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Construction of a Noncommutative Quantum Field Theory

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ABSTRACT. We review our recent successful attempt to construct the planar sector of a nonlocal scalar field model in four dimensional Euclidean deformed space-time, which needs 4 (instead of 3) relevant/marginal operators in the defining Lagrangian. As we have shown earlier, this model is renormalizable up to all orders in pertubation theory. In addition a new fixed point appears, at which the beta function for the coupling constant vanishes. This way, we were able to tame the Landau ghost.

We next discuss Ward identities and Schwinger-Dyson equations and derive integral equations for the renormalized N-point functions. They are the starting point of a nonperturbative construction of the model.

Dear Fritz! I (H.G.) almost cannot believe, that you become 60! I still remember the time, when you came from Graz to Vienna in the early 80's. I enjoyed our long standing interactions, our discussions on spectral concentration, how we handled the non-relativistic limit of the Dirac equation and especially our treatment of index problems and their connection to scattering theory. The last subject became of particular interests through the developments connected to noncommutative geometry and we enjoyed a recent Workshop at ESI on that subject together.

Here I review another outcome of using ideas from noncommutative geometry. I hope you will enjoy reading that a four dimensional quantum field theory model can be constructed on such a deformed space.

I wish you many new results for your interesting work and many happy years to come and hope for your visits to Vienna.

1. Introduction

Our present fundamental physics rests on two pillars: Quantum Field Theory and General Relativity. One of the main question in this area of physics concerns the matching of these two concepts.

In addition we hope to improve quantum field theory models by adding "gravity" effects. Constructive methods led years ago to many beautiful ideas and results, but the main goal to construct a mathematical consistent model of a four dimensional local quantum field theory has not been reached.

The requirements of local quantum field theory are easy to state and consists of quantum mechanical and relativity properties. States are supposed to be represented by vectors of a separable Hilbert space. Field operators are operator valued distribution, which should be smeared with smooth test functions in four coordinates and leads to $\Phi(f)$ acting on a

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dense domain of the Hilbert space,

$$\Phi(f) = \int d^4x \Phi(x) f^*(x). \tag{1}$$

The ground or vacuum state is unique (up to a phase) and cyclic. Space-time translations should be symmetries: This implies that the common spectrum of the energy-momentum operator $\sigma(P_\mu)$ lies in the closed forward light cone. The ground state is translation invariant. As for the relativistic properties one likes fields to transform covariantly under a unitary representation of the Poincaré group. One of the most essential postulate concerns **miscroscopic Causality** or locality. If the supports of the smearing functions f and g are space-like separated, then the field operators commute (for Bosons) or anticommute (for Fermions).

Typically one defines the expectation value of the product of smeared field operators called Wightman functions:

$$W_N(f_1 \otimes ... \otimes f_N) := \langle \Omega | \phi(f_1) \cdots \phi(f_N) | \Omega \rangle \tag{2}$$

It is not difficult to rephrase the requirements for the Wightman functions. For many purposes it is easier to go over to Euclidean Schwinger functions obtained by using analyticity of Wightman functions in the coordinate difference variables, implied by the support properties of the Wightman distributions. One has to go over to the so called extended permuted tube.

The formal definition of Schwinger functions reads:

$$S_N(z_1, ..., z_N) = \int \Phi(z_1)...\Phi(z_N) d\nu(\Phi), \qquad d\nu = \frac{1}{Z} e^{-\int L_{int}(\Phi)} d\mu(\Phi), \tag{3}$$

where $d\mu$ is the Gaussian measure corresponding to free fields with two point correlation: $\langle \phi(x_1)\phi(x_2)\rangle = C(x_1, x_2)$, or its Fourier transform: $\tilde{C}(p_1, p_2) = \delta(p_1 - p_2)\frac{1}{p_1^2 + m^2}$, ϕ above is a stochastic variable.

As for interacting fields we have to rely on **(renormalized) pertubation expansions.** We have to put first cut-offs and to expand the interacting part:

$$S_N(x_1 \dots x_N) = \sum_n \frac{(-\lambda)^n}{n!} \int d\mu(\phi) \prod_j^N \phi(x_j) \left(\int dx \, \frac{\phi^4(x)}{4!} \right)^n \tag{4}$$

$$= \sum_{\text{graph } \Gamma_N} \frac{(-\lambda)^n}{Sym_{\Gamma_N}(G)} \int_V \prod_{l \in \Gamma_N} C_\kappa(x_l - y_l) \sim \Lambda^{\omega_D(G)} \tag{5}$$

As a result we may collect contributions to the same Feynman diagram and evaluate the degree of divergence, which is given by $\omega_D(G) = (D-4)n + D - \frac{D-2}{2}N$, $\omega_2(G) = 2 - 2n$, $\omega_4(G) = 4 - N$, where n denotes the order of the graph, or the number of vertices, N the number of external lines, l the number of internal lines. Note that there are $(4n + N)!!$ number of Feynman graphs. Use Stirling formula and the factor $\frac{1}{n!}$ from the exponential, the large order behavior $K^n n!$ for the contributions result, which indicates that a naïve convergence is questionable.

