Gauge field theories in terms of graded differential Lie algebras

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Abstract

We present a mathematical framework of gauge field theories that is based upon a skew-adjoint Lie algebra and a generalized Dirac operator, both acting on a Hilbert space. We review the construction of the standard model and the derivation of its metric structure within this approach.

1 Introduction

This contribution focusses on a formulation of gauge theories that is very similar to the last chapter of Connes' noncommutative geometry [1]. The starting point is Connes' observation that the geometry of a Riemannian spin manifold M is encoded in the associative $*$-algebra $\mathcal{A} = C^\infty(M)$ of smooth functions on M and the Dirac operator $D = \mathrm{i}\partial\!\!\!/$ of the spin connection, both acting on the Hilbert space $h = L^2(\mathcal{S})$ of square integrable sections of the spinor bundle $\mathcal{S}$ over M. This is due to the fact that the operator norm of $[D, f]$ is equal to the Lipschitz norm of $f \in C^\infty(M)$ so that

$$\mathrm{dist}(p,q) = \sup_{f \in C^\infty(M)} \{ |f(p) - f(q)| \;:\; \|[D,f]\| \leq 1 \} \;, \tag{1}$$

for $p, q \in M$. This result can be generalized as follows: Replace $C^\infty(M)$ by an arbitrary $*$-algebra $\mathcal{A}$ acting via a representation π on some Hilbert space h and the Dirac operator by a selfadjoint operator D on h such that $[D, \mathcal{A}] \equiv [D, \pi(\mathcal{A})]$ is bounded. Interpret the points p, q as functionals on $\mathcal{A}$, $p(f) \equiv f(p) \in \mathbb{C}$ for $f \in \mathcal{A}$, and (1) makes the space of functionals on $\mathcal{A}$ to a metric space.

The second important observation is that $[-\mathrm{i}D, f] = \gamma^\mu \partial_\mu(f)$ can be interpreted as a differential 1-form, and 1-forms and functions generate the exterior differential algebra $\Lambda(M)$. Eventually, the exterior differential algebra enters the definition of a physical action. It is therefore interesting to ask what differential calculus is generated by $\mathcal{A}$ and $[-\mathrm{i}D, \mathcal{A}]$ in the general context. The conclusion after all is that the so-called spectral triple $(\mathcal{A}, h, D)$ is a useful set of data to define (differential) geometry on a very general level.

Remarkably, this set of data occurs naturally in gauge field theories. Here, h is the fermionic Hilbert space, D is the sum of the free Dirac operator $\mathrm{i}\partial\!\!\!/$ and the fermionic mass matrix Y (Yukawa operator), and $\mathcal{A}$ has something to do with the gauge group. This dependence is a bit complicated: If $\mathcal{G} = C^\infty(M) \otimes G$ is the gauge group associated to the (matrix) structure group G, and if G acts via a representation $\hat{\pi}_0$ on the fermionic Hilbert space h, then one must search for a matrix algebra $\mathcal{A}_M$ such that G is precisely the group of unitary elements of $\mathcal{A}_M$. If by chance $\hat{\pi}_0$ can be extended to a algebra representation $\hat{\pi}$ of $\mathcal{A}_M$ on h, then the gauge field theory can be described by a spectral triple over the algebra $\mathcal{A} = C^\infty(M) \otimes \mathcal{A}_M$.

An example of a gauge field theory which gives rise to a spectral triple is the standard model. There exists an extraordinary big number of articles where the differential calculus of the standard model is computed (mostly up to second order) and where the resulting physical action is derived. We only mention [2, 3] where real spectral triples as defined by Connes in [4] are used. It turns out however [5] that the standard model is the only realistic physical model that can be formulated in the framework of real spectral triples; without reality there are two more examples. It is a matter of belief whether one is happy about such a result or not.

There do exist numerous noncommutative formulations of other gauge field theories, where the relation between physical setting and spectral data is weaker, of course. The author has proposed [6, 7] a modification of the spectral data themselves to restore its tight connection to a bigger class of gauge field theories. In this contribution we present the basic mathematical techniques in their revised formulation [8]. The essential idea is to replace the associative $*$-algebra by a Lie algebra. This replacement relaxes the obstructions to physical models; and we demonstrate that the mathematical framework can be adapted to this situation.

