A solvable four-dimensional QFT

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Abstract. We review a sequence of papers in which we show that the quartic matrix model with an external matrix is exactly solvable in terms of the solution of a non-linear integral equation. The interacting scalar model on four-dimensional Moyal space is of this type, and our solution leads to the construction of Schwinger functions. Taking a special limit leads to a QFT on $\mathbb{R}^4$ which satisfies growth property, covariance and symmetry. There is numerical evidence for reflection positivity of the 2-point function for a certain range of the coupling constant.

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1. Prehistory

In December 1999, Minwalla, van Raamsdonk and Seiberg demonstrated [MVS99] that quantum field theories on noncommutative spaces, although nice at one-loop order [MS99, KW99], generate a severe problem in higher loops (UV/IR-mixing). The mechanism was thoroughly analysed in two papers by Chepelev and Roiban [CR99, CR00].

In summer 2002 we started a final attempt to make sense of noncommutative quantum field theories. Quantum field theory involves delicate limiting procedures and it was not completely clear that the manipulations of oscillating integrals are mathematically justified. A prerequisite to rigorous quantum field theory is to put the model into finite volume and to restrict it to finite energy. Inspired by a discussion with Thomas Krajewski, we tried to analyse the Moyal models in analogy to Polchinski's renormalisation proof [Pol84] of the $\lambda\phi^4_4$-model using exact renormalisation group equations. This approach has clear infrared and ultraviolet cut-offs and proves rigorous bounds for the amplitude of Feynman graphs. All this fails for oscillating integrals. The programme was rescued thanks to a hint by José Gracia-Bondía who introduced us to his work [GV88, VG88] on the matrix basis of the Moyal space. We briefly collect the most relevant results.

The Moyal plane is the space of Schwartz class functions equipped with the noncommutative but associative product

$$(f \star g)(x) = \int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{dy\, dk}{(2\pi)^d} f\big(x + \tfrac{1}{2}\Theta k\big)\, g\big(x+y\big)\, e^{\mathrm{i}\langle k,y\rangle} \;, \quad f,g \in \mathcal{S}(\mathbb{R}^d)\;. \quad (1)$$

Here, $\Theta = -\Theta^t \in M_d(\mathbb{R})$ is a skew-adjoint constant matrix. We assume $d = 2$ for simplicity. The Gaußian

$$f_{00}(x) = 2e^{-\frac{1}{\theta}(x_1^2+x_2^2)} \quad (2)$$

is an *idempotent* for the $\star$-product, $(f_{00} \star f_{00})(x) = f_{00}(x)$. We consider creation and annihilation operators

$$a = \frac{1}{\sqrt{2}}(x_1 + \mathrm{i}x_2)\;, \qquad \bar{a} = \frac{1}{\sqrt{2}}(x_1 - \mathrm{i}x_2)\;. \quad (3)$$

One shows $a \star f_{00} = 0$ and $f_{00} \star \bar{a} = 0$ and defines

$$f_{mn} := \frac{1}{\sqrt{n!m!\,\theta^{m+n}}}\, \bar{a}^{\star m} \star f_{00} \star a^{\star n}\;. \quad (4)$$

The f_{mn} can be expressed in terms of Laguerre polynomials. The commutation rule $[a, \bar{a}] = \theta$ leads to

$$(f_{mn} \star f_{kl})(x) = \delta_{nk} f_{ml}(x)\;. \quad (5)$$

The multiplication rule (5) identifies the $\star$-product with the ordinary matrix product; in fact there this gives an isomorphism of Fréchet algebras between Schwartz functions with $\star$-product on one hand and the product of matrices with rapidly decaying entries on the other hand [GV88]. Finally, one checks

$$\int d^2x\, f_{mn}(x) = 2\pi\theta\delta_{mn}\;. \quad (6)$$

Expanding $\phi(x_1, \dots, x_4) = \sum_{\underline{m},\underline{n}\in\mathbb{N}^2} \Phi_{\underline{mn}} f_{m_1n_1}(x_1,x_2) f_{m_2n_2}(x_3,x_4)$, now in $d = 4$ dimensions, the $\phi_4^{\star 4}$-interaction becomes a matrix product,

$$\int d^4x\; (\phi \star \phi \star \phi \star \phi)(x) = (2\pi\theta)^2 \sum_{\underline{k},\underline{l},\underline{m},\underline{n}\in\mathbb{N}^2} \Phi_{\underline{kl}}\Phi_{\underline{lm}}\Phi_{\underline{mn}}\Phi_{\underline{nk}}\;. \quad (7)$$

The prize is a complicated kinetic term

$$\int d^4x\; (\phi \star (-\Delta)\phi)(x) = (2\pi\theta)^2 \sum_{\underline{k},\underline{l},\underline{m},\underline{n}\in\mathbb{N}^2} \Delta_{\underline{kl};\underline{mn}}\Phi_{\underline{kl}}\Phi_{\underline{mn}} \quad (8)$$

for a certain integral kernel $\Delta_{\underline{kl};\underline{mn}}$ of the Laplace operator. Angular momentum conservation restricts to $\delta_{k_i+m_i,l_i+n_i}$. Then in both pairs $\underline{kn}$ and $\underline{ml}$ the kernel is, in two dimensions, a tri-diagonal band matrix, in general a sum of local interaction $\sim \delta_{\underline{lm}}\delta_{\underline{kn}}$ plus nearest neighbour interaction $\delta_{k_i,m_i\pm 1}$. To obtain Feynman rules we must invert the kernel operator $\Delta_{\underline{kl};\underline{mn}}$ and then introduce a cut-off. This was first done with a computer and small matrix cut-off $\mathcal{N}$. We found that the graphs do not behave well, for a surprising reason. The local part of $\Delta_{\underline{kl};\underline{mn}}$ was fine, but this was destroyed by the nearest-neighbour terms. In December 2002 we came up with the following working hypothesis: *Let us scale the nearest neighbours down by a factor* < 1. Then everything worked (for the computer).

We asked ourselves what operator on x-space corresponds to the weakened nearest neighbours. The answer is: an additional harmonic oscillator potential

$$\int d^4x\,(\phi\star(-\Delta)\phi)(x)\mapsto\int d^4x\,(\phi\star(-\Delta+\Omega^2\|2\Theta^{-1}x\|^2\phi)(x)\;, \qquad (9)$$

where $\Omega^2 < 1$. By January 2004 we achieved:

Theorem 1 ([GW04a]+[GW03])**.** *The action*

$$S=\int_{\mathbb{R}^4} dx\Big(\frac{Z}{2}\phi(-\Delta+\Omega^2\|2\Theta^{-1}x\|^2+\mu^2)\phi+\frac{Z^2\lambda}{4}\phi\star\phi\star\phi\star\phi\Big)(x) \qquad (10)$$

is renormalisable to all orders in perturbation theory by suitable dependence of Z,μ,λ,Ω *on cut-off and normalisation.*

A few remarks:

- Translation invariance is explicitly broken. This will be repaired in the next sections. On the other hand, the action is covariant under a duality found by Langmann and Szabo [LS02]. This duality consists in exchanging the function and its Fourier transform $\phi(x)\leftrightarrow\hat{\phi}(p)$, position and momentum $2\Theta^{-1}x\leftrightarrow p$ and then in reverting to the old variables by a modified Fourier transform with alternating $\pm$i in the phase. This transform leaves the interaction $\int dx\,(\phi\star\phi\star\phi\star\phi)(x)$ invariant, and it exchanges $\int dx\,(\phi\star(-\Delta\phi))(x)$ with $\int dx\,(\phi\star(\|2\Theta^{-1}x\|\phi))(x)$. The action (10) is Langmann-Szabo covariant: $S[\mu,\lambda,\Omega]\mapsto\Omega^2S[\frac{\mu}{\Omega},\frac{\lambda}{\Omega^2},\frac{1}{\Omega}]$.
- With some effort one can show [GW03] that all planar graphs with > 4 external legs and all non-planar graphs (amplitude $\sim\Omega^{-n}$) are finite. But this is not enough, we need the analogue of locality. For instance, in the effective action we get contributions to the 2-point function $\sum G_{\underline{kl};\underline{mn}}\Phi_{\underline{kl}}\Phi_{\underline{mn}}$ with (up to angular momentum conservation) any $\underline{k},\underline{l},\underline{m},\underline{n}$. But for renormalisability only the local terms and the nearest-neighbour terms are allowed to diverge, and a single subtraction at vanishing indices must remove the divergence. This turned out to be true. A key step in the proof was an exact diagonalisation of the $\Delta_{\underline{kl};\underline{mn}}$-kernel via Meixner polynomials. These are expressed in terms of hypergeometric functions so that we were able to control the non-local terms.

During a visit of ESI early 2004 we computed the one-loop β-function of that model. We got explicit formulae, expressed in terms of hypergeometric functions, showing that both Ω and λ flow, that $\beta_\lambda=\mathcal{N}\frac{d\lambda}{d\mathcal{N}}$ is positive but vanishes at $\Omega=1$ [GW04b]. At the and of a presentation of these results in Marseille, David Broadhurst pointed out a remarkable coincidence with these formulae: The flow is such that $\frac{\Omega^2}{\lambda}$ remains constant! Since Ω flows into its fixed point $\Omega_\infty=1$, the running coupling constant stays bounded over all scales, with finite $\lambda_\infty=\frac{\lambda_0}{\Omega_0^2}$. There is no Landau ghost [GW04c]!

