PROCEEDINGS OF SCIENCE

Renormalisation of the Grosse-Wulkenhaar model

Harald Grosse[∗†]
Faculty of Physics, University of Vienna
E-mail: harald.grosse@univie.ac.at

Raimar Wulkenhaar
Mathematisches Institut der Westfälischen Wilhelms-Universität, Münster, Germany
E-mail: raimar@math.uni-muenster.de

We give an introduction into the problems of local quantum fields and argue, that a quantization of space-time might lead to better behaviour. Next we discuss a special Euclidean ϕ^4_4-quantum field theory over quantized space-time as an example of a renormalizable field theory. Using a Ward identity, it was possible to prove the vanishing of the beta function for the coupling constant to all orders in perturbation theory. We extend this work and obtain from the Schwinger-Dyson equation a non-linear integral equation for the renormalised two-point function alone. The non-trivial renormalised four-point function fulfils a linear integral equation with the inhomogeneity determined by the two-point function. These integral equations might be the starting point of a nonperturbative construction of a Euclidean quantum field theory on a noncommutative space. We expect to learn about renormalisation from this almost solvable model.

3rd Quantum Gravity and Quantum Geometry School
February 28 - March 13, 2011
Zakopane, Poland

∗Speaker.

†H. G. thanks Prof. Jerzy Lewandowski for the kind invitation to the Zakopane Winter School and him and the many young students for the enjoyable atmosphere.

PoS(QGQGS 2011)011

1. Introduction

Our present fundamental physics rests on two pillars: Quantum Field Theory and General Relativity. One of the main question in this area of physics concerns the matching of these two concepts.
In addition we hope to improve quantum field theory models by adding "gravity" effects. Constructive methods led years ago to many beautiful ideas and results, but the main goal to construct a mathematical consistent model of a four dimensional local quantum field theory has not been reached. Renormalized pertubation expansions allow to get quantum corrections order by order in a coupling constant. The convergence of this expansion, for example as a Borel summable series, can be questioned.
In recent years, a modification of the space-time structure led to new models, which are nonlocal in a particular sense. But these models, in general suffer under an additional disease, which is called the Infrared Ultraviolet mixing [1]. Additional infrared singularities show up. A possible way to cure this problem has been found by us in previous work [2]. It led to special models, which needed 4 (instead of 3) relevant/marginal operators in the defining Lagrangian. We have been able to show that the resulting model is renormalizable up to all orders in pertubation theory. In addition a new fixed point appeared at a special value of the additional coupling constant. This way, we were able to tame the Landau ghost problem. Since the old problems of additional singularities due to partial summing up the pertubation expansion do not show up, we believe that the pertubation expansion will be Borel summable. That this new fixed point exists in pertubation theory to all orders has been shown in work by Rivasseau and collaborators.
The main open question concerns the nonpertubative construction of a nontrivial noncommutative quantum field theory. Steps in that direction will be discussed here.
Classical field theories for fundamental interactions (electroweak, strong, gravitational) are of geometrical origin. We may remind, that the Fermi interaction is non renormalisable, it needs a cutoff around 300 GeV, otherwise unitarity is violated! This is nicely resolved by adding new particles, the W^+, Z^0, W^-, and the confirmation of their existence was a great step towards consistency of quantum field theory models. On the other hand the Standard Model (electroweak+strong) of particle physics is renormalisable, while gravity is not! In the sense of renormalisation theory one might argue, that space-time should not be a smooth manifold at tiny distances, gravity would be scaled away. Or stated otherwise, the weakness of gravity determines the Planck scale, and geometry at these tiny distances should be something different.
A promising approach concerns noncommutative geometry, which allows to unify the standard model with gravity as a classical field theory.
Requirements: From Wightman axioms to Euclidean Schwinger functions:
The principles of local quantum fields are easy to state, but it was, up to now, impossible to construct nontrivial models in four space-time dimensions. The required properties can be split into quantum mechanical ones and relativity properties.
Quantum Mechanical Properties:
States are supposed to be represented by vectors of a separable Hilbert space H.
The field operator is an operator valued distribution, smearing it with smooth test functions leads to $\Phi(f)$ acting on a dense domain D, $\Phi(f) = \int d^4x \Phi(x) f^*(x)$.
The ground or vacuum state Ω is unique (up to a phase) and cyclic.