Renormalization If one imposes a finite number of renormalization conditions (here we need 3 conditions to fix a, m and λ), for example:

$$G_2(p^2 = 0) = \frac{1}{m_{phys}^2}, \frac{d}{dp^2} G_2(p^2 = 0) = -\frac{a^2}{m_{phys}^4}, G_4(p^2 = 0) = \lambda_{phys} \tag{6}$$

and no new interactions are generated order by order in pertubation theory, we call the model to be renormalizable (this is implied by the BPHZ Theorem for the scalar Φ^4 model).

The program of constructing a nontrivial interacting models was successfully done only in $D = 2, 3$ space-time dimensions. As for $D = 4$ dimensions we have to rely on renormalized pertubation theory and follow the renormalization group flow. But in addition we may add "Gravity" effects, or quantize Space-Time: This led to our program of merging **general relativity ideas with quantum physics through noncommutative geometry**.

Space-Time structure That one should limit localisation in space-time follows from a very simple old argument due to Wheeler and others:

In order to localize two events, which are a distance D apart, one has to do a scattering experiment with particles whose energy hc/λ exceed hc/D. Multiplying these quantities times G/c^4 yields the Schwarzschild radius of the appropriate energy lump. It is natural to require that this radius should be smaller than the distance between the events one started with, since otherwise the scattered particles will be captured by the black hole, which is formed. Putting both inequalities together gives a lower bound to the distance of localizability of events of the order of the Planck length.

$$D \geq R_{ss} = G/c^4 hc/\lambda \geq G/c^4 hc/D \tag{7}$$

which implies that $D \geq l_p$ = Planck length. Early ideas of modifying space-time were phrased already by Schrödinger and Heisenberg, but Snyder in 1947 was the first to formulate a deformed space-time geometry. Such ideas became popular after 1986, when Alain Connes published his work on Noncommutative Geometry. On of us (H. G.) started in 1992 (in work together with J. Madore) to use noncommutative manifolds (algebras) as a natural cut off for quantized field theory models. Doplicher, Fredenhagen and Roberts used the Wheeler argument in 1994 to formulate uncertainty relations for deformed fields and formulated deformed free fields. Filk in 1995 was the first to elaborate on Feynman rules for models defined over deformed space-time, and finally they became popular due to the work of Schomerus (1999), who observed, that such models may result from string theory after taking the zero slope limit.

Ideas: Algebra, fields, diff. calculus,...

Typically one first refers to the Gelfand - Naimark theorem, which states that the algebra of continuous functions over a manifold is isomorphic to a commutative C^* algebra. Next one studies deformations of such algebras, through associative nonlocal star products. Especially simple is the Moyal space. One may start from the algebra of smooth functions over D-dimensional Euclidean space, and define the $\star$-product as $(a \star b)(x) = \int d^D y d^D k a(x+\frac{1}{2}\Theta\cdot k) b(x+y)\, \mathrm{e}^{iky}$ where $\Theta = -\Theta^T \in M_D(\mathbb{R})$

Fields are sections of bundles, according to the Serre Swan theorem, they can be identified as projective modules over the algebra A. A very essential requirement concerns the differential calculus, which we would like to keep. Next question results:

Can we make sense of renormalisation in Noncommutative Geometric Models? As a first step we intend to construct simple quantum field theory models on simple noncommutative geometries, e.g. the Moyal space. Of course, this way we obtain models with non-local interactions.

The naïve application of this procedure to the ϕ^4-action (ϕ-real, Euclidean space) leads on Moyal plane to the action:

$$S = \int d^4x \Big(\frac{1}{2} \partial_\mu \phi \star \partial^\mu \phi + \frac{m^2}{2} \phi \star \phi + \frac{\lambda}{4} \phi \star \phi \star \phi \star \phi \Big)(x) \tag{8}$$

The Feynman rules can be obtained easily. Since we obtain only cyclic invariance at the Vertex, Graphs are best drawn as Ribbon Graphs on Riemann surfaces with a certain genus and a certain number of boundary components. We obtain planar regular contribution and non-planar graphs. The planar graphs still reveal UV divergences, the nonplanar ones are finite for generic momenta. On the other hand for exceptional momenta (if sums of incoming or outgoing momenta vanish) the contributions develop an IR singularity, which spoils **Renormalizability!** In our previous work [**1**] we realized that the UV/IR-mixing problem can be solved by adding a fourth relevant/marginal operator to the Lagrangian
Theorem: The quantum field theory defined by the action

$$S=\int d^4x\Big(\frac{1}{2}\phi\star\big(\Delta+\Omega^2\tilde{x}^2+\mu^2\big)\phi+\frac{\lambda}{4!}\phi\star\phi\star\phi\star\phi\Big)(x) \tag{9}$$

is **perturbatively renormalisable to all orders in λ.**

The additional oscillator potential $\Omega^2\tilde{x}^2$ implements mixing between large and small distance scales and results from the renormalisation proof. Maja Buric and Michael Wohlgenannt [**2**] found an interesting interpretation of this additional term: It results as the coupling of the scalar field to the scalar curvature within the truncation procedure.