2 The algebraic setting – L-cycles

Let $\mathfrak{g}$ be a skew-adjoint Lie algebra, which means $a^* = -a$ for all $a \in \mathfrak{g}$. Let h_0, h_1 be Hilbert spaces, where h_1 is dense in h_0. Denoting by $\mathcal{B}(h_0)$ and $\mathcal{B}(h_1)$ the algebras of linear continuous operators on h_0 and h_1, respectively, we define $\mathcal{B} := \mathcal{B}(h_0) \cap \mathcal{B}(h_1)$. The vector space of linear continuous mappings from h_1 to h_0 is denoted by $\mathcal{L}$. Let

π be a representation of $\mathfrak{g}$ in $\mathcal{B}$. Let $D \in \mathcal{L}$ be a generalized Dirac operator with respect to $\pi(\mathfrak{g})$. This means that D has an extension to a selfadjoint operator on h_0, that $[D, \pi(a)] \in \mathcal{L}$ even belongs to $\mathcal{B}$ for any $a \in \mathfrak{g}$ and that the resolvent of D is compact. Finally, let $\Gamma \in \mathcal{B}$ be a grading operator, i.e. Γ^2 is the identity on both h_0 and h_1, $[\Gamma, \pi(a)] = 0$ on both h_0, h_1 for any $a \in \mathfrak{g}$, and $D\Gamma + \Gamma D = 0$ on h_1 extends to 0 on h_0. We call this setting an *L-cycle*, referring to a *Lie*-algebraic version of a *K-cycle*, the former name [1] for spectral triple [4].

The standard example of an L-cycle $(\mathfrak{g}, h_0, h_1, D, \pi, \Gamma)$ is

$$\begin{aligned} \mathfrak{g} &= C^\infty(M) \otimes \mathfrak{a}\,, & h_0 &= L^2(\mathcal{S}) \otimes \mathbb{C}^F\,, \\ h_1 &= W_1^2(\mathcal{S}) \otimes \mathbb{C}^F\,, & D &= \mathrm{i}\partial\!\!\!/ \otimes \mathbf{1}_F + \gamma^5 \otimes Y\,, \\ \pi &= \mathrm{id} \otimes \hat{\pi}\,, & \Gamma &= \gamma^5 \otimes \hat{\Gamma}\,. \end{aligned} \tag{2}$$

Here, $C^\infty(M)$ denotes the algebra of real-valued smooth functions on a compact Riemannian spin manifold M, $\mathfrak{a}$ is a skew-adjoint matrix Lie algebra, $L^2(\mathcal{S})$ denotes the Hilbert space of square integrable sections of the spinor bundle $\mathcal{S}$ over M, $W_1^2(\mathcal{S})$ denotes the Sobolev space of square integrable sections of $\mathcal{S}$ with generalized first derivative, $\mathrm{i}\partial\!\!\!/$ is the Dirac operator of the spin connection, γ^5 is the grading operator on $L^2(\mathcal{S})$ anticommuting with $\mathrm{i}\partial\!\!\!/$, $\hat{\pi}$ is a representation of $\mathfrak{a}$ in $\mathrm{M}_F\mathbb{C}$ and $\hat{\Gamma}$ a grading operator on $\mathrm{M}_F\mathbb{C}$ that commutes with $\hat{\pi}(\mathfrak{a})$ and anticommutes with $Y \in \mathrm{M}_F\mathbb{C}$.

There exists a formula analogous to (1) that defines a distance on L-cycles:

Definition 1 *Let* X *be the space of linear functionals* χ *on* $\mathfrak{g}$ *whose norm equals* 1, *i.e.* $\|\chi\| = \sup_{a\in\mathfrak{g}} (|\chi(a)|/\|\pi(a)\|) = 1$. *The distance* $\mathrm{dist}(\chi_1, \chi_2)$ *between* $\chi_1, \chi_2 \in \mathrm{X}$ *is given by*

$$\mathrm{dist}(\chi_1, \chi_2) := \sup_{a\in\mathfrak{g}}\{\ |\chi_1(a) - \chi_2(a)|\ :\ \ \|\,[D, \pi(a)]\,\| \leq 1\ \}\ . \tag{3}$$