Space-time translations are symmetries: This implies that the common spectrum of the energy-momentum operator $\sigma(P_\mu)$ lies in the closed forward light cone.
The ground state $\Omega \in H$ is invariant under $e^{ia_\mu P^\mu}$.
Relativistic properties:
There is an unitary representation of the Poincaré group $U_{(a,\Lambda)}$ on H and fields transform covariantly. One of the main postulate concerns **miscroscopic Causality** or Locality. If the supports of the smearing function f and g are space-like separated, then the fields operators commute (for Bosons) or anticommute (for Fermions). $[\phi(f),\phi(g)]_\pm \Psi = 0 \qquad \text{for} \qquad supp f \subset (supp g)'$
Typically one defines the expectation value of the product of smeared field operators called Wightman functions:
$W_N(f_1 \otimes ... \otimes f_N) := \langle \Omega | \phi(f_1) \cdots \phi(f_N) | \Omega \rangle$
It is not difficult to rephrase the requirements for the Wightman functions, but we shall not do it here. For many purposes it is easier to go over to Euclidean Schwinger functions obtained by using analyticity of Wightman functions (in the so called extended permuted tube).
The formal definition of Schwinger functions reads:
$S_N(z_1,...,z_N) = \int \Phi(z_1)...\Phi(z_N) d\nu(\Phi)$
$d\nu = \frac{1}{Z} e^{-\int L_{int}(\Phi)} d\mu(\Phi)$,
where $d\mu$ is the Gaussian measure corresponding to free fields with two point correlation: $\langle \phi(x_1)\phi(x_2) \rangle = C(x_1,x_2)$, or its Fourier transform: $\tilde{C}(p_1,p_2) = \delta(p_1 - p_2) \frac{1}{p_1^2+m^2}$, ϕ above is a stochastic variable.
As for interacting fields we have to rely on **(renormalized) pertubation expansions:**

$$S_N(x_1 \ldots x_N) = \frac{1}{Z} \int [d\phi]\, e^{-\int dx \mathscr{L}(\phi)} \prod_i^N \phi(x_i) \tag{1.1}$$

We may extract the free part $d\mu(\phi) \propto [d\phi]\, e^{-\frac{m^2}{2}\int \phi^2 - \frac{a}{2}\int (\partial_\mu \phi)(\partial^\mu \phi)}$ with correlations:

$$\int d\mu(\phi)\phi(x_1)\ldots\phi(x_N) = \sum_{\text{pairings}} \prod_{l \in \gamma} C(x_{i_l} - x_{j_l}) \tag{1.2}$$

Up to now, all expressions are formal, in order to justify the procedure, we may put a cut off:
$\tilde{C}_\kappa(p) = \int_{\kappa=1/\Lambda^2}^\infty d\alpha e^{-\alpha(p^2+m^2)}$
To deal with interactions we may add $\frac{\lambda}{4!}\Phi^4$, and expand

$$S_N(x_1 \ldots x_N) = \sum_n \frac{(-\lambda)^n}{n!} \int d\mu(\phi) \prod_j^N \phi(x_j) \left(\int dx \frac{\phi^4(x)}{4!} \right)^n \tag{1.3}$$

$$= \sum_{\text{graph } \Gamma_N} \frac{(-\lambda)^n}{Sym_{\Gamma_N}(G)} \int_V \prod_{l \in \Gamma_N} C_\kappa(x_l - y_l) \sim \Lambda^{\omega_D(G)} \tag{1.4}$$

As a result we may collect contributions to the same Feynman diagram and evaluate the degree of divergence, which is given by $\omega_D(G) = (D-4)n + D - \frac{D-2}{2}N$, $\omega_2(G) = 2-2n$, $\omega_4(G) = 4-N$, where n denotes the order of the graph, or the number of vertices, N the number of external lines, l the number of internal lines. Note that there are $(4n+N)!!$ number of Feynman graphs. Use Stirling formula and the factor $\frac{1}{n!}$ from the exponential, the large order behavior $K^n n!$ for the contributions result, which indicates that there will be no Taylor or (Borel) convergence.

Renormalization

If one may impose a finite number of renormalization conditions (here we need 3: a, m, λ), for example:

$G_2(p^2 = 0) = \frac{1}{m^2_{phys}}$, $\frac{d}{dp^2} G_2(p^2 = 0) = -\frac{a^2}{m^4_{phys}}$, $G_4(p^2 = 0) = \lambda_{phys}$ and no new interactions are generated order by order in pertubation theory, we call the model to be renormalizable (this is implied by the BPHZ Theorem for the scalar Φ^4 model).