Shortly later, one of us (RW) had the chance to explain these results to Vincent Rivasseau who visited the MPI Leipzig. A simple model without Landau ghost was something he searched for long time. So we started a collaboration with the aim to construct this model non-perturbatively [Riv91]. Vincent Rivasseau infected his scientific environment with this idea: Jacques Magnen, Margherita Disertori, Jean-Christophe Wallet and a growing number of young people: Fabien Vignes-Tourneret, Razvan Gurau, Adrian Tanasa, Zhituo Wang, Axel de Goursac and in some parts also Thomas Krajewski. There was a first joint publication on multiscale analysis [RVW06], but then Rivasseau's group was much faster: they reproved the renormalisation theorem in position space [GMRV05], derived the Symanzik polynomials [GR06], extended the method to the Gross-Neveu model [Vig06] and so on [Riv07a].

The most important achievement started with a remarkable three-loop computation of the β-function by Margherita Disertori and Vincent Rivasseau [DR06] in which they confirmed that at $\Omega = 1$, β vanishes to three-loop order. The great idea was to work in the matrix basis but take advantage of the fact that the Δ-kernel is local for $\Omega = 1$. Eventually, M. Disertori, R. Gurau, J. Magnen and V. Rivasseau proved in [DGMR06] that the β-function vanishes to all orders in perturbation theory. The key step consists in an ingenious combination of Ward identities with Schwinger-Dyson equation which they borrowed from a work of Benfatto and Mastropietro [BM04] on one-dimensional Fermi systems.

We felt that the result of [DGMR06] goes much deeper: Using these tools it must be possible to solve the model!

2. Details of the solution

2.1. Reformulation as matrix model

We follow the Euclidean approach, starting from a partition function with source term $\mathcal{Z}[J]$, which itself involves the action functional of the model. For concreteness, let us look at the $\lambda\phi_4^4$-model defined by the action functional

$$S[\phi] = \int_{\mathbb{R}^4} dx \left\{ \frac{1}{2}\phi(x)\cdot(-\Delta+\mu^2)(\phi(x)) + \frac{\lambda}{4}\big(\phi(x)\big)^4 \right\} . \qquad (11)$$

It is absolutely crucial that, as in any rigorous construction, this action functional cannot be taken as the naïve action $S[\phi]$. We have to *regularise* the action, namely to place it into finite volume V and introduce an energy cut-off Λ. These must be removed in the very end to restore symmetry. For $V \to \infty$ we study the free energy density $\frac{1}{V}\log(\mathcal{Z}[J])$ and functions derived from that. The limit $\frac{1}{\Lambda} \to 0$ is achieved by the renormalisation philosophy. There is a renormalisation group flow of effective actions down in Λ, and the key step is to impose mixed boundary conditions: finitely many relevant and marginal couplings are fixed at $\Lambda_R = 0$, the infinitely many irrelevant couplings at

$\Lambda \to \infty$. This involves an inversion of the renormalsiation group flow which typically requires perturbation theory.

Finite volume is actually a compactification of the underlying geometry. A more sophisticated method than putting the model into a box is to add in (11) a harmonic oscillator potential $-\Delta \mapsto H := -\Delta + \omega^2\|x\|^2$. The result is the same, the resolvent $(H+\mathrm{i})^{-1}$ is a compact operator (with discrete spectrum), and the D-dimensional spectral volume is proportional to $\omega^{-\frac{D}{2}}$.

A well-known restriction of the energy density consists in introducing a lattice of spacing $a = \frac{1}{\Lambda}$. This is the heart of the lattice gauge theory approach [Wil74] to quantum chromodynamics. As action one takes, for example, Wilson's plaquette action, which is *non-local* but converges for $a \to 0$ to the local Yang-Mills action. We choose the Moyal product as our preferred non-locality:

$$S_{\Omega,\theta}[\phi] = \frac{1}{64\pi^2}\int d^4x\Big(\frac{Z}{2}\phi\star\big(-\Delta+\Omega^2_{bare}\|2\Theta^{-1}x\|^2 + \mu^2_{bare}\big)\phi + \frac{\lambda_{bare}Z^2}{4}\phi\star\phi\star\phi\star\phi\Big)(x)\;. \tag{12}$$

We pass to matrix representation; the energy cut-off is then the restriction to finite matrices:

$$S^{\mathcal{N}}_{\Omega,\theta}[\Phi] = \Big(\frac{\theta}{4}\Big)^2 \sum_{\underline{k},\underline{l},\underline{m},\underline{n}\in\mathbb{N}^2_{\mathcal{N}}} \Big(\frac{Z}{2}\Phi_{\underline{k}\underline{l}}(\Delta^{(\Omega)}_{\underline{k}\underline{l};\underline{m}\underline{n}} + \mu^2_{bare}\delta_{\underline{k}\underline{n}}\delta_{\underline{l}\underline{m}})\Phi_{\underline{m}\underline{n}} + \frac{Z^2\lambda_{bare}}{4}\Phi_{\underline{k}\underline{l}}\Phi_{\underline{l}\underline{m}}\Phi_{\underline{m}\underline{n}}\Phi_{\underline{n}\underline{k}}\Big)\;. \tag{13}$$

This action is used to define the partition function. We should perform the renormalisation and then take the limits $\mathcal{N} \to \infty$, $\theta \to 0$ and $\Omega \to 0$ to rigorously define the $\lambda\phi^4_4$-model. This is not what we do; we take $\Omega = 1$. At first sight, this cuts all ties to translation invariance. The only chance to kill the oscillator potential for $\Omega = 1$ is to let $\theta \to \infty$. This is a highly singular limit in (1), although already mentioned in [MVS99] as 'stringy'. We were able to make sense of this limit first for matrices [GW12] but later in position space [GW13], and surprisingly it not only restores translation invariance but also full rotation invariance.

At $\Omega = 1$ the kernel of the Schrödinger operator is local,

$$\Delta^{(\Omega=1)}_{\underline{k}\underline{l};\underline{m}\underline{n}} = \frac{|\underline{m}| + |\underline{n}| + 2}{\sqrt{V}}\;, \quad |\underline{m}| := m_1 + m_2 \text{ for } \underline{m} = (m_1, m_2)\;, \quad V = \Big(\frac{\theta}{4}\Big)^2\;.$$

This allows us to write the action as

$$S[\Phi] = V\,\mathrm{tr}\Big(E\Phi^2 + \frac{Z^2\lambda}{4}\Phi^4\Big)\;, \quad E = (E_{\underline{m}}\delta_{\underline{m}\underline{n}})\;, \quad E_{\underline{m}} = Z\Big(\frac{|\underline{m}|}{\sqrt{V}} + \frac{\mu^2_{bare}}{2}\Big)\;. \tag{14}$$

We view E as unbounded on Hilbert space which is positive, selfadjoint and has compact resolvent. Adding a source term to the action, we define the partition function as

$$\mathcal{Z}[J] = \int \mathcal{D}[\Phi]\; \exp(-S[\Phi] + V\,\mathrm{tr}(\Phi J))\;, \tag{15}$$

where $\mathcal{D}[\Phi]$ is the extension of the Lebesgue measure from finite-rank operators to the Hilbert-Schmidt class and J a test function matrix. In the sequel we use simplified notation $\underline{m} \mapsto m$ and $\mathbb{N}^2_{\mathcal{N}} \mapsto I$.

2.2. Ward identity and topological expansion

There is a subgroup of unitary operators U on Hilbert space such that the transformed operator $\tilde{\Phi} = U\Phi U^*$ belongs to the same class as Φ. This implies

$$\int \mathcal{D}[\Phi]\; \exp(-S[\Phi] + V\,\mathrm{tr}(\Phi J)) = \int \mathcal{D}[\tilde{\Phi}]\; \exp(-S[\tilde{\Phi}] + V\,\mathrm{tr}(\tilde{\Phi} J))\;.$$

Unitary invariance $\mathcal{D}[\tilde{\Phi}] = \mathcal{D}[\Phi]$ of the Lebesgue measure leads to

$$0 = \int \mathcal{D}[\Phi]\; \Big\{\exp(-S[\Phi] + V\,\mathrm{tr}(\Phi J)) - \exp(-S[\tilde{\Phi}] + V\,\mathrm{tr}(\tilde{\Phi} J))\Big\}\;.$$

Note that the integrand $\{\dots\}$ itself does not vanish because $\mathrm{tr}(E\Phi^2)$ and $\mathrm{tr}(\Phi J)$ are not unitarily invariant; we only have $\mathrm{tr}(\Phi^4) = \mathrm{tr}(\tilde{\Phi}^4)$ due to $UU^* = U^*U = \mathrm{id}$ together with the trace property. Linearisation of U about the identity operator leads to the *Ward identity*

$$0 = \int \mathcal{D}[\Phi]\; \Big\{E\Phi\Phi - \Phi\Phi E - J\Phi + \Phi J\Big\} \exp(-S[\Phi] + V\,\mathrm{tr}(\Phi J))\;. \tag{16}$$