3 The universal graded differential Lie algebra Ω

For $\mathfrak{g}$ being a real Lie algebra we consider the real vector space $\mathfrak{g}^2 = \mathfrak{g} \times \mathfrak{g}$, with the linear operations given by $\lambda_1(a_1, a_2) + \lambda_2(a_3, a_4) = (\lambda_1 a_1 + \lambda_2 a_3, \lambda_1 a_2 + \lambda_2 a_4)$, for $a_i \in \mathfrak{g}$ and $\lambda_i \in \mathbb{R}$. Let T be the tensor algebra of $\mathfrak{g}^2$, equipped with the $\mathbb{N}$–grading structure $\deg((a,0)) = 0$ and $\deg((0,a)) = 1$, and linear extension to higher degrees, $\deg(t_1 \otimes t_2) = \deg(t_1) + \deg(t_2)$, for $t_i \in T$. Defining $T^n = \{t \in T : \deg(t) = n\}$, we have $T = \bigoplus_{n\in\mathbb{N}} T^n$ and $T^k \otimes T^l \subset T^{k+l}$. We regard T as a graded Lie algebra with graded commutator given by $[t^k, t^l] := t^k \otimes t^l - (-1)^{kl}\, t^l \otimes t^k$, for $t^n \in T^n$.

Let $\tilde{\Omega} = \bigoplus_{n\in\mathbb{N}} \tilde{\Omega}^n = \sum [\mathfrak{g}^2, [\ldots[\mathfrak{g}^2, \mathfrak{g}^2]\ldots]]$ be the $\mathbb{N}$-graded Lie subalgebra of T [due to the graded Jacobi identity] given by the set of sums of repeated commutators of elements of $\mathfrak{g}^2$. Let I' be the vector subspace of $\tilde{\Omega}$ of sums of elements of the following type:

$$[(a,0),(b,0)] - ([a,b],0)\ , \qquad [(a,0),(0,b)] + [(0,a),(b,0)] - (0,[a,b])\ , \tag{4}$$

for $a, b \in \mathfrak{g}$. The first part extends the Lie algebra structure of $\mathfrak{g}$ to the first component of $\mathfrak{g}^2$ and the second part plays the rôle of a Leibniz rule, see below. Obviously, $I := I' + [\mathfrak{g}^2, I'] + [\mathfrak{g}^2, [\mathfrak{g}^2, I']] + \ldots$ is an $\mathbb{N}$-graded ideal of $\tilde{\Omega}$ so that $\Omega := \bigoplus_{n\in\mathbb{N}} \Omega^n$, $\Omega^n := \tilde{\Omega}^n/(I \cap \Omega^n)$, is an $\mathbb{N}$-graded Lie algebra.

On T we define recursively a graded differential as an $\mathbb{R}$-linear map $d : T^n \to T^{n+1}$ by

$$\begin{aligned} d(a,0) &= (0,a) \;, & d(0,a) &= 0 \;, \\ d((a,0)\otimes t) &= d(a,0)\otimes t + (a,0)\otimes dt \;, & d((0,a)\otimes t) &= -(0,a)\otimes dt \;, \end{aligned} \tag{5}$$

for $a \in \mathfrak{g}$ and $t \in T$. One easily verifies $d^2 = 0$ on T and the graded Leibniz rule $d(t^k \otimes t^l) = dt^k \otimes t^l + (-1)^k t^k \otimes dt^l$, for $t^n \in T^n$. Thus, d defined by (5) is a graded differential of the tensor algebra T and of the graded Lie algebra T as well, $d[t^k, t^l] = [dt^k, t^l] + (-1)^k [t^k, dt^l]$.

Due to $d\mathfrak{g}^2 \subset \mathfrak{g}^2$ we conclude that d is also a graded differential of the graded Lie subalgebra $\tilde{\Omega} \subset T$. Moreover, one easily checks $dI' \subset I'$, giving $dI \subset I$. Therefore, $(\Omega\,, [\;,\;]\,, d)$ is a graded differential Lie algebra, with the graded differential d given by $d(\varpi + I) := d\varpi + I$, for $\varpi \in \tilde{\Omega}$.