According to a different point of view, we may follow Wilsons Renormalization Group Flow ideas and divide the covariance for free Euclidean scalar field into slices:

$$\Phi_m = \sum_{j=0}^{m} \phi_j, \quad C_j = \int_{M^{-2j}}^{M^{-2(j-1)}} d\alpha \frac{e^{-m^2\alpha - x^2/4\alpha}}{\alpha^{D/2}} \tag{1.5}$$

We may integrate out degrees of freedom:

$$Z_{m-1}(\Phi_{m-1}) = \int d\mu_m(\phi_m) e^{-S_m(\phi_m + \Phi_{m-1})} \tag{1.6}$$

and obtain a relation between the actions at different scales:

$$Z_{m-1}(\Phi_{m-1}) = e^{-S_{m-1}(\Phi_{m-1})} \tag{1.7}$$

Of course, in order to evaluate these expressions we have to use some expansion.

As a matter of fact, in all these expansions a certain chain of finite subgraphs (for example with m bubbles) grows like $\simeq C^m m!$, and indicates that this expansion will not be Borel summable. An easy estimate shows that the scale dependence of the coupling constant will be given by:

$$\lambda_j \simeq \frac{\lambda_0}{1 - \beta \lambda_0 j}$$

If the sign of β is positive it indicates the appearance of the so called Landau ghost, or phrased differently triviality of this model may result. A negative sign of β indicates asymptotic freedom. The program of constructing a nontrivial interacting models was successfully done only in $D = 2, 3$ space-time dimensions. As for $D = 4$ dimensions we have to rely first on renormalized pertubation theory and follow the renormalization group flow. In addition we may add "Gravity" effects, or quantize Space-Time: This led to our program of merging **general relativity ideas with quantum physics through noncommutative geometry**.

Space-Time structure

That one should limit localisation in space-time follows from a very simple old argument due to Wheeler:

In order to localize two events, which are a distance D apart, one has to do a scattering experiment with particles whose energy hc/λ exceed hc/D. Multiplying these quantities times G/c^4 yields the Schwarzschild radius of the appropriate energy lump. It is natural to require that this radius should be smaller than the distance between the events one started with, since otherwise the scattered particles will be captured by the black hole, which is formed. Putting both inequalities together

gives a lower bound to the distance of localizability of events of the order of the Planck length. $D \geq R_{ss} = G/c^4 hc/\lambda \geq G/c^4 hc/D$ which implies that $D \geq l_p \quad 10^{-35} m$.
Early ideas of modifying space-time were phrased already by Riemann, Schrödinger and Heisenberg, but Snyder in 1947 was the first to formulate a deformed space-time geometry. Such ideas become popular after 1986, when Alain Connes published his work on Noncommutative Geometry. On of us (H. G.) started in 1992 (in work together with J. Madore) to use noncommutative manifolds (algebras) as a natural cut off for quantized field theory models. Doplicher, Fredenhagen and Roberts used the Wheeler argument in 1994 to formulate uncertainty relations for deformed fields and formulated deformed free fields. Filk in 1995 was the first to elaborate on Feynman rules for models defined over deformed space-time, and finally they became popular due to the work of Schomerus (1999), who observed, that such models may result from string theory after taking the zero slope limit.
Ideas: Algebra, fields, diff. calculus,...
Typically one first refers to the Gelfand - Naimark theorem, which states that the algebra of continuous functions over a manifold is isomorphic to a commutative C^* algebra. Next one studies deformations of such algebras, through associative nonlocal star products. Especially simple is the Moyal space. One may start from the algebra of smooth functions over D-dimensional Euclidean space, and define the $\star$-product as
$(a \star b)(x) = \int d^D y d^D k a(x + \frac{1}{2}\Theta \cdot k) b(x+y) \, \mathrm{e}^{iky}$ where $\Theta = -\Theta^T \in M_D(\mathbb{R})$
Fields are sections of bundles, according to the Serre Swan theorem, they can be identified as projective modules over the algebra A.

A very essential requirement concerns the differential calculus. We would like to have a differential, which obeys the Leibniz rule and squares to zero. By duality vector fields can be defined. Since we will be dealing in the following only with the canonical deformed space, there is no problem with having a differential calculus.
Next question results: **Can we make sense of renormalisation in Noncommutative Geometric Models?**
As a first step we intend to construct simple quantum field theory models on simple noncommutative geometries, e.g. the Moyal space. Of course, this way we obtain models with non-local interaction.
The naïve application of this procedure to the ϕ^4-action (ϕ-real, Euclidean space) leads on Moyal plane to the action:

$$S = \int d^4x \Big(\frac{1}{2} \partial_\mu \phi \star \partial^\mu \phi + \frac{m^2}{2} \phi \star \phi + \frac{\lambda}{4} \phi \star \phi \star \phi \star \phi \Big)(x) \tag{1.8}$$