We can always choose an orthonormal basis where E is diagonal (but J is not). Since E is of compact resolvent, E has eigenvalues $E_a > 0$ of finite multiplicity μ_a. We thus label the matrices by an enumeration of the (necessarily discrete) eigenvalues of E and an enumeration of the basis vectors of the finite-dimensional eigenspaces. Writing Φ in $\{\dots\}$ of (16) as functional derivative $\Phi_{ab} = \frac{\partial}{V\partial J_{ba}}$, we have thus proved (first obtained in [DGMR06]):

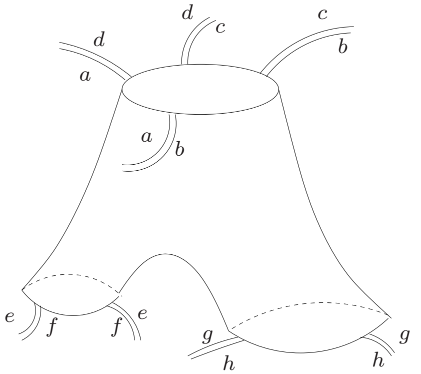
Proposition 2. *The partition function $\mathcal{Z}[J]$ of the matrix model defined by the external matrix E satisfies the $|I| \times |I|$ Ward identities*

$$0 = \sum_{n\in I} \Big(\frac{(E_a - E_p)}{V}\frac{\partial^2 \mathcal{Z}}{\partial J_{an}\partial J_{np}} + J_{pn}\frac{\partial \mathcal{Z}}{\partial J_{an}} - J_{na}\frac{\partial \mathcal{Z}}{\partial J_{np}}\Big)\;. \tag{17}$$

The compactness of the resolvent of E implies that at the expense of adding a measure $\mu_{[m]} = \dim\ker(E - E_m\mathrm{id})$, we can assume that $m \mapsto E_m$ is injective.

In a perturbative expansion, Feynman graphs in matrix models are *ribbon graphs.* Viewed as simplicial complexes, they encode the topology (B, g) of a genus-g Riemann surface with B boundary components. The k^{th} boundary face is characterised by $N_k \geq 1$ external double lines to which we attach the source matrices J. See e.g. [GW03]. Since E is diagonal, the matrix index is conserved along each strand of the ribbon graph. Therefore, the right index of J_{ab} coincides with the left index of another J_{bc}, or of the same J_{bb}. Accordingly, the k^{th} boundary component carries a cycle $J_{p_1\dots p_{N_k}} := \prod_{j=1}^{N_k} J_{p_j p_{j+1}}$ of

N_k external sources, with $N_k+1 \equiv 1$. Here is a drawing for a $(B=3, g=0)$ Riemann surface with cycles of lengths $N_1=4$, $N_2=2$, $N_3=2$:



$$(J_{ab}J_{bc}J_{cd}J_{da})\,(J_{ef}J_{fe})\,(J_{gh}J_{hg})$$

We had discussed that only $\frac{1}{V}\log \mathcal{Z}[J]$, not $\mathcal{Z}[J]$ itself, can have an infinite volume limit. Consequently we *define* $\log \mathcal{Z}[J]$ as an expansion according to the cycle structure:

$$\log \frac{\mathcal{Z}[J]}{\mathcal{Z}[0]} = \sum_{B=1}^{\infty} \sum_{1\leq N_1 \leq \cdots \leq N_B}^{\infty} \sum_{p_1^\beta,\dots,p_{N_\beta}^\beta \in I} \frac{V^{2-B}}{S_{N_1\dots N_B}} G_{|p_1^1\dots p_{N_1}^1|\dots|p_1^B\dots p_{N_B}^B|} \times \prod_{\beta=1}^{B} \Big(\frac{J_{p_1^\beta\dots p_{N_\beta}^\beta}}{N_\beta}\Big) . \qquad (18)$$

The symmetry factor $S_{N_1\dots N_B}$ is obtained as follows: If ν_i of the B numbers N_β in a given tuple $(N_1,\dots,N_B)$ are equal to i, then $S_{N_1\dots N_B} = \prod_{i=1}^{N_B} \nu_i!$.

We stress that these weight functions $G_{|p_1^1\dots p_{N_1}^1|\dots|p_1^B\dots p_{N_B}^B|}$ constitute the QFT; we construct these functions rigorously and not a measure. The formal relation to the partition function gives identities (Ward + Schwinger-Dyson) between the weight functions. These identities, whenever defined, are used to extend the weight functions into regions for parameters where the measure does not exist. In the very end, QFT is understood in terms of Schwinger or Wightman functions and scattering amplitudes, not in terms of a measure.

Now comes the crucial step for the construction of the weight functions. We turn the Ward identity (17) into a formula for the second derivative $\sum_{n\in I} \frac{\partial^2 \mathcal{Z}[J]}{\partial J_{an}\partial J_{np}}$ of the partition function, thus giving new relations for $G_{\dots}$. We have to identify the kernel of multiplication by $(E_p - E_a)$. For injective $m \mapsto E_m$ this kernel is given by $W_a[J]\delta_{ap}$ for some function $W_a[J]$. This function is identified by inserting (18) into $\sum_{n\in I} \frac{\partial^2 \exp(\log \mathcal{Z}[J])}{\partial J_{an}\partial J_{np}}$ and carefully registering the possibilities which give rise to a factor δ_{ap}. We find [GW12]:

Theorem 3.

$$\sum_{n\in I} \frac{\partial^2 \mathcal{Z}[J]}{\partial J_{an}\partial J_{np}} = \delta_{ap}\Big\{ V^2 \sum_{(K)} \frac{J_{P_1}\cdots J_{P_K}}{S_{(K)}} \Big(\sum_{n\in I} \frac{G_{|an|P_1|\dots|P_K|}}{V^{|K|+1}} + \frac{G_{|a|a|P_1|\dots|P_K|}}{V^{|K|+2}}$$

$$+\sum_{r\geq 1}\sum_{q_1,\dots,q_r\in I}\frac{G_{|q_1aq_1\dots q_r|P_1|\dots|P_K|}J^r_{q_1\dots q_r}}{V^{|K|+1}}\Big)$$

$$+V^4\sum_{(K),(K')}\frac{J_{P_1}\cdots J_{P_K}J_{Q_1}\cdots J_{Q_{K'}}}{S_{(K)}S_{(K')}}\frac{G_{|a|P_1|\dots|P_K|}}{V^{|K|+1}}\frac{G_{|a|Q_1|\dots|Q_{K'}|}}{V^{|K'|+1}}\Big\}\mathcal{Z}[J]$$

$$+\frac{V}{E_p-E_a}\sum_{n\in I}\Big(J_{pn}\frac{\partial\mathcal{Z}[J]}{\partial J_{an}}-J_{na}\frac{\partial\mathcal{Z}[J]}{\partial J_{np}}\Big)\,. \tag{19}$$

Formula (19) is the core of our approach. It is a consequence of the unitary group action and the cycle structure of the partition function. The importance lies in the fact that the formula allows to kill two J-derivatives in the partition function. As we describe below, this is the key step in breaking up the tower of Schwinger-Dyson equations.

2.3. Schwinger-Dyson equations

We can write the action as $S=\frac{V}{2}\sum_{a,b}(E_a+E_b)\Phi_{ab}\Phi_{ba}+VS_{int}[\Phi]$, where E_a are the eigenvalues of E. Functional integration yields, up to an irrelevant constant,

$$\mathcal{Z}[J]=e^{-VS_{int}[\frac{\partial}{V\partial J}]}e^{\frac{V}{2}\langle J,J\rangle_E}\,,\qquad \langle J,J\rangle_E:=\sum_{m,n\in I}\frac{J_{mn}J_{nm}}{E_m+E_n}\,. \tag{20}$$

Acually, this formula is the *definition* of the partition function in any rigorous approach, and $\frac{1}{E_m+E_n}$ is the covariance. Instead of a perturbative expansion of $e^{-VS_{int}[\frac{\partial}{V\partial J}]}$ we apply those J-derivatives to (20) which give rise to a correlation function $G_{\dots}$ on the lhs. On the rhs of (20), these external derivatives combine with internal derivatives from $S_{int}[\frac{\partial}{V\partial J}]$ to certain identities for $G_{\dots}$. These Schwinger-Dyson equations are often of little use because they express an N-point function in terms of $(N{+}2)$-point functions. But thanks to (19) we can express the $(N{+}2)$-point function on the rhs in terms of N'-point functions with $N'\leq N$.