Here, $\star$ refers to the Moyal product parametrised by the antisymmetric 4×4-matrix Θ, and $\tilde{x}=2\Theta^{-1}x$. The model is covariant under the Langmann-Szabo duality transformation [**3**] and becomes self-dual at $\Omega=1$. Certain variants have also been treated, see [**4**] for a review. Evaluation of the β-functions for the coupling constants Ω,λ in first order of perturbation theory leads to a coupled dynamical system which indicates a fixed-point at $\Omega=1$, while λ remains bounded [**5**, **6**]. The vanishing of the β-function at $\Omega=1$ was next proven in [**7**] at three-loop order and finally in [**8**] to all orders of perturbation theory. It implies that there is no infinite renormalisation of λ, which makes the nonperturbative construction simpler. The Landau ghost problem is solved. The vanishing of the β-function to all orders has been obtained using a Ward identity [**8**]. We extend this work and derive an integral equation for the planar sector of the two-point function alone by using the Ward identity and Schwinger-Dyson equations. Usually, Schwinger-Dyson equations couple the two-point function to the four-point function. In our model, we show that the Ward identity allows to express the four-point function in terms of the two-point function, resulting in an equation for the two-point function alone. This is achieved in the first step for the bare two-point function. We are able to perform the mass and wavefunction renormalisation directly in the integral equation, giving a *self-consistent non-linear equation for the renormalised two-point function alone*. Higher n-point functions fulfil a *linear* (inhomogeneous) Schwinger-Dyson equation, with the inhomogeneity given by m-point functions with $m<n$. This means that solving our equation for the two-point function leads to a non-perturbative construction of the planar sector of this interacting quantum field theory in four dimensions. Recently we reduced the question of solving this model to solving one nonlinear integral equation in one variable [**11**]. Of course, the next question concerns the nonplanar sector of this model. We know the appropriate Ward identities, an extension of the reviewed ideas to this sector is under discussion.

In the case of the Φ^4 model with negavite coupling constant, it was possible to sum up the planar graphs, but the nonplanar graphs cannot be summed up, due to lack of stability, see [**12**] and [**13**]. In the present model, we have a positive coupling constant and stability is not a problem. Nevertheless the construction of the full model is still a hard task. A new summation technique has been invented recently for such a situation [**14**]. It has been applied to the two-dimensional model already [**15**].

We hope that a detailed analysis of our model will help for a non-perturbative treatment of more realistic quantum field theories. We expect that we can learn much about non-perturbative renormalization of Euclidean quantum field theories in four dimensions from this almost solvable model.

2. Matrix Model

It is convenient to write the action (9) in the matrix base of the Moyal space, see [**1**, **9**]. It simplifies enormously at the self-duality point $\Omega = 1$. We write down the resulting action functionals for the *bare* quantities, which involves the bare mass μ_{bare} and the wave function renormalisation $\phi \mapsto Z^{\frac{1}{2}}\phi$. For simplicity we fix the length scale to $\theta = 4$. This gives

$$S = \sum_{m,n\in\mathbb{N}^2_\Lambda} \frac{1}{2}\phi_{mn}H_{mn}\phi_{nm} + V(\phi), \tag{10}$$

$$H_{mn} = Z\big(\mu^2_{bare} + |m| + |n|\big)\ , \qquad V(\phi) = \frac{Z^2\lambda}{4}\sum_{m,n,k,l\in\mathbb{N}^2_\Lambda} \phi_{mn}\phi_{nk}\phi_{kl}\phi_{lm}\ , \tag{11}$$

It is already used that this model has no renormalisation of the coupling constant [**8**]. All summation indices $m, n, \ldots$ belong to $\mathbb{N}^2$, with $|m| := m_1 + m_2$. The symbol $\mathbb{N}^2_\Lambda$ refers to a cut-off in the matrix size. The scalar field is real, $\phi_{mn} = \overline{\phi_{nm}}$.

3. Ward Identity

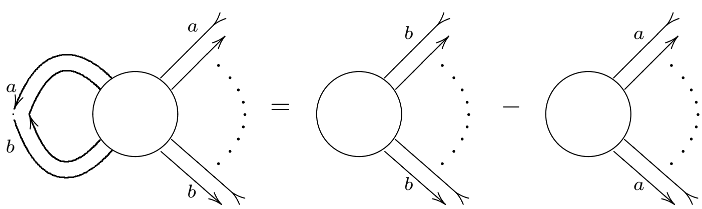
The key step in the proof [**8**] that the β-function vanishes is the discovery of a Ward identity induced by inner automorphisms $\phi \mapsto U\phi U^\dagger$. Inserting into the connected graphs the special insertion vertex

$$V^{ins}_{ab} := \sum_n (H_{an} - H_{nb})\phi_{bn}\phi_{na} \tag{12}$$

is the same as the difference of graphs with external indices b and a, respectively, $Z(|a| - |b|)G^{ins}_{[ab]\ldots} = G_{b\ldots} - G_{a\ldots}$:

We write Feynman graphs in the self-dual ϕ^4_4-model as ribbon graphs on a genus-g Riemann surface with B external faces. Adding for each external face an external vertex to get a closed surface, the matrix index is constant at every face. Inserting the special vertex V^{ins}_{ab} leads, however, to an index jump from a to b in an external face which meets an external vertex. The corresponding external sources at the jumped face are thus J_{na} and J_{bm} for some other indices m, n. According to the Ward identity, this is the same as the difference between the graphs with face index b and a, respectively:

(13) $Z(|a| - |b|)$



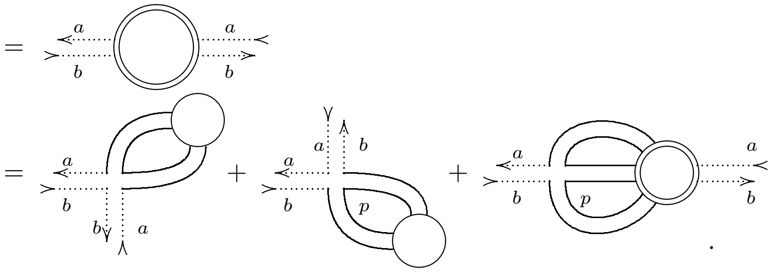
$$Z(|a| - |b|)G^{ins}_{[ab]\ldots} = G_{b\ldots} - G_{a\ldots}\ . \tag{14}$$

The dots in (14) stand for the remaining face indices. We have used $H_{an} - H_{nb} = Z(|a| - |b|)$.

4. Schwinger-Dyson equation

The Schwinger-Dyson equation for the one-particle irreducible two-point function Γ^{ab} reads

(15) $\Gamma_{ab} =$



The sum of the last two graphs can be reexpressed in terms of the two-point function with insertion vertex. Adding the left tadpole and using the Ward identity yields

$$
\begin{aligned}
\Gamma_{ab} &= Z^2\lambda \sum_p \Big(G_{ap} + G_{ab}^{-1} G^{ins}_{[ap]b}\Big) = Z^2\lambda \sum_p \Big(G_{ap} - G_{ab}^{-1} \frac{G_{bp} - G_{ba}}{Z(|p| - |a|)}\Big) \\
&= Z^2\lambda \sum_p \Big(\frac{1}{H_{ap} - \Gamma_{ap}} + \frac{1}{H_{bp} - \Gamma_{bp}} - \frac{1}{H_{bp} - \Gamma_{bp}} \frac{(\Gamma_{bp} - \Gamma_{ab})}{Z(|p| - |a|)}\Big) .
\end{aligned}
\tag{16}
$$

This is a closed equation for the two-point function alone. It involves the divergent quantities Γ_{bp} and Z, μ_{bare}.

5. Renormalization

Introducing the renormalised planar two-point function Γ^{ren}_{ab} by Taylor expansion $\Gamma_{ab} = Z\mu^2_{bare} - \mu^2 + (Z-1)(|a|+|b|) + \Gamma^{ren}_{ab}$ and imposing the renormalization condition $\Gamma^{ren}_{00} = 0$ and $(\partial\Gamma^{ren})_{00} = 0$, we obtain a coupled system of equations for Γ^{ren}_{ab}, Z and μ_{bare}. It leads to a closed equation for the renormalised function Γ^{ren}_{ab} alone, which is further analysed in the integral representation.

We replace the indices in $a, b, \dots \mathbb{N}$ by continuous variables in $\mathbb{R}_+$. Equation (16) depends only on the length $|a| = a_1 + a_2$ of indices. The Taylor expansion respects this feature, so that we replace $\sum_{p\in\mathbb{N}^2_\Lambda}$ by $\int_0^\Lambda |p|\, dp$. After a convenient change of variables $|a| =: \mu^2 \frac{\alpha}{1-\alpha}$, $|p| =: \mu^2 \frac{\rho}{1-\rho}$ and

$$
\Gamma^{ren}_{ab} =: \mu^2 \frac{1 - \alpha\beta}{(1-\alpha)(1-\beta)} \Big(1 - \frac{1}{G_{\alpha\beta}}\Big) , \tag{17}
$$

and using an identity resulting from the symmetry $G_{0\alpha} = G_{\alpha 0}$, we arrive at [**10**]:

THEOREM 1. *The renormalised planar connected two-point function $G_{\alpha\beta}$ of the self-dual noncommutative ϕ^4_4-theory satisfies the integral equation*

$$
\begin{aligned}
(18)\qquad G_{\alpha\beta} = 1 + \lambda\Big(& \frac{1-\alpha}{1-\alpha\beta}\big(\mathcal{M}_\beta - \mathcal{L}_\beta - \beta\mathcal{Y}\big) + \frac{1-\beta}{1-\alpha\beta}\big(\mathcal{M}_\alpha - \mathcal{L}_\alpha - \alpha\mathcal{Y}\big) \\
& + \frac{1-\beta}{1-\alpha\beta}\Big(\frac{G_{\alpha\beta}}{G_{0\alpha}} - 1\Big)\big(\mathcal{M}_\alpha - \mathcal{L}_\alpha + \alpha\mathcal{N}_{\alpha 0}\big) \\
& - \frac{\alpha(1-\beta)}{1-\alpha\beta}\big(\mathcal{L}_\beta + \mathcal{N}_{\alpha\beta} - \mathcal{N}_{\alpha 0}\big) + \frac{(1-\alpha)(1-\beta)}{1-\alpha\beta}(G_{\alpha\beta} - 1)\mathcal{Y}\Big) ,
\end{aligned}
$$

where $\alpha, \beta \in [0, 1)$,

$$
\mathcal{L}_\alpha := \int_0^1 d\rho\, \frac{G_{\alpha\rho} - G_{0\rho}}{1-\rho} , \quad \mathcal{M}_\alpha := \int_0^1 d\rho\, \frac{\alpha\, G_{\alpha\rho}}{1-\alpha\rho} , \quad \mathcal{N}_{\alpha\beta} := \int_0^1 d\rho\, \frac{G_{\rho\beta} - G_{\alpha\beta}}{\rho - \alpha} ,
$$

and $\mathcal{Y} = \lim_{\alpha\to 0} \frac{\mathcal{M}_\alpha - \mathcal{L}_\alpha}{\alpha}$.

Recently we related the construction of this noncommutative quantum field theory to the problem of solving a nonlinear integral equation in one variable [**11**], which we review next.

6. Nonperturbative Construction of this model

We rewrite equation (18) in terms of $D_{\alpha\beta} := \frac{\alpha(1-\beta)}{\beta(1-\alpha)}\Big(\frac{(1-\beta)}{1-\alpha\beta}G_{\alpha\beta} - G_{\alpha 0}\Big)$ and obtain after simple manipulations the integral equation

$$
(19)\qquad \frac{\beta(1-\alpha)}{\alpha(1-\beta)} + \frac{1 + \lambda\mathcal{Y} + \lambda\pi\alpha\mathcal{H}_\alpha\big[G_{\bullet 0}\big]}{\alpha G_{\alpha 0}} D_{\alpha\beta} - \lambda\pi\mathcal{H}_\alpha\big[D_{\bullet\beta}\big] = -G_{\alpha 0},
$$

which is of the Carleman type. $\mathcal{Y}$ is defined as

$$
(20)\qquad -\lambda\pi\mathcal{H}_0[D_{\bullet 0}] = \frac{\lambda\mathcal{Y}}{1+\lambda\mathcal{Y}}.
$$

Here we assume that $D_{\alpha\beta}$ is Hölder continuous. The finite Hilbert transform is given by $\mathcal{H}_\alpha[f(\bullet)] := \frac{1}{\pi}\lim_{\epsilon\to 0}\Big(\int_0^{\alpha-\epsilon} + \int_{\alpha+\epsilon}^1\Big)\frac{f(\rho)}{\rho - \alpha}$.
Equation (19) is a singular linear integral equation of the Carleman type. We quote its solution [Carleman 1922, Tricomi 1957]
Theorem: The singular linear integral equation $a(x)y(x) - \lambda\pi\mathcal{H}_x[y] = f(x)$, $x \in [-1, 1]$ is for $a(x)$ continuous and Hölder continuous near ± 1 and $f \in L^p$ is solved by

$$
(21)\qquad y(x) = \frac{\sin(\theta(x))}{\lambda\pi}\Big(f(x)\cos(\theta(x)) + e^{\mathcal{H}_x[\theta]}\mathcal{H}_x\Big[e^{-\mathcal{H}_\bullet[\theta]}f(\bullet)\sin(\theta(\bullet))\Big] + \frac{Ce^{\mathcal{H}_x[\theta]}}{1-x}\Big)
$$

$$
(22)\qquad \theta(x) = \operatorname*{arctan}_{[0,\,\pi]}\Big(\frac{\lambda\pi}{a(x)}\Big) , \; \sin(\theta(x)) = \frac{|\lambda\pi|}{\sqrt{(a(x))^2 + (\lambda\pi)^2}}
$$

where C is an arbitrary constant. We assume first: $C = 0$
We apply the solution of the Carleman equation to (19) and obtain for its solution:

$$
(23)\qquad \frac{(1-\beta)}{1-\alpha\beta}\frac{G_{\alpha\beta}}{1+\lambda\mathcal{Y}} = \frac{\sin(\theta_\beta(\alpha))}{|\lambda|\pi\alpha}e^{\mathcal{H}_\alpha[\theta_\beta(\bullet)] - \mathcal{H}_0[\theta_0(\bullet)] + \mathcal{H}_1[\theta_0(\bullet) - \theta_\beta(\bullet)]}
$$

$$\frac{\lambda\mathcal{Y}}{1+\lambda\mathcal{Y}} = \int_0^1 d\rho\, \frac{\sin^2(\theta_0(\rho))}{\lambda\pi^2\rho^2}, \tag{24}$$

$$\theta_\beta(\alpha) = \underset{[0,\,\pi]}{\arctan}\left(\frac{\lambda\pi\alpha}{\frac{\beta(1-\alpha)}{1-\beta} + \frac{1+\lambda\mathcal{Y}+\lambda\pi\alpha\mathcal{H}_\alpha[G_{\bullet 0}]}{G_{\alpha 0}}}\right) \tag{25}$$

