, n. According to the Ward identity, this is the same as the difference between the graphs with face index b and a, respectively.

4. Schwinger-Dyson equation

The Schwinger-Dyson equation for the one-particle irreducible two-point function Γ_{ab} can be reexpressed in terms of the two-point function with insertion vertex. Adding the left tadpole and using the Ward identity yields

$$\begin{aligned}\Gamma_{ab} &= Z^2(-\lambda)\sum_p \Big(G_{ap} + G_{ab}^{-1} G_{[ap]b}^{ins}\Big) = Z^2(-\lambda)\sum_p \Big(G_{ap} - G_{ab}^{-1}\frac{G_{bp}-G_{ba}}{Z(|p|-|a|)}\Big) \\ &= -Z^2\lambda\sum_p\Big(\frac{1}{H_{ap}-\Gamma_{ap}} + \frac{1}{H_{bp}-\Gamma_{bp}} - \frac{1}{H_{bp}-\Gamma_{bp}}\frac{(\Gamma_{bp}-\Gamma_{ab})}{Z(|p|-|a|)}\Big) \,. \qquad (6)\end{aligned}$$

This is a closed equation for the two-point function alone. It involves the divergent quantities Γ_{bp} and Z, μ_{bare}.

5. Renormalization

Introducing the renormalized planar two-point function Γ_{ab}^{ren} by Taylor expansion $\Gamma_{ab} = Z\mu_{bare}^2 - \mu^2 + (Z-1)(|a|+|b|) + \Gamma_{ab}^{ren}$ and imposing the renormalization condition $\Gamma_{00}^{ren} = 0$ and $(\partial\Gamma^{ren})_{00} = 0$, we obtain a coupled system of equations for Γ_{ab}^{ren}, Z and μ_{bare}. It leads to a closed equation for the renormalized function Γ_{ab}^{ren} alone, which is further analyzed in the integral representation.

We replace the indices in $a, b, \ldots \mathbb{N}$ by continuous variables in $\mathbb{R}_+$. Equation (6) depends only on the length $|a| = a_1 + a_2$ of indices. The Taylor expansion respects this feature, so that we replace $\sum_{p\in\mathbb{N}_\Lambda^2}$ by $\int_0^\Lambda |p|\,dp$. After a convenient change of variables $|a| =: \mu^2\frac{\alpha}{1-\alpha}$, $|p| =: \mu^2\frac{\rho}{1-\rho}$ and

$$\Gamma_{ab}^{ren} =: \mu^2\frac{1-\alpha\beta}{(1-\alpha)(1-\beta)}\Big(1 - \frac{1}{G_{\alpha\beta}}\Big) \,, \qquad (7)$$

and using an identity resulting from the symmetry $G_{0\alpha} = G_{\alpha 0}$, we arrive at:[9]

Theorem 5.1. *The renormalized planar connected two-point function $G_{\alpha\beta}$ of the self-dual noncommutative ϕ_4^4-theory satisfies the integral equation*

$$\begin{aligned}G_{\alpha\beta} = 1 - \lambda\Bigg(&\frac{1-\alpha}{1-\alpha\beta}\big(\mathcal{M}_\beta - \mathcal{L}_\beta - \beta\mathcal{Y}\big) + \frac{1-\beta}{1-\alpha\beta}\big(\mathcal{M}_\alpha - \mathcal{L}_\alpha - \alpha\mathcal{Y}\big) \qquad (8)\\ &+ \frac{1-\beta}{1-\alpha\beta}\Big(\frac{G_{\alpha\beta}}{G_{0\alpha}} - 1\Big)\big(\mathcal{M}_\alpha - \mathcal{L}_\alpha + \alpha\mathcal{N}_{\alpha 0}\big) \\ &- \frac{\alpha(1-\beta)}{1-\alpha\beta}\big(\mathcal{L}_\beta + \mathcal{N}_{\alpha\beta} - \mathcal{N}_{\alpha 0}\big) + \frac{(1-\alpha)(1-\beta)}{1-\alpha\beta}(G_{\alpha\beta}-1)\mathcal{Y}\Bigg) \,,\end{aligned}$$