Let us look at this mechanism for the 2-point function $G_{|ab|}$ for $a\neq b$. According to (18), $G_{|ab|}$ is obtained by deriving (20) with respect to J_{ba} and J_{ab}:

$$\begin{aligned}G_{|ab|}&=\frac{1}{V\mathcal{Z}[0]}\frac{\partial^2\mathcal{Z}[J]}{\partial J_{ba}\partial J_{ab}}\Big|_{J=0} \qquad \text{(disconnected part of } \mathcal{Z} \text{ does not contribute for } a\neq b)\\
&=\frac{1}{V\mathcal{Z}[0]}\Big\{\frac{\partial}{\partial J_{ba}}e^{-VS_{int}\left[\frac{\partial}{V\partial J}\right]}\frac{\partial}{\partial J_{ab}}e^{\frac{V}{2}\langle J,J\rangle_E}\Big\}_{J=0}\\
&=\frac{1}{(E_a+E_b)\mathcal{Z}[0]}\Big\{\frac{\partial}{\partial J_{ba}}e^{-VS_{int}\left[\frac{\partial}{V\partial J}\right]}J_{ba}e^{\frac{V}{2}\langle J,J\rangle_E}\Big\}_{J=0}\\
&=\frac{1}{E_a+E_b}+\frac{1}{(E_a+E_b)\mathcal{Z}[0]}\Big\{\Big(\Phi_{ab}\frac{\partial(-VS_{int})}{\partial\Phi_{ab}}\Big)\Big[\frac{\partial}{V\partial J}\Big]\Big\}\mathcal{Z}[J]\Big|_{J=0}\,.\end{aligned} \tag{21}$$

Now observe that $\frac{\partial(-VS_{int})}{\partial\Phi_{ab}}$ contains, for any polynomial interaction, the derivative $\sum_n \frac{\partial^2}{\partial J_{an}\partial J_{np}}$ which we know from (19). In case of the quartic matrix model with interaction $\frac{\lambda_4}{4}\Phi^4$ we have $\frac{\partial(-VS_{int})}{\partial\Phi_{ab}} = -\lambda_4 V \sum_{n,p\in I}\Phi_{bp}\Phi_{pn}\Phi_{na}$, hence

$$\Big(\Phi_{ab}\frac{\partial(-VS_{int})}{\partial\Phi_{ab}}\Big)\Big[\frac{\partial}{V\partial J}\Big] = -\frac{\lambda_4}{V^3}\sum_{p,n\in I}\frac{\partial^2}{\partial J_{pb}\partial J_{ba}}\frac{\partial^2}{\partial J_{an}\partial J_{np}}\;.$$

This is inserted into (21), then we insert (19) up to $\mathcal{O}(J^2)$. Finally one needs the expansion $\frac{\partial^2\mathcal{Z}[J]}{\partial J_{pb}\partial J_{bp}} = (VG_{|pb|}+\delta_{pb}G_{|p|b|})\mathcal{Z}[0]+\mathcal{O}(J)$ and $\frac{\partial J_{rr}}{\partial J_{ab}} = 0$ for $a\neq b$. One arrives at [GW12]:

$$\left.G_{|ab|} = \frac{1}{E_a+E_b} - \frac{\lambda_4}{E_a+E_b}\frac{1}{V}\sum_{p\in I}\Big(G_{|ab|}G_{|ap|} - \frac{G_{|pb|}-G_{|ab|}}{E_p-E_a}\Big)\quad\right\}\quad (22a)$$

$$\left.\begin{aligned}&-\frac{\lambda_4}{V^2(E_a+E_b)}\Big(G_{|a|a|}G_{|ab|}+\frac{1}{V}\sum_{n\in I}G_{|an|ab|}\\&\qquad\qquad +G_{|aaab|}+G_{|baba|}-\frac{G_{|b|b|}-G_{|a|b|}}{E_b-E_a}\Big)\end{aligned}\quad\right\}\quad (22b)$$

$$\left.-\frac{\lambda_4}{V^4(E_a+E_b)}G_{|a|a|ab|}\;.\quad\right\}\quad (22c)$$

It can be checked [GW12] that in a genus expansion $G_{\dots} = \sum_{g=0}^{\infty}V^{-2g}G^{(g)}_{\dots}$ (which is probably not convergent but Borel summable), precisely the line (22a) preserves the genus, the lines (22b) increase $g\mapsto g+1$ and the line (22c) increases $g\mapsto g+2$.

We will not rely on a genus expansion. Instead we consider a scaling limit $V\to\infty$ such that the densitised index summation $\frac{1}{V}\sum_{p\in I}$ remains finite. Then the exact Schwinger-Dyson equation for $G_{|ab|}$ coincides with its restriction (22a) to the planar sector $g=0$, a closed non-linear equation for $G^{(0)}_{|ab|}$ alone. There might exist other reasonable limits which take (22b) and (22c) into account, similar to the double scaling limit [BK90, DS90, GM90] in matrix models. Here we choose a planar limit, but even here a non-trivial topology survives: The higher boundary components $B\geq 2$ are not suppressed; and in fact these contributions from $B\geq 2$ make the model interesting!

By similar calculation we derive the Schwinger-Dyson equation for higher N-point functions. This expresses the N-point function $G_{|ab_1\dots b_{N-1}|}$ in terms of its summation $\frac{\lambda_4}{E_a+E_{b_1}}\frac{1}{V}\sum_{p\in I}\Big(G_{|ap|}G_{|ab_1\dots b_{N-1}|} - \frac{G_{|pb_1\dots b_{N-1}|}-G_{|ab_1\dots b_{N-1}|}}{E_p-E_a}\Big)$ and several other functions [GW12]. It turns out that a real theory with $\Phi=\Phi^*$ admits a short-cut which directly gives the higher N-point functions without any index summation. Since the equations for $G_{\dots}$ are real and $\overline{J_{ab}} = J_{ba}$, the reality $\mathcal{Z} = \overline{\mathcal{Z}}$ implies (in addition to invariance under cyclic permutations) invariance under orientation reversal

$$G_{|p_1^1 p_2^1 \dots p_{N_1}^1|\dots|p_1^B p_2^B \dots p_{N_B}^B|} = G_{|p_1^1 p_{N_1}^1 \dots p_2^1|\dots|p_1^B p_{N_B}^B \dots p_2^B|} . \tag{23}$$

Whereas empty for $G_{|ab|}$, in $(E_a+E_{b_1})G_{ab_1b_2\dots b_{N-1}} - (E_a+E_{b_{N-1}})G_{ab_{N-1}\dots b_2b_1}$ the identities (23) lead to many cancellations which result in a universal algebraic recursion formula [GW12]:

Proposition 4. *Given a quartic matrix model $S[\Phi] = V\,\mathrm{tr}(E\Phi^2 + \frac{\lambda_4}{4}\Phi^4)$ with E of compact resolvent. Then in a scaling limit $V \to \infty$ with $\frac{1}{V}\sum_{i\in I}$ finite, the $(B=1)$-sector of $\log\mathcal{Z}$ is given by*

$$G_{|ab|} = \frac{1}{E_a+E_b} - \frac{\lambda_4}{E_a+E_b}\frac{1}{V}\sum_{p\in I}\Big(G_{|ab|}G_{|ap|} - \frac{G_{|pb|}-G_{|ab|}}{E_p-E_a}\Big) , \tag{24a}$$

$$\begin{aligned} &G_{|b_0b_1\dots b_{N-1}|} \qquad\qquad (24b)\\ &= (-\lambda_4)\sum_{l=1}^{\frac{N-2}{2}} \frac{G_{|b_0b_1\dots b_{2l-1}|}G_{|b_{2l}b_{2l+1}\dots b_{N-1}|} - G_{|b_{2l}b_1\dots b_{2l-1}|}G_{|b_0b_{2l+1}\dots b_{N-1}|}}{(E_{b_0}-E_{b_{2l}})(E_{b_1}-E_{b_{N-1}})} . \end{aligned}$$

The self-consistency equation (24a) was first obtained in [GW09] for the Moyal model by the graphical method proposed by [DGMR06]. There we also solved the renormalisation problem resulting from the divergent summation $\sum_{p\in I}$. The non-linearity of (24a) was a considerable challenge which we successfully addressed in [GW12, GW14].

The other topological sectors $B \geq 2$ made of $(N_1+\dots+N_B)$-point functions $G_{|b_1^1\dots b_{N_1}^1|\dots|b_1^1\dots b_{N_B}^B|}$ are similar in the following sense [GW12]: The basic functions with all $N_i \leq 2$ satisfy an equation with index summation as (24a), but in contrast to the 2-point function these equations are linear. The other functions with one $N_i \geq 3$ are purely algebraic.

We make the following key observation: An affine transformation $E \mapsto ZE+C$ together with a corresponding rescaling $\lambda_4 \mapsto Z^2\lambda_4$ leaves the algebraic equations invariant:

Theorem 5. *Given a real quartic matrix model with $S = V\,\mathrm{tr}(E\Phi^2 + \frac{\lambda_4}{4}\Phi^4)$ and $m \mapsto E_m$ injective, which determines the set $G_{|p_1^1\dots p_{N_1}^1|\dots|p_1^B\dots p_{N_B}^B|}$ of $(N_1+\dots+N_B)$-point functions. Assume that the basic functions with all $N_i \leq 2$ are turned finite by $E_a \mapsto Z(E_a + \frac{\mu^2}{2} - \frac{\mu_{bare}^2}{2})$ and $\lambda_4 \mapsto Z^2\lambda_4$. Then all functions with one $N_i \geq 3$*

1. *are finite without further need of a renormalisation of λ, i.e. all renormalisable quartic matrix models have vanishing β-function,*
2. *are given by universal algebraic recursion formulae in terms of renormalised basic functions with $N_i \leq 2$.*

The theorem tells us that vanishing of the β-function for the self-dual Φ^4_4-model on Moyal space (proved in [DGMR06] to all orders in perturbation theory) is generic to all quartic matrix models, and the result even holds non-perturbatively!