For the proof one uses the Carleman equation $\lambda\pi\cot\theta_0(\alpha)G_{\alpha 0} - \lambda\pi\mathcal{H}_\alpha[G_{\bullet 0}] = \frac{1+\lambda\mathcal{Y}}{\alpha}$ and Tricomi's identity $e^{-\mathcal{H}_\alpha[\theta_\beta]}\cos(\theta_\beta(\alpha)) + \mathcal{H}_\alpha\Big[e^{-\mathcal{H}_\bullet[\theta_\beta]}\sin(\theta_\beta(\bullet)\Big] = 1$.
The Carleman equation computes $G_{\alpha\beta}$, as a consequence it implies that $G_{\alpha\beta} \geq 0$! Therefore $G_{0\beta}$ can be evaluated and this implies also a self-consistency equation for $G_{\beta 0}$, since symmetry forces $G_{\beta 0} = G_{0\beta}$.
This leads to the Master equation, whose solution determines the theory completely:

$$G_{\beta 0} = \frac{1+\lambda\mathcal{Y}}{1+(1-\beta)\lambda\mathcal{Y}} exp(-\lambda\int_0^{\frac{\beta}{1-\beta}} dt \int_0^1 \frac{d\rho}{(\lambda\pi\rho)^2 + (t(1-\rho) + \frac{1+\lambda\mathcal{Y}+\lambda\pi\rho\mathcal{H}_\rho[G_{\bullet 0}]}{G_{\rho 0}})^2}) \tag{26}$$

) provided it exists. Of course, together with $\lambda\mathcal{Y}$, which has to be determined from equation (24). Up to now, we deduced various non-perturbative results from this system of equations and used computer calculations for the visualization of the solution of (26). As expected, there is a big difference between the case $\lambda > 0$ and $\lambda < 0$. For positive $\lambda > 0$ we deduce, that $\frac{(1+(1-\beta)\lambda\mathcal{Y})}{1+\lambda\mathcal{Y}}G_{\beta 0} \in \mathcal{C}^1([0,1[)$, is monotonously decreasing and positive. Therefore the limiting value G_{10} exists and $G_{\beta 0} \in \mathcal{C}[0,1]$. For $\lambda < 0$ $\frac{(1+(1-\beta)\lambda\mathcal{Y})}{1+\lambda\mathcal{Y}}G_{\beta 0} \in \mathcal{C}^1([0,1[)$ is monotonously increasing and positive, therefore $G_{\beta 0}$ is unbounded at $\beta = 1$.
Let $\lambda > 0$, $G = TG$ be the master equation and F be within the Hölder class of index λ. Recall $Z^{-1}(G) = 1 + \lambda\mathcal{Y}_G - \lambda\int_0^1 d\rho\, \frac{G_{\rho 0}}{1-\rho}$. We can prove, that if $F(1) \neq 0$, then $(TF)(1) = 0$, if $Z^{-1}(F) \geq \delta > 0$, then $(TF)(1) \geq \epsilon > 0$. If $Z^{-1}(F) < 0$, then $(TF)(1) = 0$. As a consequence we deduce, that $G_{10} = 0$ and $Z^{-1}(G) \leq 0$. But this means, that if

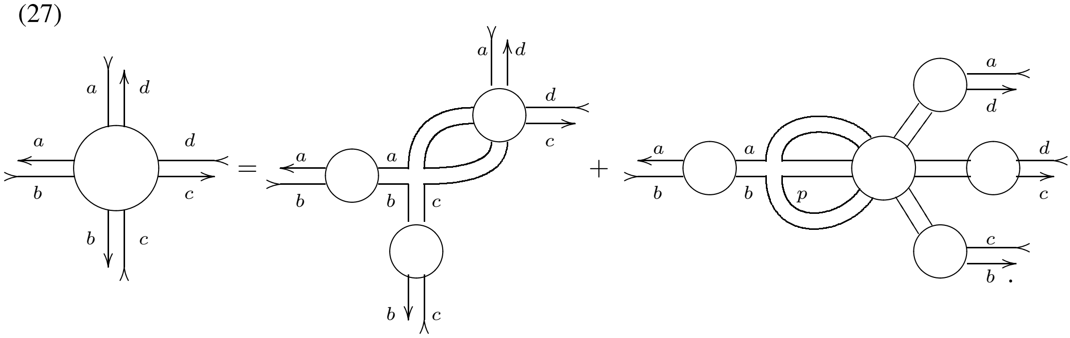
$$G_{\alpha 0} = 0 \quad \Rightarrow \quad 1 + \lambda\mathcal{Y} + \lambda\pi\alpha\mathcal{H}_\alpha\big[G_{\bullet 0}\big] = 0$$

For $\alpha = 1$ this means $Z^{-1}(G) = 0$.