where $\alpha, \beta \in [0,1)$,

$$\mathcal{L}_\alpha := \int_0^1 d\rho\, \frac{G_{\alpha\rho} - G_{0\rho}}{1-\rho}\ , \quad \mathcal{M}_\alpha := \int_0^1 d\rho\, \frac{\alpha\, G_{\alpha\rho}}{1-\alpha\rho}\ , \quad \mathcal{N}_{\alpha\beta} := \int_0^1 d\rho\, \frac{G_{\rho\beta} - G_{\alpha\beta}}{\rho - \alpha}\ ,$$

and $\mathcal{Y} = \lim_{\alpha\to 0} \frac{\mathcal{M}_\alpha - \mathcal{L}_\alpha}{\alpha}$.

Recently we related the construction of this noncommutative quantum field theory to the problem of solving a non-linear integral equation in one variable,[10] which we review next.

6. Non-perturbative Construction of this model

We rewrite equation (8) in terms of $D_{\alpha\beta} := \frac{\alpha(1-\beta)}{\beta(1-\alpha)}\Big(\frac{(1-\beta)}{1-\alpha\beta}G_{\alpha\beta} - G_{\alpha 0}\Big)$ and obtain after simple manipulations the integral equation

$$\frac{\beta(1-\alpha)}{\alpha(1-\beta)} + \frac{1+\lambda\mathcal{Y} + \lambda\pi\alpha\mathcal{H}_\alpha\big[G_{\bullet 0}\big]}{\alpha G_{\alpha 0}} D_{\alpha\beta} - \lambda\pi\mathcal{H}_\alpha\big[D_{\bullet\beta}\big] = -G_{\alpha 0}\ , \qquad (9)$$

$$-\lambda\pi\mathcal{H}_0[D_{\bullet 0}] = \frac{\lambda\mathcal{Y}}{1+\lambda\mathcal{Y}}\ ,$$

which is of the Carleman type. Here we assume that $D_{\alpha\beta}$ is Hölder continuous. The finite Hilbert transform is given by $\mathcal{H}_\alpha[f(\bullet)] := \dfrac{1}{\pi}\lim_{\epsilon\to 0}\Big(\int_0^{\alpha-\epsilon} + \int_{\alpha+\epsilon}^1\Big)\dfrac{f(\rho)}{\rho-\alpha}$. Equation (9) is a singular linear integral equation of the Carleman type. We quote its solution:[15,16]

Theorem 6.1. *The singular linear integral equation* $a(x)y(x) - \lambda\pi\mathcal{H}_x[y] = f(x)$, *for* $x \in [-1,1]$, *is for* $a(x)$ *continuous and Hölder continuous near* ± 1 *and* $f \in L^p[-1,1]$ *solved by*

$$y(x) = \frac{\sin(\theta(x))}{\lambda\pi}\Big(f(x)\cos(\theta(x)) + e^{\mathcal{H}_x[\theta]}\mathcal{H}_x\Big[e^{-\mathcal{H}_\bullet[\theta]}f(\bullet)\sin(\theta(\bullet))\Big] + \frac{C e^{\mathcal{H}_x[\theta]}}{1-x}\Big) \qquad (10)$$

$$\theta(x) = \operatorname*{arctan}_{[0,\,\pi]}\Big(\frac{\lambda\pi}{a(x)}\Big)\ , \quad \sin(\theta(x)) = \frac{|\lambda\pi|}{\sqrt{(a(x))^2 + (\lambda\pi)^2}} \qquad (11)$$

where C *is an arbitrary constant.*

We assume first $C = 0$ and apply the solution of the Carleman equation to (9):

$$\frac{(1-\beta)}{1-\alpha\beta}\frac{G_{\alpha\beta}}{1+\lambda\mathcal{Y}} = \frac{\sin(\theta_\beta(\alpha))}{|\lambda|\pi\alpha} e^{\mathcal{H}_\alpha[\theta_\beta(\bullet)] - \mathcal{H}_0[\theta_0(\bullet)] + \mathcal{H}_1[\theta_0(\bullet) - \theta_\beta(\bullet)]}\ , \qquad (12)$$