The Feynman rules can be obtained easily. Since we obtain only cyclic invariance at the Vertex, Graphs are best drawn as Ribbon Graphs on Riemann surfaces with a certain genus and a certain number of boundary components. We obtain planar regular contribution and non-planar graphs. The planar graphs still reveal UV divergences, the nonplanar ones are finite for generic momenta. On the other hand for exceptional momenta (if sums of incoming or outgoing momenta vanish) the contributions develop a IR singularity, which spoils **Renormalizability!** In our previous work

[2] we realized that the UV/IR-mixing problem can be solved by adding a fourth relevant/marginal operator to the Lagrangian **Theorem:** The quantum field theory defined by the action

$$S = \int d^4x \Big(\frac{1}{2} \phi \star \big(\Delta + \Omega^2 \tilde{x}^2 + \mu^2 \big) \phi + \frac{\lambda}{4!} \phi \star \phi \star \phi \star \phi \Big)(x) \tag{1.9}$$

is **perturbatively renormalisable to all orders in λ.**
The additional oscillator potential $\Omega^2 \tilde{x}^2$ implements mixing between large and small distance scales and results from the renormalisation proof. Maja Buric and Michael Wohlgenannt [3] found an interesting interpretation of this additional term: It results as the coupling of the scalar field to the scalar curvature within the truncation procedure (see also the contribution of Maja Buric to these Proceedings).
Here, $\star$ refers to the Moyal product parametrised by the antisymmetric 4×4-matrix Θ, and $\tilde{x} = 2\Theta^{-1}x$. The model is covariant under the Langmann-Szabo duality transformation [4] and becomes self-dual at $\Omega = 1$. Certain variants have also been treated, see [5] for a review. Evaluation of the β-functions for the coupling constants Ω, λ in first order of perturbation theory leads to a coupled dynamical system which indicates a fixed-point at $\Omega = 1$, while λ remains bounded [6, 7]. The vanishing of the β-function at $\Omega = 1$ was next proven in [8] at three-loop order and finally in [9] to all orders of perturbation theory. It implies that there is no infinite renormalisation of λ, and a non-perturbative construction seems possible [10]. The Landau ghost problem is solved. The vanishing of the β-function to all orders has been obtained using a Ward identity [9]. We extend this work and derive an integral equation for the two-point function alone by using the Ward identity and Schwinger-Dyson equations. Usually, Schwinger-Dyson equations couple the two-point function to the four-point function. In our model, we show that the Ward identity allows to express the four-point function in terms of the two-point function, resulting in an equation for the two-point function alone. This is achieved in the first step for the bare two-point function. We are able to perform the mass and wavefunction renormalisation directly in the integral equation, giving a *self-consistent non-linear equation for the renormalised two-point function alone*. Higher n-point functions fulfil a *linear* (inhomogeneous) Schwinger-Dyson equation, with the inhomogeneity given by m-point functions with $m < n$. This means that solving our equation for the two-point function leads to a full non-perturbative construction of this interacting quantum field theory in four dimensions. So far we treated our equation perturbatively up to third order in λ. The solution shows an interesting number-theoretic structure.
We hope that a detailed analysis of our model will help for a non-perturbative treatment of more realistic Euclidean quantum field theories. We expect that we can learn much about non-perturbative renormalization of Euclidean quantum field theories in four dimensions from this almost solvable model.

2. Matrix Model

It is convenient to write the action (1.9) in the matrix base of the Moyal space, see [2, 11]. It simplifies enormously at the self-duality point $\Omega = 1$. We write down the resulting action functionals for the *bare* quantities, which involves the bare mass μ_{bare} and the wave function renormalisation

$\phi \mapsto Z^{\frac{1}{2}}\phi$. For simplicity we fix the length scale to $\theta = 4$. This gives

$$S = \sum_{m,n \in \mathbb{N}^2_\Lambda} \frac{1}{2}\phi_{mn} H_{mn} \phi_{nm} + V(\phi)\,, \tag{2.1}$$

$$H_{mn} = Z\big(\mu^2_{bare} + |m| + |n|\big)\,, \qquad V(\phi) = \frac{Z^2\lambda}{4} \sum_{m,n,k,l \in \mathbb{N}^2_\Lambda} \phi_{mn}\phi_{nk}\phi_{kl}\phi_{lm}\,, \tag{2.2}$$

It is already used that this model has no renormalisation of the coupling constant [9]. All summation indices $m,n,\ldots$ belong to $\mathbb{N}^2$, with $|m| := m_1 + m_2$. The symbol $\mathbb{N}^2_\Lambda$ refers to a cut-off in the matrix size. The scalar field is real, $\phi_{mn} = \overline{\phi_{nm}}$.