We remark that the algebraic equations for $N_i \geq 3$ have a graphical realisation in terms of non-crossing chord diagrams with additional decoration which describe the denominators $\frac{1}{E_{b_i}-E_{b_j}}$. The different chord structures are counted by the Catalan numbers. These functions alone would make the higher N-point functions very close to trivial. It is the inclusion of the (2+2+...+2)-point functions which gives a rich structure.

2.4. Infinite volume limit and renormalisation

We return to the Moyal-space regularisation of the $\lambda\phi^4_4$-model. We know that the *unrenormalised* 2-point function $G_{|\underline{a}\underline{b}|}$ satisfies the self-consistency equation (24a) for $E_{\underline{m}} = Z\Big(\frac{|\underline{m}|}{\sqrt{V}} + \frac{\mu^2_{bare}}{2}\Big)$ and $\lambda_4 = Z^2\lambda$. Because of the vanishing β-function (Theorem 5), there is no need to introduce a bare coupling λ_{bare}. The matrix indices have ranges $\underline{a}, \dots \in I := \mathbb{N}^2_{\mathcal{N}}$, i.e. pairs of natural numbers with certain cut-off. The index sum diverges for $\mathbb{N}^2_{\mathcal{N}} \mapsto \mathbb{N}^2$.

It is important that all functions only depend on the spectrum of $E_{\underline{m}}$, i.e. on the norms $|\underline{m}| = m_1 + m_2$ and not on m_1, m_2 separately. It turns out that also renormalisation respects this degeneracy. Therefore, all index sums reduce to $\sum_{\underline{p}\in\mathbb{N}^2_{\mathcal{N}}} f(|\underline{p}|) = \sum_{|\underline{p}|=0}^{\mathcal{N}} (|\underline{p}|+1) f(|\underline{p}|)$. The equations (24a) result from (22) in a scaling limit $V \to \infty$ and $\frac{1}{V}\sum_{|\underline{p}|=0}^{\mathcal{N}} (|\underline{p}|+1) f(|\underline{p}|)$ finite. The most natural way to achieve this is to keep the ratio $\frac{\mathcal{N}}{\sqrt{V\mu^4}} = \Lambda^2(1+\mathcal{Y})$ fixed. Note that $V = \big(\frac{\theta}{4}\big)^2 \to \infty$ is a limit of extreme noncommutativity! The new parameter $(1+\mathcal{Y})$ corresponds to a finite wavefunction renormalisation, identified later to decouple our equations, and μ will be the renormalised mass. The parameter Λ^2 represents an ultraviolet cut-off which is sent to $\Lambda \to \infty$ in the very end (continuum limit). In the scaling limit, functions of $\frac{|\underline{p}|}{\sqrt{V}} =: \mu^2(1+\mathcal{Y})p$ converge to functions of 'continuous matrix indices' $p \in [0, \Lambda^2]$, and the densitised index summation converges to a Riemann integral. After all these steps, the unrenormalised function $G^{(ur)}_{ab} := \lim_{V\to\infty} \mu^2 G_{|\underline{a}\underline{b}|}$ satisfies the following equation resulting from (24a):

$$G^{(ur)}_{ab} = \frac{1}{Z\big(\frac{\mu^2_{bare}}{\mu^2} + (a+b)(1+\mathcal{Y})\big)} - \frac{Z^2\lambda(1+\mathcal{Y})^2}{Z\big(\frac{\mu^2_{bare}}{\mu^2} + (a+b)(1+\mathcal{Y})\big)} \int_0^{\Lambda^2} p\,dp\,\Big(G^{(ur)}_{ab}G^{(ur)}_{ap} - \frac{G^{(ur)}_{pb} - G^{(ur)}_{ab}}{(1+\mathcal{Y})Z(p-a)}\Big)\,. \tag{25}$$

The next step is the renormalisation which constructs the limit $\Lambda^2 \to \infty$. We pass to the 1PI function defined by

$$\big(G^{(ur)}_{ab}\big)^{-1} =: Z\big(\frac{\mu^2_{bare}}{\mu^2} + (a+b)(1+\mathcal{Y})\big) - \Gamma^{(ur)}_{ab}\,. \tag{26a}$$

We *define* the renormalised 1PI function Γ_{ab} via second-order Taylor formula with remainder

$$\Gamma^{(ur)}_{ab} = Z\frac{\mu^2_{bare}}{\mu^2} - 1 + (Z-1)(a+b)(1+\mathcal{Y}) - \Gamma_{ab} \;, \\ \Gamma_{00} := 0 \;, \quad (\partial\Gamma)_{00} := 0 \tag{26b}$$

and then the renormalised connected function G_{ab} as

$$\big(G_{ab}\big)^{-1} = 1 + (a+b)(1+\mathcal{Y}) - \Gamma_{ab} \;.$$

We stress that this procedure would be completely wrong for other models. Renormalisation is a recursive procedure [BS59, Hep66, Zim69] which has to take subdivergences into account. Of course our 4-point subfunctions diverge, but these divergences are exactly cancelled by the divergent vertex factors $(-Z^2\lambda)$ because β is zero. So there only remain divergent 2-point subfunctions, but inductively they are already renormalised. In conclusion, $\beta = 0$ permits a non-perturbative renormalisation prescription, in some sense a renormalisation of all Feynman graphs at once. In detail, we view (25) via (26) as an equation for Γ_{ab} and have together with $\Gamma_{00} = 0$ and $(\partial\Gamma)_{00} = 0$ three equations for the three functions Γ_{ab}, $\frac{\mu_{bare}}{\mu}$ and Z which allows us to eliminate μ_{bare} and Z.

Eliminating μ_{bare} is easy; eliminating Z is difficult due to the non-linearity in (25). We propose the following trick which postpones the non-linearity: If we multiply (25) by $\frac{Z\big(\frac{\mu^2_{bare}}{\mu^2}+(a+b)(1+\mathcal{Y})\big)}{G^{(ur)}_{ab}}$, then the previously non-linear term is independent of b. So we subtract from that equation the equation at $b = 0$. Our problem is then equivalent to the difference equation plus (25) at $b = 0$. The difference equation reads after elimination of μ_{bare}, but before elimination of Z,

$$\frac{Z^{-1}}{(1+\mathcal{Y})}\Big(\frac{1}{G_{ab}} - \frac{1}{G_{a0}}\Big) = b - \lambda\int_0^{\Lambda^2} p\,dp\;\frac{\frac{G_{pb}}{G_{ab}} - \frac{G_{p0}}{G_{a0}}}{p-a} \;. \tag{27}$$

Differentiation $\frac{d}{db}\big|_{a=b=0}$ of (27) yields Z^{-1} in terms of G_{ab} and its derivative. The resulting derivative G' can be avoided by adjusting $\mathcal{Y} := -\lambda\lim_{b\to 0}\int_0^{\Lambda^2} dp\,\frac{G_{pb}-G_{p0}}{b}$. This choice leads to $\frac{Z^{-1}}{(1+\mathcal{Y})} = 1 - \lambda\int_0^{\Lambda^2} dp\,G_{p0}$, which is a perturbatively divergent integral for $\Lambda\to\infty$. Inserting Z^{-1} and $\mathcal{Y}$ back into (27) we end up in a *linear* integral equation for the difference function $D_{ab} := \frac{a}{b}(G_{ab} - G_{a0})$ to the boundary. The non-linearity restricts to the boundary function G_{a0} where the second index is put to zero. Assuming $a \mapsto G_{ab}$ Hölder-continuous, we can pass to Cauchy principal values. In terms of the *finite Hilbert transform*

$$\mathcal{H}^{\Lambda}_a[f(\bullet)] := \frac{1}{\pi}\lim_{\epsilon\to 0}\Big(\int_0^{a-\epsilon} + \int_{a+\epsilon}^{\Lambda^2}\Big)\frac{f(q)\,dq}{q-a} \;, \tag{28}$$

the integral equation becomes

$$\Big(\frac{b}{a} + \frac{1+\lambda\pi a \mathcal{H}^\Lambda_a\big[G_{\bullet 0}\big]}{aG_{a0}}\Big) D_{ab} - \lambda\pi\mathcal{H}^\Lambda_a\big[D_{\bullet b}\big] = -G_{a0}\;. \tag{29}$$

Equation (29) is a well-known singular integral equation of Carleman type [Car22, Tri57]:

Theorem 6 ([Tri57]**, transformed from** $[-1,1]$ **to** $[0,\Lambda^2]$**).** *The singular linear integral equation*

$$h(a)y(a) - \lambda\pi\mathcal{H}^\Lambda_a[y] = f(a)\;, \qquad a\in\,]0,\Lambda^2[\;,$$

is for $h(a)$ *continuous on* $]0,\Lambda^2[$*, Hölder-continuous near* $0,\Lambda^2$*, and* $f\in L^p$ *for some* $p>1$ *(determined by* $\vartheta(0)$ *and* $\vartheta(\Lambda^2)$*) solved by*

$$\begin{aligned} y(a) &= \frac{\sin(\vartheta(a))e^{-\mathcal{H}^\Lambda_a[\pi-\vartheta]}}{\lambda\pi a}\Big(a\,f(a)e^{\mathcal{H}^\Lambda_a[\pi-\vartheta]}\cos(\vartheta(a)) \\ &\qquad\qquad + \mathcal{H}^\Lambda_a\Big[e^{\mathcal{H}^\Lambda_\bullet[\pi-\vartheta]}\bullet f(\bullet)\sin(\vartheta(\bullet))\Big] + C\Big) & \text{(30a)} \\ &\overset{*}{=} \frac{\sin(\vartheta(a))e^{\mathcal{H}^\Lambda_a[\vartheta]}}{\lambda\pi}\Big(f(a)e^{-\mathcal{H}^\Lambda_a[\vartheta]}\cos(\vartheta(a)) \\ &\qquad\qquad + \mathcal{H}^\Lambda_a\Big[e^{-\mathcal{H}^\Lambda_\bullet[\vartheta]}f(\bullet)\sin(\vartheta(\bullet))\Big] + \frac{C'}{\Lambda^2-a}\Big)\;, & \text{(30b)} \end{aligned}$$

where $\vartheta(a) = \operatorname*{arctan}_{[0,\pi]}\Big(\dfrac{\lambda\pi}{h(a)}\Big)$ *and* C, C' *are arbitrary constants.*

The possibility of $C, C'\neq 0$ is due to the fact that the finite Hilbert transform has a kernel, in contrast to the infinite Hilbert transform with integration over $\mathbb{R}$. The two formulae (30a) and (30b) are formally equivalent, but the solutions belong to different function classes and normalisation conditions may (and will) make a choice.

A lengthy discussion [GW14] shows that such a constant C, C' arises for $\lambda>0$ but not for $\lambda<0$. The key step in this analysis is to regard the defining equation for ϑ as a Carleman type singular integral equation for G_{a0}. This allows to express G_{a0} in terms of ϑ, and various identities in [Tri57] and trigonometric addition theorems give the result:

Theorem 7 ([GW14]**).** *The matrix* 2*-point function* G_{ab} *of the* $\lambda\phi^{\star 4}_4$*-model is in infinite volume limit given in terms of the boundary* 2*-point function* G_{0a} *by the equation*

$$G_{ab} = \frac{\sin(\tau_b(a))}{|\lambda|\pi a}\mathrm{e}^{\mathrm{sign}(\lambda)(\mathcal{H}^\Lambda_0[\tau_0(\bullet)]-\mathcal{H}^\Lambda_a[\tau_b(\bullet)])}\begin{cases} 1 & \textit{for } \lambda<0\;,\\ \big(1+\frac{Ca+bF(b)}{\Lambda^2-a}\big) & \textit{for } \lambda>0\;, \end{cases} \tag{31}$$

$$\tau_b(a) := \operatorname*{arctan}_{[0,\pi]}\left(\frac{|\lambda|\pi a}{b+\frac{1+\lambda\pi a\mathcal{H}^\Lambda_a[G_{\bullet 0}]}{G_{a0}}}\right)\;, \tag{32}$$

where C *is a undetermined constant and* $b\,F(b)$ *an undetermined function of* b *vanishing at* $b=0$*.*

Some remarks:

- We proved this theorem in 2012 for $\lambda > 0$ under the assumption $C' = 0$ in (30b), but knew that non-trivial solutions of the homogeneous Carleman equation parametrised by $C' \neq 0$ are possible. That no such term arises for $\lambda < 0$ (if angles are redefined $\vartheta \mapsto \tau$) is a recent result [GW14].
- We expect C, F to be Λ-dependent so that $\big(1+\frac{Ca+bF(b)}{\Lambda^2-a}\big) \stackrel{\Lambda\to\infty}{\longrightarrow} 1+\tilde{C}a+b\tilde{F}(b)$.
- An important observation is $G_{ab} \geq 0$, at least for $\lambda < 0$. This is a truly non-perturbative result; individual Feynman graphs show no positivity at all!
- As in [GW09], the equation for G_{ab} can be solved perturbatively. This reproduces exactly [GW12] the Feynman graph calculation! Matching at $\lambda = 0$ requires C, F to be flat functions of λ (all derivatives vanish at zero).
- Because of $\mathcal{H}^\Lambda_a[G_{\bullet 0}] \stackrel{a\to\Lambda^2}{\longrightarrow} -\infty$, the naïve arctan series is dangerous for $\lambda > 0$. Unless there are cancellations, we expect zero radius of convergence!
- From (31) we deduce the finite wavefunction renormalisation

$$\mathcal{Y} := -1 - \frac{dG_{ab}}{db}\Big|_{a=b=0} = \int_0^{\Lambda^2} \frac{dp}{(\lambda\pi p)^2 + \big(\frac{1+\lambda\pi p\mathcal{H}^\Lambda_p[G_{\bullet 0}]}{G_{p0}}\big)^2} - \begin{cases} 0 & \text{for } \lambda < 0\,, \\ F(0) & \text{for } \lambda > 0\,. \end{cases} \tag{33}$$

- The partition function $\mathcal{Z}$ is undefined for $\lambda < 0$. But the Schwinger-Dyson equations for G_{ab} and for higher functions, and with them $\log \mathcal{Z}$, extend to $\lambda < 0$. These extensions are unique but probably not analytic in a neighbourhood of $\lambda = 0$.

It remains to identify the boundary function G_{a0}. It is determined by (25) at $b = 0$. The equation involves subtle cancellations which so far we did not succeed to control. As substitute we use a symmetry argument. Given the boundary function G_{a0}, the Carleman theory computes the full 2-point function G_{ab} via (31). In particular, we get G_{0b} as function of G_{a0}. But the 2-point function is symmetric, $G_{ab} = G_{ba}$, and the special case $a = 0$ leads to the following self-consistency equation:

Proposition 8. *The limit $\theta \to \infty$ of $\lambda\phi^4_4$-theory on Moyal space is for $\lambda \leq 0$ determined by the solution of the fixed point equation $G = TG$,*

$$G_{b0} \equiv G_{0b} = \frac{1}{1+b} \exp\left(-\lambda \int_0^b dt \int_0^{\Lambda^2} \frac{dp}{(\lambda\pi p)^2 + \big(t+\frac{1+\lambda\pi p\mathcal{H}^\Lambda_p[G_{\bullet 0}]}{G_{p0}}\big)^2}\right). \tag{34}$$

At this point we can eventually send $\Lambda \to \infty$. Any solution of (34) is automatically smooth and monotonously decreasing. Any solution of the true equation (25) (without the difference to $b = 0$) also solves the master equation (34), but not necessarily conversely. In case of a unique solution of (34), it is enough to check one candidate. Existence of a solution of (34) is established by the Schauder fixed point theorem. This was done in [GW12] for $\lambda > 0$, and is work in progress for $\lambda < 0$.

This solution provides G_{ab} via (31) and all higher correlation functions via the universal algebraic recursion formulae (24b), or via the linear equations for the basic $(N_1+\ldots+N_B)$-point functions [GW12].

3. Schwinger functions and reflection positivity

3.1. Reverting the matrix representation

In the previous section we have constructed the connected matrix correlation functions $G_{|\underline{q}_1^1\cdots\underline{q}_{N_1}^1|\cdots|\underline{q}_1^B\cdots\underline{q}_{N_B}^B|}$ of the $(\theta\to\infty)$-limit of $\lambda\phi_4^4$-theory on Moyal space. These functions arise from the topological expansion (18) of the free energy. Now we revert the introduction of the matrix basis (4) to obtain Schwinger functions [Sch59] in position space:

$$S_c(\mu x_1,\ldots,\mu x_N) := \lim_{V\mu^4\to\infty}\lim_{\Lambda\to\infty}\frac{1}{64\pi^2}\sum_{N_1+\cdots+N_B=N}\sum_{\underline{q}_i^\beta\in\mathbb{N}^2_{\mathcal{N}}} G_{|\underline{q}_1^1\cdots\underline{q}_{N_1}^1|\cdots|\underline{q}_1^B\cdots\underline{q}_{N_B}^B|}$$
$$\times\sum_{\sigma\in\mathcal{S}_N}\prod_{\beta=1}^{B}\frac{f_{\underline{q}_1\underline{q}_2}(x_{\sigma(s_\beta+1)})\cdots f_{\underline{q}_{N_\beta}\underline{q}_1}(x_{\sigma(s_\beta+N_\beta)})}{V\mu^4 N_\beta}, \tag{35}$$

where $s_\beta := N_1+\ldots+N_{\beta-1}$ and $\mathcal{N} = \Lambda^2(1+\mathcal{Y})\sqrt{V\mu^4}$. The $G_{\ldots}$ are made dimensionless by appropriate rescaling in μ. These Schwinger functions are fully symmetric in $\mu x_1,\ldots,\mu x_N$. We recall that the prefactor of $G_{\ldots}$ in (18) was V^{2-B}. The factor V^{-B} is distributed over the B cycles. We have thus defined the density as $V^{-2}\log\frac{\mathcal{Z}[J]}{\mathcal{Z}[0]}$ in agreement with the spectral geometry of the Moyal plane with harmonic propagation [GW11]. There is one delicate point with this definition: We perform the limits $\lim_{V\mu^4\to\infty}, \lim_{\Lambda\to\infty}$ in different order than before and leave the justification as an open problem.