$$\frac{\lambda\mathcal{Y}}{1+\lambda\mathcal{Y}} = \int_0^1 d\rho\, \frac{\sin^2(\theta_0(\rho)}{\lambda\pi^2\rho^2}\ , \qquad (13)$$

$$\theta_\beta(\alpha) = \operatorname*{arctan}_{[0,\,\pi]}\left(\frac{\lambda\pi\alpha}{\frac{\beta(1-\alpha)}{1-\beta} + \frac{1+\lambda\mathcal{Y}+\lambda\pi\alpha\mathcal{H}_\alpha[G_{\bullet 0}]}{G_{\alpha 0}}}\right).$$

For the proof one uses the Carleman equation $\lambda\pi\cot\theta_0(\alpha)G_{\alpha 0}-\lambda\pi\mathcal{H}_\alpha[G_{\bullet 0}]=\frac{1+\lambda\mathcal{Y}}{\alpha}$ and Tricomi's identity[16] $e^{-\mathcal{H}_\alpha[\theta_\beta]}\cos(\theta_\beta(\alpha))+\mathcal{H}_\alpha\Big[e^{-\mathcal{H}_\bullet[\theta_\beta]}\sin(\theta_\beta(\bullet)\Big]=1$.

The Carleman equation computes $G_{\alpha\beta}$, as a consequence it implies that $G_{\alpha\beta}\geq 0$! Therefore $G_{0\beta}$ can be evaluated and this implies also a self-consistency equation for $G_{\beta 0}$, since symmetry forces $G_{\beta 0}=G_{0\beta}$. This leads to the Master Equation, whose solution determines the theory completely:

$$G_{\beta 0}=\frac{1+\lambda\mathcal{Y}}{1+(1-\beta)\lambda\mathcal{Y}}\exp\Big(-\lambda\int_0^{\frac{\beta}{1-\beta}}dt\int_0^1\frac{d\rho}{(\lambda\pi\rho)^2+\big(t(1-\rho)+\frac{1+\lambda\mathcal{Y}+\lambda\pi\rho\mathcal{H}_\rho[G_{\bullet 0}]}{G_{\rho 0}}\big)^2}\Big), \tag{14}$$

provided it exists, together with $\lambda\mathcal{Y}$ which has to be determined from equation (13)).

Up to now, we deduced various non-perturbative results from this system of equations and used computer calculations for the visualization of the solution of (14). As expected, there is a big difference between the case $\lambda>0$ and $\lambda<0$. For positive $\lambda>0$ we deduce that $\frac{(1+(1-\beta)\lambda\mathcal{Y})}{1+\lambda\mathcal{Y}}G_{\beta 0}\in\mathcal{C}^1([0,1[)$, is monotonously decreasing and positive. Therefore the limiting value G_{10} exists and $G_{\beta 0}\in\mathcal{C}[0,1]$. For $\lambda<0$ $\frac{(1+(1-\beta)\lambda\mathcal{Y})}{1+\lambda\mathcal{Y}}G_{\beta 0}\in\mathcal{C}^1([0,1[)$ is monotonously increasing and positive, therefore $G_{\beta 0}$ is unbounded at $\beta=1$. Moreover, we can prove that the equation for the inverse wavefunction renormalization Z^{-1} which results from (6) leads to $\lim_{\Lambda\to\infty}Z^{-1}=0$ for $\lambda>0$ but $\lim_{\Lambda\to\infty}Z^{-1}=\infty$ for $\lambda<0$.