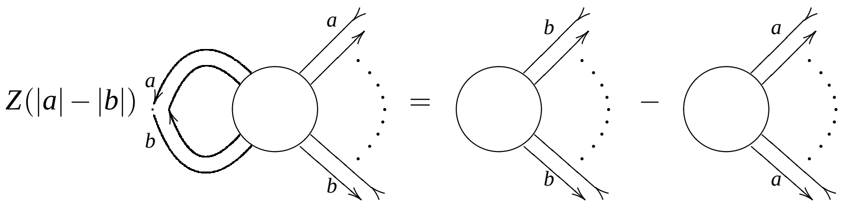
3. Ward Identity

The key step in the proof [9] that the β-function vanishes is the discovery of a Ward identity induced by inner automorphisms $\phi \mapsto U\phi U^\dagger$. Inserting into the connected graphs the special insertion vertex

$$V^{ins}_{ab} := \sum_n (H_{an} - H_{nb})\phi_{bn}\phi_{na} \tag{3.1}$$

is the same as the difference of graphs with external indices b and a, respectively, $Z(|a|-|b|)G^{ins}_{[ab]\ldots} = G_{b\ldots} - G_{a\ldots}$:

We write Feynman graphs in the self-dual ϕ^4_4-model as ribbon graphs on a genus-g Riemann surface with B external faces. Adding for each external face an external vertex to get a closed surface, the matrix index is constant at every face. Inserting the special vertex V^{ins}_{ab} leads, however, to an index jump from a to b in an external face which meets an external vertex. The corresponding external sources at the jumped face are thus J_{na} and J_{bm} for some other indices m,n. According to the Ward identity, this is the same as the difference between the graphs with face index b and a, respectively:



(3.2)

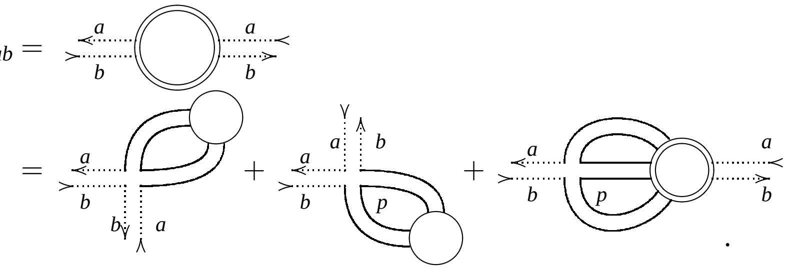
$$Z(|a|-|b|)G^{ins}_{[ab]\ldots} = G_{b\ldots} - G_{a\ldots}\,. \tag{3.3}$$

The dots in (3.3) stand for the remaining face indices. We have used $H_{an} - H_{nb} = Z(|a| - |b|)$.

4. Schwinger-Dyson equation

The Schwinger-Dyson equation for the one-particle irreducible two-point function Γ^{ab} reads

$\Gamma_{ab} =$ (4.1)



The sum of the last two graphs can be reexpressed in terms of the two-point function with insertion vertex,

$$\begin{aligned}\Gamma_{ab} &= Z^2\lambda\sum_p\Big(G_{ap}+G^{-1}_{ab}G^{ins}_{[ap]b}\Big) = Z^2\lambda\sum_p\Big(G_{ap}-G^{-1}_{ab}\frac{G_{bp}-G_{ba}}{Z(|p|-|a|)}\Big) \\ &= Z^2\lambda\sum_p\Big(\frac{1}{H_{ap}-\Gamma_{ap}}+\frac{1}{H_{bp}-\Gamma_{bp}}-\frac{1}{H_{bp}-\Gamma_{bp}}\frac{(\Gamma_{bp}-\Gamma_{ab})}{Z(|p|-|a|)}\Big)\;.\end{aligned} \tag{4.2}$$

This is a closed equation for the two-point function alone. It involves the divergent quantities Γ_{bp} and Z,μ_{bare}.

5. Renormalization

Introducing the renormalised planar two-point function Γ^{ren}_{ab} by Taylor expansion $\Gamma_{ab} = Z\mu^2_{bare} - \mu^2 + (Z-1)(|a|+|b|) + \Gamma^{ren}_{ab}$, with $\Gamma^{ren}_{00} = 0$ and $(\partial\Gamma^{ren})_{00} = 0$, we obtain a coupled system of equations for Γ^{ren}_{ab}, Z and μ_{bare}. It leads to a closed equation for the renormalised function Γ^{ren}_{ab} alone, which is further analysed in the integral representation.