We extend the involution $* : a \mapsto -a$ on $\mathfrak{g}$ to an involution of T by $(a,0)^* = -(a,0)$, $(0,a)^* = -(0,a)$ and $(t_1 \otimes t_2)^* = t_2^* \otimes t_1^*$, giving $[t^k, t^l]^* = -(-1)^{kl}[t^{k*}, t^{l*}]$. Clearly, this involution extends to Ω. The identity $a = -a^*$ yields $\omega^{k*} = -(-1)^{k(k-1)/2}\omega^k$, for any $\omega^k \in \Omega^k$.

The graded differential Lie algebra Ω is universal in the following sense:

Proposition 2 *Let $\Lambda = \bigoplus_{n\in\mathbb{N}} \Lambda^n$ be an $\mathbb{N}$-graded Lie algebra with graded differential* d *such that*

i) $\Lambda^0 = \pi(\mathfrak{g})$ *for a surjective homomorphism π of Lie algebras,*

ii) Λ *is generated by $\pi(\mathfrak{g})$ and $\mathrm{d}\pi(\mathfrak{g})$ as the set of repeated commutators.*

Then there exists a differential ideal $I_\Lambda \subset \Omega$ fulfilling $\Lambda \cong \Omega/I_\Lambda$.

Proof. Define a linear surjective mapping $\mathrm{p} : \Omega \to \Lambda$ by

$$\mathrm{p}((a,0)) = \pi(a) \;, \qquad \mathrm{p}(d\omega) = \mathrm{d}(\mathrm{p}(\omega)) \;, \qquad \mathrm{p}([\omega, \tilde{\omega}]) := [\mathrm{p}(\omega), \mathrm{p}(\tilde{\omega})] \;,$$

for $a \in \mathfrak{g}$ and $\omega, \tilde{\omega} \in \Omega$. Because of $d \ker \mathrm{p} \subset \ker \mathrm{p}$, $I_\Lambda = \ker \mathrm{p}$ is the desired differential ideal of Ω. □

4 The graded differential Lie algebra Ω_D

Using the grading operator Γ we define on $\mathcal{L}$ and $\mathcal{B}$ a $\mathbb{Z}_2$–grading structure, the even subspaces carry the subscript 0 and the odd subspaces the subscript 1. Then, the graded commutator $[\,.\,,\,.\,]_g : \mathcal{L}_i \times \mathcal{B}_j \to \mathcal{L}_{(i+j) \bmod 2}$ is defined by

$$[A_i, B_j]_g := A_i \circ B_j - (-1)^{ij} B_j \circ A_i \equiv -(-1)^{ij}[B_j, A_i]_g \;, \tag{6}$$

where $B_j \in \mathcal{B}_j$ and $A_i \in \mathcal{L}_i$. If $A_i \in \mathcal{B}_i$ then $[\,.\,,\,.\,]_g$ maps h_1 to h_1 and h_0 to h_0.

Using the components π and D of the L-cycle we define a linear map $\pi : \Omega \to \mathcal{B}$ by

$$\begin{aligned} \pi((a,0)) &:= \pi(a)\;, & \pi((0,a)) &:= [-\mathrm{i}D, \pi(a)]_g\;, \\ \pi([\omega^k,\omega^l]) &:= [\pi(\omega^k),\pi(\omega^l)]_g\;, \end{aligned} \tag{7}$$

for $a \in \mathfrak{g}$ and $\omega^n \in \Omega^n$. The selfadjointness of D on h_0 implies that π is involutive, $\pi(\omega^*) = (\pi(\omega))^*$.

Note that $\pi(\Omega)$ is not a graded *differential* Lie algebra. The standard procedure to construct such an object is to define $\mathcal{J} = \ker\pi + d\ker\pi \subset \Omega$. It is easy to show that $\mathcal{J}$ is a graded differential ideal of Ω, providing the graded differential Lie algebra

$$\Omega_D = \textstyle\bigoplus_{n\in\mathbb{N}} \Omega_D^n\;, \qquad \Omega_D^n := \Omega^n/\mathcal{J}^n \cong \pi(\Omega^n)/\pi(\mathcal{J}^n)\;, \tag{8}$$

with $\mathcal{J}^n = \mathcal{J} \cap \Omega^n$. One has $\Omega_D^0 \cong \pi(\Omega^0) \equiv \pi(\mathfrak{g})$ and $\Omega_D^1 \cong \pi(\Omega^1)$. By construction, the differential d on Ω_D is given by $d(\pi(\omega^n) + \pi(\mathcal{J}^n)) := \pi(d\omega^n) + \pi(\mathcal{J}^{n+1})$, for $\omega^n \in \Omega^n$.