The next step consists in representing $G_{\ldots|\underline{p}_1\cdots\underline{p}_{N_\beta}|\ldots}$, for every boundary component, as a Laplace transform in $\frac{1}{\sqrt{V\mu^4}}(|\underline{p}_1|+\cdots+|\underline{p}_{N_\beta}|)$ and Fourier transform in $\frac{1}{\sqrt{V\mu^4}}(|\underline{p}_{i+1}|-|\underline{p}_i|)$. For example,

$$G_{|\underline{a}\underline{b}|} = \int_0^\infty dt\int_{-\infty}^{\infty} d\omega\;\mathcal{G}(t,\omega)e^{-\frac{t}{\sqrt{V\mu^4}}(|a|+|b|)-\mathrm{i}\frac{\omega}{\sqrt{V\mu^4}}(|a|-|b|)}\,. \tag{36}$$

Compatibility with the infinite volume limit to continuous matrix indices $\frac{|a|}{\sqrt{V\mu^4}}\to(1+\mathcal{Y})a$ is assumed.

The f_{mn} are products of associated Laguerre polynomials with a Gaußian [GV88, GW13]. The summation over the matrix indices is performed (at finite volume V) with the help of generating functions for the Laguerre polynomials. In two dimensions one has [GW13]:

$$\sum_{m_1,\ldots,m_L=0}^{\infty}\frac{1}{\theta}\prod_{i=1}^{L} f_{m_i m_{i+1}}(x_i)z_i^{m_i}$$

$$= \frac{2^L}{\theta(1-\prod_{i=1}^L(-z_i))} \exp\Big(- \frac{\sum_{i=1}^L \|x_i\|^2}{\theta} \frac{1+\prod_{i=1}^L(-z_i)}{1-\prod_{i=1}^L(-z_i)}\Big)$$
$$\times \exp\Big(-\frac{2}{\theta}\sum_{1\leq k<l\leq L}\Big((\langle x_k,x_l\rangle - \mathrm{i}x_k\times x_l)\frac{\prod_{j=k+1}^l(-z_j)}{1-\prod_{i=1}^L(-z_i)}$$
$$+ (\langle x_k,x_l\rangle + \mathrm{i}x_k\times x_l)\frac{\prod_{j=l+1}^{L+k}(-z_j)}{1-\prod_{i=1}^L(-z_i)}\Big)\Big). \qquad (37)$$

The z_i are of the form $z\sim \exp(-\frac{t+\mathrm{i}\omega}{\sqrt{V\mu^4}})$ as in (36). At this point the limit $V\mu^4 = \frac{\mu^2\theta^2}{16} \to \infty$ can be taken where z_i converges to 1. Thus for odd L the limit is zero, whereas for L even one has $\lim_{\theta\to\infty}\theta(1-\prod_{i=1}^L(-z_i)) = \frac{4Lt}{\mu^2}$. The vector product and all Fourier variables ω drop out, and the scalar products (37) arrange with the the norms to $\mu^2\|x_1 - x_2 + \cdots - x_L\|^2$. Absence of the Fourier variables means that all matrix indices per boundary component are equal. The Laplace transform is easily reverted after introduction of an auxiliary p-integration per boundary component. The final result is:

Theorem 9. *The connected N-point Schwinger functions of the $\lambda\phi^4_4$-model on extreme Moyal space $\theta\to\infty$ are given by*

$$S_c(\mu x_1,\dots,\mu x_N)$$
$$= \frac{1}{64\pi^2}\sum_{\substack{N_1+\ldots+N_B=N\\ N_\beta \text{ even}}}\sum_{\sigma\in\mathcal{S}_N}\Big(\prod_{\beta=1}^B \frac{4^{N_\beta}}{N_\beta}\int_{\mathbb{R}^4}\frac{dp_\beta}{4\pi^2\mu^4}\, e^{\mathrm{i}\big\langle \frac{p_\beta}{\mu},\sum_{i=1}^{N_\beta}(-1)^{i-1}\mu x_{\sigma(s_\beta+i)}\big\rangle}\Big)$$
$$\times G_{\underbrace{\frac{\|p_1\|^2}{2\mu^2(1+\mathcal{Y})},\dots,\frac{\|p_1\|^2}{2\mu^2(1+\mathcal{Y})}}_{N_1}|\dots|\underbrace{\frac{\|p_B\|^2}{2\mu^2(1+\mathcal{Y})},\dots,\frac{\|p_B\|^2}{2\mu^2(1+\mathcal{Y})}}_{N_B}}. \qquad (38)$$

Some comments:

- Only a restricted sector of the underlying matrix model contributes to position space: All strands of the same boundary component carry the same matrix index.
- Schwinger functions are symmetric and invariant under the full Euclidean group. This comes truly surprising since $\theta\neq 0$ breaks both translation invariance and manifest rotation invariance. The limit $\theta\to\infty$ was expected to make this symmetry violation even worse!
- The most interesting sector is the case where every boundary component has $N_\beta = 2$ indices. It is described by the $(2+\ldots+2)$-point functions $G_{\frac{\|p_1\|^2}{2\mu^2(1+\mathcal{Y})}\frac{\|p_1\|^2}{2\mu^2(1+\mathcal{Y})}|\dots|\frac{\|p_B\|^2}{2\mu^2(1+\mathcal{Y})}\frac{\|p_B\|^2}{2\mu^2(1+\mathcal{Y})}}$. The corresponding matrix functions $G_{a_1a_1|\dots|a_Ba_B}$ satisfy more complicated singular (but linear!) integral equations. The solution techniques of the Carleman problem can be used in a first step to regularise these equations to linear integral equations of Fredholm type. These have always a unique solution for $|\lambda|$ small enough.

- This (2+ ... +2)-sector describes the propagation and interaction of B (at the moment Euclidean) particles without any momentum exchange. Such a behaviour is necessary in any integrable model [Mos75, Kul76]. It is tempting to speculate that there might be an integrable structure behind that is responsible for the model being solvable and for absence of momentum transfer.
- We are aware of the problem that the absence of momentum transfer in four dimensions is a sign of *triviality*. Typical triviality proofs rely on clustering, analyticity in Mandelstam representation or absence of bound states. All this needs verification.
- That the $\theta \to \infty$ limit is so close to an ordinary field theory expected for $\theta \to 0$ can be seen from the following observation: The interaction term in momentum space

  $$\frac{\lambda}{4}\int_{(\mathbb{R}^4)^4}\Big(\prod_{i=1}^4 \frac{dp_i}{(2\pi)^4}\Big)\,\delta(p_1+\dots+p_4)\,\exp\Big(\mathrm{i}\sum_{i<j}\langle p_i,\Theta p_j\rangle\Big)\,\prod_{i=1}^4\hat{\phi}(p_i)$$

  leads to the Feynman rule $\lambda\exp\big(\mathrm{i}\sum_{i<j}\langle p_i,\Theta p_j\rangle\big)$, plus momentum conservation. For $\theta \to \infty$, this converges to zero almost everywhere by the Riemann-Lebesgue lemma, *unless p_i, p_j are linearly dependent.* This case of linearly dependent momenta might be protected for topological reasons, and these are precisely the boundary components $B > 1$ which guarantee full Lebesgue measure!

3.2. Osterwalder-Schrader axioms

Under conditions identified by Osterwalder-Schrader [OS73, OS75], Schwinger functions of a Eulidean quantum field theory permit an analytical continuation to Wightman functions [Wig56, SW64] of a true relativistic quantum field theory. In simplified terms, the reconstruction theorem of Osterwalder-Schrader for a field theory on $\mathbb{R}^D$ says:

Theorem 10 ([OS73, OS75]). *Assume the Schwinger functions $S(x_1,\dots,x_N)$ satisfy*

(OS0) factorial growth,
(OS1) Euclidean invariance,
(OS2) reflection positivity[1],
(OS3) permutation symmetry.

Then the $S(\xi_1,\dots\xi_{N-1})\big|_{\xi_i^0>0}$, with $\xi_i = x_i - x_{i+1}$, are Laplace-Fourier transforms of Wightman functions in a relativistic quantum field theory. If in addition the $S(x_1,\dots,x_N)$ satisfy

(OS4) clustering

[1]For each assignment $N \mapsto f_N \in \mathcal{S}^N$ of test functions, one has

$$\sum_{M,N}\int dx\,dy\;S(x_1,\dots,x_N,y_1,\dots,y_M)\overline{f_N(x_1^r,\dots,x_N^r)}f_M(y_1,\dots,y_M) \geq 0\,,$$

where $(x^0,x^1,\dots x^{D-1})^r := (-x^0,x^1,\dots x^{D-1})$

then the Wightman functions satisfy clustering, too.

The Schwinger functions (38) clearly satisfy (OS1)+(OS3). Clustering (OS4) is not realised. Bounds on $S_c(\mu x_1,\dots,\mu x_N)$ for large N follow from bounds on (24b) at coinciding indices and eventually from bounds on derivatives of the 2-point function. Formulae for $\frac{\partial^{n+\ell}G_{ab}}{\partial a^n\,\partial b^\ell}$ were derived in [GW14] and one has indeed a bound $\leq C(n+\ell-1)!$.