We replace the indices in $a,b,\ldots\mathbb{N}$ by continuous variables in $\mathbb{R}_+$. Equation (4.2) depends only on the length $|a| = a_1 + a_2$ of indices. The Taylor expansion respects this feature, so that we replace $\sum_{p\in\mathbb{N}^2_\Lambda}$ by $\int_0^\Lambda |p|\,dp$. After a convenient change of variables $|a| =: \mu^2\frac{\alpha}{1-\alpha}$, $|p| =: \mu^2\frac{\rho}{1-\rho}$ and

$$\Gamma^{ren}_{ab} =: \mu^2\frac{1-\alpha\beta}{(1-\alpha)(1-\beta)}\Big(1-\frac{1}{G_{\alpha\beta}}\Big)\;, \tag{5.1}$$

and using an identity resulting from the symmetry $G_{0\alpha} = G_{\alpha 0}$, we arrive at [12]:

Theorem 1. *The renormalised planar connected two-point function $G_{\alpha\beta}$ of the self-dual noncommutative ϕ^4_4-theory satisfies the integral equation*

$$\begin{aligned}G_{\alpha\beta} = 1+\lambda\Bigg(&\frac{1-\alpha}{1-\alpha\beta}\big(\mathscr{M}_\beta-\mathscr{L}_\beta-\beta\mathscr{Y}\big)+\frac{1-\beta}{1-\alpha\beta}\big(\mathscr{M}_\alpha-\mathscr{L}_\alpha-\alpha\mathscr{Y}\big) \\ &+\frac{1-\beta}{1-\alpha\beta}\Big(\frac{G_{\alpha\beta}}{G_{0\alpha}}-1\Big)\big(\mathscr{M}_\alpha-\mathscr{L}_\alpha+\alpha\mathscr{N}_{\alpha 0}\big) \\ &-\frac{\alpha(1-\beta)}{1-\alpha\beta}\big(\mathscr{L}_\beta+\mathscr{N}_{\alpha\beta}-\mathscr{N}_{\alpha 0}\big)+\frac{(1-\alpha)(1-\beta)}{1-\alpha\beta}(G_{\alpha\beta}-1)\mathscr{Y}\Bigg)\;,\end{aligned} \tag{5.2}$$

where $\alpha, \beta \in [0,1)$,

$$\mathscr{L}_\alpha := \int_0^1 d\rho \, \frac{G_{\alpha\rho} - G_{0\rho}}{1-\rho}\,, \qquad \mathscr{M}_\alpha := \int_0^1 d\rho \, \frac{\alpha\, G_{\alpha\rho}}{1-\alpha\rho}\,, \qquad \mathscr{N}_{\alpha\beta} := \int_0^1 d\rho \, \frac{G_{\rho\beta} - G_{\alpha\beta}}{\rho-\alpha}\,,$$

and $\mathscr{Y} = \lim_{\alpha\to 0} \frac{\mathscr{M}_\alpha - \mathscr{L}_\alpha}{\alpha}$.

6. Perturbation expansion

These integral equations are the starting point for a perturbative solution. In this way, the renormalised correlation functions are directly obtained, without Feynman graph computation and further renormalisation steps. We obtain

$$\begin{aligned} G_{\alpha\beta} = 1 &+ \lambda\Big\{A(I_\beta - \beta) + B(I_\alpha - \alpha)\Big\} \\ &+ \lambda^2\Big\{AB\big((I_{\underset{\bullet}{\alpha}} - \alpha) + (I_{\underset{\bullet}{\beta}} - \beta) + (I_\alpha - \alpha)(I_\beta - \beta) + \alpha\beta(\zeta(2)+1)\big) \\ &\quad + A\big(\beta I_{\underset{\bullet}{\beta}} - \beta I_\beta\big) - \alpha AB\big((I_\beta)^2 - 2\beta I_\beta + I_\beta\big) \\ &\quad + B\big(\alpha I_{\underset{\bullet}{\alpha}} - \alpha I_\alpha\big) - \beta BA\big((I_\alpha)^2 - 2\alpha I_\alpha + I_\alpha\big)\Big\} + \mathscr{O}(\lambda^3)\,, \end{aligned} \tag{6.1}$$

where $A := \frac{1-\alpha}{1-\alpha\beta}$, $B := \frac{1-\beta}{1-\alpha\beta}$ and the following iterated integrals appear:

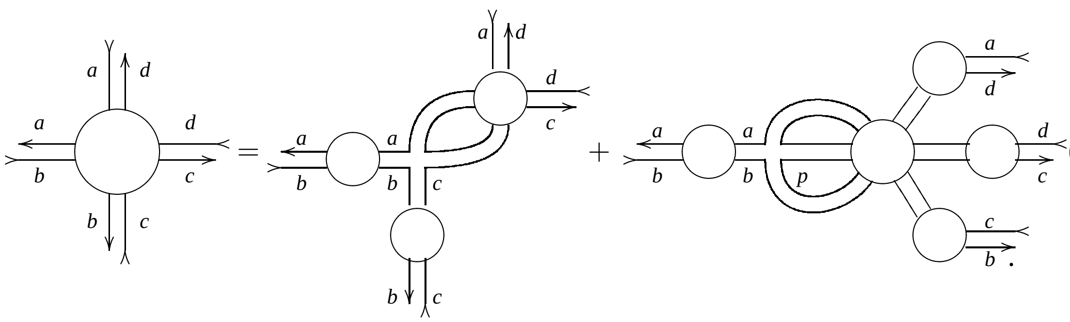
$$\begin{aligned} I_\alpha &:= \int_0^1 dx \, \frac{\alpha}{1-\alpha x} = -\ln(1-\alpha)\,, \\ I_{\underset{\bullet}{\alpha}} &:= \int_0^1 dx \, \frac{\alpha\, I_x}{1-\alpha x} = \mathrm{Li}_2(\alpha) + \frac{1}{2}\big(\ln(1-\alpha)\big)^2\,. \end{aligned} \tag{6.2}$$

We conjecture that $G_{\alpha\beta}$ is at any order a polynomial with rational coefficients in α, β, A, B and iterated integrals labelled by rooted trees.

7. Four-point Schwinger-Dyson equation

The knowledge of the two-point function allows a successive construction of the whole theory. As an example we treat the planar connected four-point function G_{abcd}.

Following the a-face in direction of an arrow, there is a distinguished vertex at which the first ab-line starts. For this vertex there are two possibilities for the matrix index of the diagonally opposite corner to the a-face: either c or a summation vertex p:



(7.1)

We write the first contribution as a product of the vertex $Z^2\lambda$, the left connected two-point function, the downward two-point function and an insertion, which is reexpressed by means of the Ward-identity. After amputation of the external two-point functions we obtain the Schwinger-Dyson equation for the *renormalised* 1PI four-point function $G_{abcd} = G_{ab}G_{bc}G_{cd}G_{da}\Gamma^{ren}_{abcd}$ as follows:

$$\Gamma^{ren}_{abcd} = Z\lambda \frac{1}{|a|-|c|}\Big(\frac{1}{G_{ad}} - \frac{1}{G_{cd}}\Big) + Z\lambda \sum_p \frac{1}{|a|-|p|} G_{pb}\Big(\frac{G_{dp}}{G_{ad}}\Gamma^{ren}_{pbcd} - \Gamma^{ren}_{abcd}\Big)\,. \tag{7.2}$$

In terms of the 1PI function we have

$$\begin{aligned} Z^{-1}\Gamma^{ren}_{abcd} &= \lambda\Big(1 - \frac{\Gamma^{ren}_{ad} - \Gamma^{ren}_{cd}}{|a|-|c|}\Big) \\ &\quad + \lambda \sum_p \frac{|a|+|d|+\mu^2-\Gamma^{ren}_{ad}}{|p|+|b|+\mu^2-\Gamma^{ren}_{pb}} \frac{\dfrac{\Gamma^{ren}_{pbcd} - \Gamma^{ren}_{abcd}}{|p|-|a|}}{|p|+|d|+\mu^2-\Gamma^{ren}_{pd}} \\ &\quad + \lambda\Gamma^{ren}_{abcd} \sum_p \frac{1 - \dfrac{\Gamma^{ren}_{ad} - \Gamma^{ren}_{pd}}{|a|-|p|}}{(|p|+|b|+\mu^2-\Gamma^{ren}_{pb})(|p|+|d|+\mu^2-\Gamma^{ren}_{pd})}\,. \end{aligned} \tag{7.3}$$

Passing to the integral representation and the variables α and β, we find for $\Gamma_{\alpha\beta\gamma\delta} := \Gamma^{ren}_{abcd}$ an integral equation, which manipulated appropriately allows again to take the limit $\xi \to 1$ after insertion of the expression for the wave function renormalisation constant.