It is very useful to consider an extension of the second formula of (7), $\pi(d(a,0)) := [-\mathrm{i}D, \pi((a,0))]_g$, to higher degrees:

$$\pi(d\omega^n) = [-\mathrm{i}D, \pi(\omega^n)]_g + \sigma(\omega^n)\;, \quad \omega^n \in \Omega^n\;. \tag{9}$$

It turns out [7] that $\sigma : \Omega \to \mathcal{L}$ is a linear mapping recursively given by

$$\begin{aligned} \sigma((a,0)) &= 0\;, \qquad\qquad \sigma((0,a)) = [D,[D,\pi(a)]_g]_g\;, \\ \sigma([\omega^k,\omega^l]) &= [\sigma(\omega^k),\pi(\omega^l)]_g + (-1)^k[\pi(\omega^k),\sigma(\omega^l)]_g\;. \end{aligned} \tag{10}$$

Equation (9) has an important consequence: Putting $\omega^n \in \ker\pi$ we get

$$\pi(\mathcal{J}^{n+1}) = \{\sigma(\omega^n) \;:\; \omega^n \in \Omega^n \cap \ker\pi\}\;. \tag{11}$$

The point is that $\sigma(\Omega)$ can be computed from the last equation (10) once $\sigma(\Omega^1)$ is known. Then one can compute $\pi(\mathcal{J})$ and obtains with (9) an explicit formula for the differential on Ω_D.

5 Connections

We define the graded normalizer $N_{\mathcal{L}}(\pi(\Omega))$ of $\pi(\Omega)$ in $\mathcal{L}$, its vector subspace $\mathcal{H}$ compatible with $\pi(\mathcal{J})$ and the graded centralizer $\mathcal{C}$ of $\pi(\Omega)$ in $\mathcal{L}$ by

$$\begin{aligned} N_{\mathcal{L}}^k(\pi(\Omega)) &= \{\eta^k \in \mathcal{L}_{k \bmod 2} \;:\; \eta^k \text{ has} \left\{\begin{matrix}\text{selfadjoint}\\ \text{skew-adjoint}\end{matrix}\right\} \text{extension for } \tfrac{k(k-1)}{2} \left\{\begin{matrix}\text{odd}\\ \text{even}\end{matrix}\right\}, \\ &\qquad [\eta^k,\pi(\omega^n)]_g \in \pi(\Omega^{k+n}) \;\; \forall\omega^n \in \Omega^n\,,\; \forall n \in \mathbb{N}\}\,, \\ \mathcal{H}^k &= \{\eta^k \in N_{\mathcal{L}}^k(\pi(\Omega)) \;:\; [\eta^k,\pi(j^n)] \in \pi(\mathcal{J}^{k+n}) \;\; \forall j^n \in \mathcal{J}^n\}\,, \\ \mathcal{C}^k &= \{c^k \in N_{\mathcal{L}}^k(\pi(\Omega)) \;:\; [c^k,\pi(\omega)]_g = 0 \;\; \forall\omega \in \Omega\}\,. \end{aligned} \tag{12}$$

Here, the linear continuous operator $[\eta^k, \pi(\omega^n)]_g : h_1 \to h_0$ must have its image even in the subspace $h_1 \subset h_0$ and must have an extension to a linear continuous operator on h_0. For each degree n we have the following system of inclusions:

$$\begin{array}{ccccccc} \mathcal{L} & \supset & \mathcal{H}^n & \supset & \pi(\Omega^n) & \supset & \pi(\mathcal{J}^n) \\ & & \cup & & \cap & & \\ & & \mathcal{C}^n & & \mathcal{B} & \subset & \mathcal{L} \end{array} \tag{13}$$