7. Four-point Schwinger-Dyson equation

The knowledge of the two-point function allows a successive construction of the whole theory. As an example we mention the planar connected four-point function G_{abcd}. The Ward identity gives rise to the following Schwinger-Dyson equation for G_{abcd}:[10]

$$Z^{-1}G_{abcd}=\lambda\sum_{p\in\mathbb{N}^2_\lambda}\frac{G_{ab}G_{pbcd}-G_{pb}G_{abcd}}{Z(|p|-|a|)}+\lambda G_{ab}G_{bc}\frac{G_{cd}-G_{ad}}{(|c|-|a|)}\,. \tag{15}$$

After amputation of the external two-point functions we obtain the Schwinger-Dyson equation for the *renormalized* 1PI four-point function $G_{abcd}=G_{ab}G_{bc}G_{cd}G_{da}\Gamma^{ren}_{abcd}$ as follows:

$$Z^{-1}\Gamma^{ren}_{abcd}=\frac{\lambda}{|a|-|c|}\Big(\frac{1}{G_{ad}}-\frac{1}{G_{cd}}\Big)+\sum_{p\in\mathbb{N}^2_\Lambda}\frac{\lambda}{|a|-|p|}G_{pb}\Big(\frac{G_{dp}}{G_{ad}}\Gamma^{ren}_{pbcd}-\Gamma^{ren}_{abcd}\Big)\,. \tag{16}$$

We pass to the integral representation and to the continuous variables $\alpha,\beta,\gamma,\delta$ and find for $\Gamma_{\alpha\beta\gamma\delta}:=\Gamma^{ren}_{abcd}$ an integral equation which is linear in the four-point function and non-linear in the (by the previous work known) two-point function $G_{\alpha\beta}$. Manipulated appropriately, the limit $\Lambda\to\infty$ exists after insertion of the equation for the inverse wavefunction renormalization Z^{-1}. We find:[9]

Theorem 7.1. *The renormalized planar 1PI four-point function* $\Gamma_{\alpha\beta\gamma\delta}$ *of self-dual noncommutative* ϕ_4^4*-theory (with continuous indices* $\alpha, \beta, \gamma, \delta \in [0,1)$*) satisfies the integral equation*

$$\Gamma_{\alpha\beta\gamma\delta} = \lambda \cdot \frac{\begin{array}{l}\Big(1 - \dfrac{(1-\alpha)(1-\gamma\delta)(G_{\alpha\delta}-G_{\gamma\delta})}{G_{\gamma\delta}(1-\delta)(\alpha-\gamma)} \\ \qquad + \displaystyle\int_0^1 \rho\, d\rho \frac{(1-\beta)(1-\alpha\delta)G_{\beta\rho}G_{\delta\rho}}{(1-\beta\rho)(1-\delta\rho)} \frac{\Gamma_{\rho\beta\gamma\delta}-\Gamma_{\alpha\beta\gamma\delta}}{\rho-\alpha}\Big)\end{array}}{\begin{array}{l}G_{\alpha\delta} + \lambda\Big((\mathcal{M}_\beta - \mathcal{L}_\beta - \mathcal{Y})G_{\alpha\delta} + \displaystyle\int_0^1 d\rho \frac{G_{\alpha\delta}G_{\beta\rho}(1-\beta)}{(1-\delta\rho)(1-\beta\rho)} \\ \qquad + \displaystyle\int_0^1 \rho\, d\rho \frac{(1-\beta)(1-\alpha\delta)G_{\beta\rho}}{(1-\beta\rho)(1-\delta\rho)} \frac{(G_{\rho\delta}-G_{\alpha\delta})}{(\rho-\alpha)}\Big)\end{array}} . \tag{17}$$

In our recent work,[10] we have been able to solve equation (17) in terms of the two-point function and a remarkable simple expression results:

$$\Gamma_{\alpha\beta\gamma\delta} = \frac{\lambda}{(\alpha-\gamma)(\beta-\delta)} \Big(\frac{(1-\alpha\delta)}{G_{\alpha\delta}} \frac{(1-\gamma\beta)}{G_{\gamma\beta}} - \frac{(1-\alpha\beta)}{G_{\alpha\beta}} \frac{(1-\gamma\delta)}{G_{\gamma\delta}} \Big) . \tag{18}$$

It was now possible to evaluate the effective coupling in terms of the bare coupling constant. Although the scale is changed by an infinite amount, a finite coupling constant renormalization results. This means that the β-function vanishes non-perturbatively!