Theorem 2. *The renormalised planar 1PI four-point function $\Gamma_{\alpha\beta\gamma\delta}$ of self-dual noncommutative ϕ^4_4-theory (with continuous indices $\alpha,\beta,\gamma,\delta \in [0,1)$) satisfies the integral equation*

$$\Gamma_{\alpha\beta\gamma\delta} = \lambda \cdot \frac{\begin{aligned}&\Big(1 - \frac{(1-\alpha)(1-\gamma\delta)(G_{\alpha\delta}-G_{\gamma\delta})}{G_{\gamma\delta}(1-\delta)(\alpha-\gamma)} \\ &\qquad + \int_0^1 \rho\, d\rho \frac{(1-\beta)(1-\alpha\delta)G_{\beta\rho}G_{\delta\rho}}{(1-\beta\rho)(1-\delta\rho)} \frac{\Gamma_{\rho\beta\gamma\delta} - \Gamma_{\alpha\beta\gamma\delta}}{\rho-\alpha}\Big)\end{aligned}}{\begin{aligned}&G_{\alpha\delta} + \lambda\Big((\mathscr{M}_\beta - \mathscr{L}_\beta - \mathscr{Y})G_{\alpha\delta} + \int_0^1 d\rho \frac{G_{\alpha\delta}G_{\beta\rho}(1-\beta)}{(1-\delta\rho)(1-\beta\rho)} \\ &\qquad + \int_0^1 \rho\, d\rho \frac{(1-\beta)(1-\alpha\delta)G_{\beta\rho}}{(1-\beta\rho)(1-\delta\rho)} \frac{(G_{\rho\delta}-G_{\alpha\delta})}{(\rho-\alpha)}\Big)\end{aligned}}\,. \tag{7.4}$$

In lowest order we find

$$\begin{aligned} \Gamma_{\alpha\beta\gamma\delta} = \lambda - \lambda^2\Big(&\frac{(1-\gamma)(I_\alpha-\alpha)-(1-\alpha)(I_\gamma-\gamma)}{\alpha-\gamma} \\ &+ \frac{(1-\delta)(I_\beta-\beta)-(1-\beta)(I_\delta-\delta)}{\beta-\delta}\Big) + \mathscr{O}(\lambda^3)\,. \end{aligned} \tag{7.5}$$

Note that $\Gamma_{\alpha\beta\gamma\delta}$ is cyclic in the four indices, and that $\Gamma_{0000} = \lambda + \mathscr{O}(\lambda^3)$.
These integral equations might be the starting point of a nonperturbative construction of a Euclidean quantum field theory on a noncommutative space.

References

[1] S. Minwalla, M. Van Raamsdonk and N. Seiberg, "Noncommutative perturbative dynamics," JHEP **0002** (2000) 020 [arXiv:hep-th/9912072].

[2] H. Grosse and R. Wulkenhaar, "Renormalisation of ϕ^4-theory on noncommutative $\mathbb{R}^4$ in the matrix base," Commun. Math. Phys. **256** (2005) 305 [arXiv:hep-th/0401128].

[3] M. Buric and M. Wohlgenannt, "Geometry of the Grosse-Wulkenhaar Model," JHEP **1003** (2010) 053 [arXiv:0902.3408 [hep-th]].

[4] E. Langmann and R. J. Szabo, "Duality in scalar field theory on noncommutative phase spaces," Phys. Lett. B **533** (2002) 168 [arXiv:hep-th/0202039].

[5] V. Rivasseau, "Non-commutative renormalization" In: *Quantum Spaces (Séminaire Poincaré X)*, eds. B. Duplantier and V. Rivasseau, Birkhäuser Verlag Basel (2007) 19–109 [arXiv:0705.0705 [hep-th]].

[6] H. Grosse and R. Wulkenhaar, "The β-function in duality-covariant noncommutative ϕ^4-theory," Eur. Phys. J. C **35** (2004) 277 [arXiv:hep-th/0402093].

[7] H. Grosse and R. Wulkenhaar, "Renormalisation of ϕ^4-theory on non-commutative $\mathbb{R}^4$ to all orders," Lett. Math. Phys. **71** (2005) 13.

[8] M. Disertori and V. Rivasseau, "Two and three loops beta function of non commutative ϕ^4_4 theory," Eur. Phys. J. C **50** (2007) 661 [arXiv:hep-th/0610224].

[9] M. Disertori, R. Gurau, J. Magnen and V. Rivasseau, "Vanishing of beta function of non commutative ϕ^4_4 theory to all orders," Phys. Lett. B **649** (2007) 95 [arXiv:hep-th/0612251].

[10] V. Rivasseau, "Constructive Matrix Theory," JHEP **0709** (2007) 008 [arXiv:0706.1224 [hep-th]].

[11] H. Grosse and R. Wulkenhaar, "Renormalisation of ϕ^4 theory on noncommutative $\mathbb{R}^2$ in the matrix base," JHEP **0312**, 019 (2003) [arXiv:hep-th/0307017].

[12] H. Grosse and R. Wulkenhaar, "Progress in solving a noncommutative quantum field theory in four dimensions," arXiv:0909.1389 [hep-th].