Thus the remaining problem is (OS2) reflection positivity. We will not discuss the axiom itself. Instead we rely on the Källén-Lehmann spectral representation [Käl52, Leh54] of a Wightman 2-point function according to which the Wightman 2-point function is a superposition of free fields with certain mass spectrum. Wightman functions always have an analytic continuation to Schwinger functions. Comparing with $S_c(\mu\xi,0)$ we find that the *diagonal matrix 2-point function is a Stieltjes function*, i.e. a function from $\mathbb{R}_+$ to $\mathbb{R}_+$ of the form

$$G_{aa}=\int_0^\infty \frac{d(\rho(t))}{a+t}\;, \tag{39}$$

where ρ is a positive measure. Indeed, using the residue theorem it is straightforward to check

$$\begin{aligned} S_c(\mu\xi,0)\big|_{\xi^0>0} &= \int_0^\infty \frac{2(1+\mathcal{Y})\,d\rho(t)}{\mu^4}\int_0^\infty dq^0\int_{\mathbb{R}^3} d\vec{q}\;\hat{W}_t(q)e^{-q^0\xi^0+\mathrm{i}\vec{q}\cdot\vec{\xi}}\;, \qquad (40)\\ \hat{W}_t(q) &:= \frac{\theta(q^0)}{(2\pi)^3}\delta\Big(\frac{(q^0)^2-\vec{q}^2-2\mu^2(1+\mathcal{Y})t}{\mu^2}\Big)\;. \end{aligned}$$

Stieltjes functions form an important subclass of the class of completely monotonic functions. We refer to [Ber08] for an overview about completely monotonic functions and their relations to other important classes of functions. The Stieltjes integral (39) provides a unique analytic continuation of a Stieltjes function to the cut plane $\mathbb{C}\setminus\,]-\infty,0[$. But the difficulty is to decide whether or not a given function is Stieltjes. Widder found criteria [Wid38] for the real function alone which guarantee the existence of the measure and gave a sequence which weakly converges to the measure. If the extension to the complex plane is known, then a function $f:\mathbb{C}\setminus\,]-\infty,0]\to\mathbb{C}$ is Stieltjes if [Ber08]

1. $f(x)\geq 0$ for $x>0$ (Euclidean positivity),
2. f is holomorphic on $\mathbb{C}\setminus\,]-\infty,0]$,
3. f is anti-Herglotz, i.e. $\mathrm{Im}(f(x+\mathrm{i}y)\leq 0$ for $y>0$ (Minkowskian positivity).

There is a complex inversion formula already due to Stieltjes (see [Wid38]) which recovers the measure from the boundary values at both sides of the cut.

From the anti-Herglotz property one concludes that $p^2\mapsto f(p^2)=\frac{1}{(p^2+\mu^2)^{1-\frac{\eta}{2}}}$ is Stieltjes precisely for $0\leq\eta\leq 2$, i.e. slower decay than

p^{-2}. In this sense, renormalisation (good decay of the propagator) contradicts positivity, and the only chance to construct a renormalisable theory in 4 dimensions is by continuation of a theory with good decay beyond its domain of definition, here to negative λ. Fourier transform implies $G(x-y) \sim \frac{e^{-\mu|x-y|}}{|x-y|^{4-(2-\eta)}}$ so that η is the anomalous dimension, which must be positive. What happens is that the *bare* anomalous dimension is positive for positive λ, but the leading term diverges. Renormalisation oversubtracts so that the renormalised anomalous dimension gets the opposite sign of λ.

3.3. Recent numerical and analytical results

A first hint about the two-point function and reflection positivity can be obtained from a numerical solution of the fixed point equation (34). This was done in [GW14] using *Mathematica*^{TM}. The idea is to approximate G_{0b} as a piecewise linear function on $[0, \Lambda^2]$ sampled according to a geometric progression and view (34) as iteration $G^{i+1}_{0b} = (TG^i)_{0b}$ for some initial function G^0. We confirmed the convergence of this iteration in Lipschitz norm for any $\lambda \in \mathbb{R}$. It turned out that the required symmetry $G_{ab} = G_{ba}$ does not hold for $\lambda > 0$, which is a clear hint that the winding number contributions $C, F(b)$ are present for $\lambda > 0$. For $\lambda < 0$ everything is consistent within small numerical errors. This allows us to compute for $\lambda < 0$ all quantities of the model with sufficient precision.

We find clear evidence for a second-order phase transition at $\lambda_c \approx -0.39$, which is a common critical value in several independent problems. The first one is the derivative $1 + \mathcal{Y} := -\frac{dG_{0b}}{db}\big|_{b=0}$, viewed as function of λ (fig. 1). More precisely, for $\lambda < \lambda_c$ we have $G_{0b} = 1$ in a whole neighbourhood of $b = 0$, and the length of this neighbourhood serves as an order parameter. Since also $1+\mathcal{Y} = 0$ for $\lambda < \lambda_c$, the infinite volume limit is ill-defined; i.e. the model is inconsistent beyond λ_c. Another phase transition occurs at $\lambda = 0$. It is not visible in G_{0b} but in the full 2-point function G_{ab} which looses its symmetry for $\lambda > 0$. This means that the 'good' phase is $\lambda_c < \lambda \leq 0$.

Of paramount importance is the question whether or not $a \mapsto G_{aa}$ is a Stieltjes function. We cannot expect a definite answer from a numerical simulation because a discrete approximation, here a piecewise linear function, cannot be analytic. The criteria for complete monotonicity [Ber08] and Widder's criterium for the Stieltjes property [Wid38] must fail for some order n. But refining the approximation, i.e. increasing the number L of sample points, the failure should occur at larger n, with no failure in the limit. This is precisely what we observe: The critical index where complete monotonicity or Stieltjes property fails increases with the resolution, but this increase slows down for larger $|\lambda|$ and stops at precisely the same value $\lambda_c \approx -0.39$ that located the discontinuity in fig. 1! We find clear evidence for

- a mass gap $\rho(t) = 0$ for $0 \leq t < \frac{m_0^2}{\mu^2}$,
- absence of a further mass gap, i.e. scattering right away from m_0^2 and not only from the two-particle threshold on.

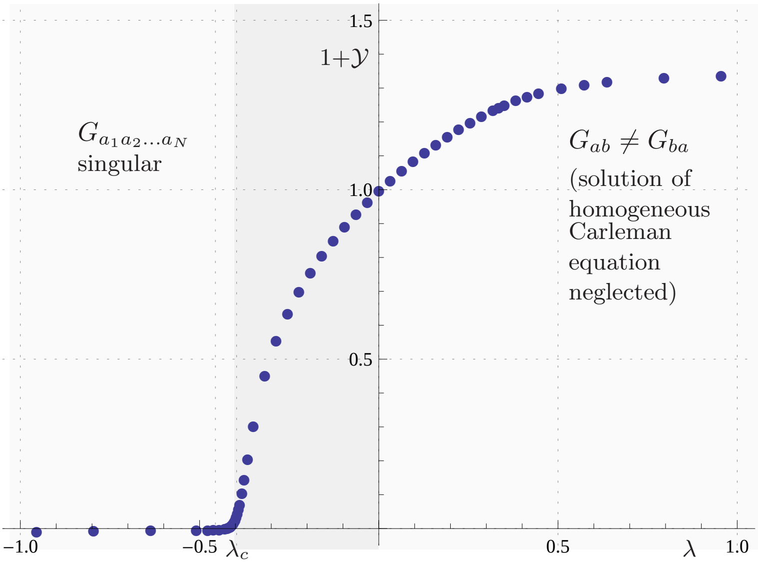


FIGURE 1. $1+\mathcal{Y} := -\frac{dG_{0b}}{db}\Big|_{b=0}$ as function of λ, based on G_{0b} computed for $\Lambda^2{=}10^7$ with $L = 2000$ sample points.

This provided enough motivation for an analytic treatment of the question. In a work in progress, we view (34) as a fixed point problem for $\log G_{b0} = T(\log G_{\bullet 0})(b)$, identify a Banach space $(X, \|\ \|)$ and a closed convex subset $\mathcal{K}_\lambda \subseteq X$ with the following properties for $-\frac{1}{6} \leq \lambda \leq 0$:

1. T maps $\mathcal{K}_\lambda$ into itself.
2. $\|Tf - Tg\| \leq (1 + \frac{1}{e} + \mathcal{O}(|\lambda|))\|f - g\|$ for any $f, g \in \mathcal{K}_\lambda$.
3. $T\mathcal{K}_\lambda$ is $\|\ \|$-compact in $\mathcal{K}_\lambda$ when restricted to $[0, \Lambda^2]$.

Property 2. excludes the Banach fixed point theorem, but together with 3. the Schauder fixed point theorem guarantees existence of a solution G_{b0}.

As by-product of these investigations we obtain an analytic continuation of G_{z0} to the complex plane. We prove that any function in $\exp(T\mathcal{K}_\lambda)$, in particular any fixed point of (34), is anti-Herglotz. The function G_{z0} is holomorphic outside the negative reals and outside a certain curve in left half-plane. If we could prove that G_{z0} is differentiable on that curve, then G_{b0} would be a Stieltjes. From here it is still some work to prove that G_{aa} is Stieltjes, but the compatibilities established so far are an extremely encouraging sign that this should also be true. We are thus convinced that the model will define a true relativistic quantum field theory in four dimensions.

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