7. Four-point Schwinger-Dyson equation

The knowledge of the two-point function allows a successive construction of the whole theory. As an example we mention the planar connected four-point function G_{abcd}.
Following the a-face in direction of an arrow, there is a distinguished vertex at which the first ab-line starts. For this vertex there are two possibilities for the matrix index of the diagonally opposite corner to the a-face: either c or a summation vertex p:

(27)



We write the first contribution as a product of the vertex $Z^2\lambda$, the left connected two-point function, the downward two-point function and an insertion, which is reexpressed by means of the Ward-identity. After amputation of the external two-point functions we obtain the Schwinger-Dyson equation for the *renormalised* 1PI four-point function $G_{abcd} = G_{ab}G_{bc}G_{cd}G_{da}\Gamma^{ren}_{abcd}$ as follows:

(28)
$$\Gamma^{ren}_{abcd} = Z\lambda\frac{1}{|a|-|c|}\Big(\frac{1}{G_{ad}} - \frac{1}{G_{cd}}\Big) + Z\lambda\sum_p \frac{1}{|a|-|p|}G_{pb}\Big(\frac{G_{dp}}{G_{ad}}\Gamma^{ren}_{pbcd} - \Gamma^{ren}_{abcd}\Big)\,.$$

We introduce the 1PI function and pass to the integral representation and to the variables α and β and find for $\Gamma_{\alpha\beta\gamma\delta} := \Gamma^{ren}_{abcd}$ an integral equation, which manipulated appropriately allows again to take the limit $\xi \to 1$ after insertion of the expression for the wave function renormalisation constant.

THEOREM 2. *The renormalised planar 1PI four-point function $\Gamma_{\alpha\beta\gamma\delta}$ of self-dual noncommutative ϕ^4_4-theory (with continuous indices $\alpha,\beta,\gamma,\delta \in [0,1)$) satisfies the integral equation*

$$\Gamma_{\alpha\beta\gamma\delta} = \lambda\cdot\frac{\begin{aligned}\Big(1 - \frac{(1-\alpha)(1-\gamma\delta)(G_{\alpha\delta}-G_{\gamma\delta})}{G_{\gamma\delta}(1-\delta)(\alpha-\gamma)}\\ + \int_0^1 \rho\,d\rho\frac{(1-\beta)(1-\alpha\delta)G_{\beta\rho}G_{\delta\rho}}{(1-\beta\rho)(1-\delta\rho)}\frac{\Gamma_{\rho\beta\gamma\delta}-\Gamma_{\alpha\beta\gamma\delta}}{\rho-\alpha}\Big)\end{aligned}}{\begin{aligned}G_{\alpha\delta} + \lambda\Big((\mathcal{M}_\beta - \mathcal{L}_\beta - \mathcal{Y})G_{\alpha\delta} + \int_0^1 d\rho\frac{G_{\alpha\delta}G_{\beta\rho}(1-\beta)}{(1-\delta\rho)(1-\beta\rho)}\\ + \int_0^1 \rho\,d\rho\frac{(1-\beta)(1-\alpha\delta)G_{\beta\rho}}{(1-\beta\rho)(1-\delta\rho)}\frac{(G_{\rho\delta}-G_{\alpha\delta})}{(\rho-\alpha)}\Big)\end{aligned}}\,. \tag{29}$$

In lowest order we find

$$\begin{aligned}\Gamma_{\alpha\beta\gamma\delta} = \lambda - \lambda^2\Big(&\frac{(1-\gamma)(I_\alpha-\alpha)-(1-\alpha)(I_\gamma-\gamma)}{\alpha-\gamma}\\ &+\frac{(1-\delta)(I_\beta-\beta)-(1-\beta)(I_\delta-\delta)}{\beta-\delta}\Big) + \mathcal{O}(\lambda^3)\,.\end{aligned} \tag{30}$$

Note that $\Gamma_{\alpha\beta\gamma\delta}$ is cyclic in the four indices, and that $\Gamma_{0000} = \lambda + \mathcal{O}(\lambda^3)$.
In our recent work, we have been able to solve equation (29) in terms of the two point function and a remarkable simple expression results:

$$\Gamma_{\alpha\beta\gamma\delta} = \frac{\lambda}{(\alpha-\gamma)(\beta-\delta)}\Big(\frac{(1-\alpha\delta)}{G_{\alpha\delta}}\frac{(1-\gamma\beta)}{G_{\gamma\beta}} - \frac{(1-\alpha\beta)}{G_{\alpha\beta}}\frac{(1-\gamma\delta)}{G_{\gamma\delta}}\Big)\,. \tag{31}$$

It was now possible to evaluate the effective coupling in terms of the bare coupling constant. Although the scale is changes ba an infinite amount, a finite coupling constant renormalization results.