8. Conclusions

A remarkable result concerns the appearance of the nontrivial fixed point at $\Omega = 1$, proved to all orders in perturbation theory. We used Ward identities and Schwinger-Dyson equations to deduce integral equations for the renormalized N-point functions. We reduced the construction of this non-trivial noncommutative quantum field theory to the solution of a non-linear integral equation for a function of one variable. A survey of this construction is given in a recent work of us.[10] This is the first non-trivial four-dimensional quantum field theory model. We achieved much more than just proving that the Feynman perturbation expansion can be resummed: *The non-perturbative planar two- and four-point functions are solved exactly*!

9. Acknowledgment:

I (H.G.) would like to thank Abdelmalek Abdesselam and Edwin Langmann for the kind invitation to present our results at this conference. Many thanks go to Prof. Arne Jensen and his team for the perfect organization of this event.

References

1. H. Grosse and R. Wulkenhaar, "Renormalisation of ϕ^4-theory on noncommutative $\mathbb{R}^4$ in the matrix base," Commun. Math. Phys. **256** (2005) 305 [arXiv:hep-th/0401128].

2. M. Buric and M. Wohlgenannt, "Geometry of the Grosse-Wulkenhaar Model," JHEP **1003** (2010) 053 [arXiv:0902.3408 [hep-th]].
3. E. Langmann and R. J. Szabo, "Duality in scalar field theory on noncommutative phase spaces," Phys. Lett. B **533** (2002) 168 [arXiv:hep-th/0202039].
4. V. Rivasseau, "Non-commutative renormalization" In: *Quantum Spaces (Séminaire Poincaré X)*, eds. B. Duplantier and V. Rivasseau, Birkhäuser Verlag Basel (2007) 19–109 [arXiv:0705.0705 [hep-th]].
5. H. Grosse and R. Wulkenhaar, "The β-function in duality-covariant noncommutative ϕ^4-theory," Eur. Phys. J. C **35** (2004) 277 [arXiv:hep-th/0402093].
6. H. Grosse and R. Wulkenhaar, "Renormalisation of ϕ^4-theory on non-commutative $\mathbb{R}^4$ to all orders," Lett. Math. Phys. **71** (2005) 13. [arXiv:hep-th/0403232].
7. M. Disertori and V. Rivasseau, "Two and three loops beta function of non commutative ϕ^4_4 theory," Eur. Phys. J. C **50** (2007) 661 [arXiv:hep-th/0610224].
8. M. Disertori, R. Gurau, J. Magnen and V. Rivasseau, "Vanishing of beta function of non commutative ϕ^4_4 theory to all orders," Phys. Lett. B **649** (2007) 95 [arXiv:hep-th/0612251].
9. H. Grosse and R. Wulkenhaar, "Progress in solving a noncommutative quantum field theory in four dimensions," arXiv:0909.1389 [hep-th].
10. H. Grosse and R. Wulkenhaar, "Self-dual noncommutative ϕ^4-theory in four dimensions is a non-perturbatively solvable and non-trivial quantum field theory," arXiv:1205.0465 [math-ph].
11. G. 't Hooft, "Rigorous Construction Of Planar Diagram Field Theories in Four-dimensional Euclidean Space," Commun. Math. Phys. **88**, 1 (1983).
12. V. Rivasseau, "Construction And Borel Summability Of Planar Four-dimensional Euclidean Field Theory," Commun. Math. Phys. **95**, 445 (1984).
13. V. Rivasseau, "Constructive Matrix Theory," JHEP **0709**, 008 (2007) [arXiv:0706.1224 [hep-th]].
14. Z. Wang, "Constructive Renormalization of 2-dimensional Grosse-Wulkenhaar Model," arXiv:1205.0196 [hep-th].
15. T. Carleman, "Sur la résolution de certaines équations intégrales," Arkiv for Mat., Astron. och Fysik **16** (1922), 19pp.
16. F. G. Tricomi, *Integral equations*, Interscience, New York (1957).