8. Conclusions

A remarkable result concerns the appearance of the nontrivial fixed point at $\Omega = 1$, proven to all orders in pertubation theory. We used Ward identities and Schwinger-Dyson equations to deduce integral equations for the renormalized N-point functions. We reduced the construction of this nontrivial noncommutative quantum field theory to solving one nonlinear integral equation for a function of one variable. A survey of this construction is given in [**11**].

We believe, that the first nontrivial four dimensional quantum field theory model, where one is able to sum up the Feynman pertubation expansion, will allow to learn a lot about renormalization.

The zero of the beta function occurs in the one-loop calculation for the degenerate model too. There are attempts to deduce implications for cosmology from space-time noncommutativity. But, of course, there is, up to now, no effect known, which allows a check by experiments in the near future.

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References

[1] Harald Grosse and Raimar Wulkenhaar, *Renormalisation of ϕ^4-theory on noncommutative $\mathbb{R}^4$ in the matrix base*, Comm. Math. Phys. **256** (2005), no. 2, 305–374, DOI 10.1007/s00220-004-1285-2. MR2160797 (2006g:81133)

[2] Maja Burić and Michael Wohlgenannt, *Geometry of the Grosse-Wulkenhaar model*, J. High Energy Phys. **3** (2010), 053, 17, DOI 10.1007/JHEP03(2010)053. MR2653469 (2011j:81168)

[3] Edwin Langmann and Richard J. Szabo, *Duality in scalar field theory on noncommutative phase spaces*, Phys. Lett. B **533** (2002), no. 1-2, 168–177, DOI 10.1016/S0370-2693(02)01650-7. MR1911944 (2003d:81305)

[4] Vincent Rivasseau, *Non-commutative renormalization*, Quantum spaces, Prog. Math. Phys., vol. 53, Birkhäuser, Basel, 2007, pp. 19–107, DOI 10.1007/978-3-7643-8522-4_2. MR2382235 (2009e:81143)

[5] H. Grosse and R. Wulkenhaar, *The β-function in duality-covariant non-commutative ϕ^4-theory*, Eur. Phys. J. C Part. Fields **35** (2004), no. 2, 277–282, DOI 10.1140/epjc/s2004-01853-x. MR2075076 (2005b:81213)

[6] Harald Grosse and Raimar Wulkenhaar, *Renormalisation of ϕ^4-theory on non-commutative $\mathbb{R}^4$ to all orders*, Lett. Math. Phys. **71** (2005), no. 1, 13–26, DOI 10.1007/s11005-004-5116-3. MR2136734 (2006d:81172)

[7] M. Disertori and V. Rivasseau, “Two and three loops beta function of non commutative ϕ^4_4 theory,” Eur. Phys. J. C **50** (2007) 661 [arXiv:hep-th/0610224].

[8] Margherita Disertori, Razvan Gurau, Jacques Magnen, and Vincent Rivasseau, *Vanishing of beta function of non-commutative Φ^4_4 theory to all orders*, Phys. Lett. B **649** (2007), no. 1, 95–102, DOI 10.1016/j.physletb.2007.04.007. MR2308801 (2008f:81261)

[9] Harald Grosse and Raimar Wulkenhaar, *Renormalisation of ϕ^4-theory on noncommutative $\mathbb{R}^2$ in the matrix base*, J. High Energy Phys. **12** (2003), 019, 26 pp. (electronic), DOI 10.1088/1126-6708/2003/12/019. MR2041157 (2004m:81162)

[10] H. Grosse and R. Wulkenhaar, “Progress in solving a noncommutative quantum field theory in four dimensions,” arXiv:0909.1389 [hep-th].

[11] H. Grosse and R. Wulkenhaar, “Self-dual noncommutative ϕ^4-theory in four dimensions is a non-perturbatively solvable and non-trivial quantum field theory,” arXiv:1205.0465 [math-ph].

[12] Gerard ’t Hooft, *Rigorous construction of planar diagram field theories in four-dimensional Euclidean space*, Comm. Math. Phys. **88** (1983), no. 1, 1–25. MR691195 (84h:81065)

[13] Vincent Rivasseau, *Construction and Borel summability of planar* 4*-dimensional Euclidean field theory*, Comm. Math. Phys. **95** (1984), no. 4, 445–486. MR767190 (86a:81046)
[14] Vincent Rivasseau, *Constructive matrix theory*, J. High Energy Phys. **9** (2007), 008, 13, DOI 10.1088/1126-6708/2007/09/008. MR2342423 (2008k:81187)
[15] Z. Wang, "Constructive Renormalization of 2-dimensional Grosse-Wulkenhaar Model," arXiv:1205.0196 [hep-th].

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