The graded Jacobi identity and Leibniz rule define the structure of a graded differential Lie algebra on $\hat{\mathcal{H}} = \bigoplus_{n\in\mathbb{N}} \hat{\mathcal{H}}^n$, with $\hat{\mathcal{H}}^n = \mathcal{H}^n/(\mathcal{C}^n + \pi(\mathcal{J}^n))$:

$$\begin{aligned} &[[\eta^k+\mathcal{C}^k+\pi(\mathcal{J}^k)\,,\,\eta^l+\mathcal{C}^l+\pi(\mathcal{J}^l)]_g\,,\,\pi(\omega^n)+\pi(\mathcal{J}^n)]_g \\ &\qquad := [\eta^k,[\eta^l,\pi(\omega^n)]_g]_g - (-1)^{kl}[\eta^l,[\eta^k,\pi(\omega^n)]_g]_g + \pi(\mathcal{J}^{k+l+n})\,, \\ &[d(\eta^k+\mathcal{C}^k+\pi(\mathcal{J}^k))\,,\,\pi(\omega^n)+\pi(\mathcal{J}^n)]_g \\ &\qquad := \pi\circ d\circ\pi^{-1}([\eta^k,\pi(\omega^n)]_g) - (-1)^k[\eta^k,\pi(d\omega^n)]_g + \pi(\mathcal{J}^{k+n+1})\,, \end{aligned} \tag{14}$$

for $\eta^n \in \mathcal{H}^n$ and $\omega^n \in \Omega^n$.

The lesson is that $\pi(\Omega)$ and its ideal $\pi(\mathcal{J})$ give rise not only to the graded differential Lie algebra Ω_D but also to $\hat{\mathcal{H}}$, both being natural. It turns out that it is the differential Lie algebra $\hat{\mathcal{H}}$ which occurs in our connection theory:

Definition 3 *A connection on an L-cycle is a pair* (∇, ∇_h), *where*

i) $\nabla_h \in \mathcal{L}_1$ *with skew-adjoint extension,*
ii) $\nabla : \Omega_D^n \to \Omega_D^{n+1}$ *is linear,*
iii) $\nabla(\pi(\omega^n)+\pi(\mathcal{J}^n)) = [\nabla_h, \pi(\omega^n)]_g + \sigma(\omega^n) + \pi(\mathcal{J}^{n+1})\,, \quad \omega^n \in \Omega^n.$

The operator $\nabla^2 : \Omega_D^n \to \Omega_D^{n+2}$ *is called the curvature of the connection.*

Proposition 4 *Any connection has the form* $(d+[\rho+\mathcal{C}^1\,,\,.\,]_g\,,\,-\mathrm{i}D+\rho)$, *for* $\rho \in \mathcal{H}^1$. *Its curvature is* $\nabla^2 = [\theta,\,.\,]$, *with* $\theta = d\hat{\rho} + \frac{1}{2}[\hat{\rho},\hat{\rho}]_g \in \hat{\mathcal{H}}^2$, *where* $\hat{\rho} = \rho + \mathcal{C}^1 \in \hat{\mathcal{H}}^1$.

Proof. There is a canonical connection given by $(\nabla = d, \nabla_h = -\mathrm{i}D)$. If $(\nabla^{(1)}, \nabla_h^{(1)})$ and $(\nabla^{(2)}, \nabla_h^{(2)})$ are two connections, we get from iii)

$$(\nabla^{(1)}-\nabla^{(2)})(\pi(\omega^n)+\pi(\mathcal{J}^n)) = [\nabla_h^{(1)}-\nabla_h^{(2)}, \pi(\omega^n)]_g + \pi(\mathcal{J}^{n+1}).$$

This means that $\rho := \nabla_h^{(1)}-\nabla_h^{(2)} \in \mathcal{H}^1$ is a concrete representative and $\nabla^{(1)}-\nabla^{(2)} = [\hat{\rho},\,.\,]_g$, where $\hat{\rho} = \rho + \mathcal{C}^1 \in \hat{\mathcal{H}}^1$. The formula for θ is a direct consequence of (14). □

6 Gauge transformations and physical action

The exponential mapping defines a unitary group

$$\begin{aligned} \mathcal{U} := \{\textstyle\prod_{\alpha=1}^N \exp(v_\alpha) : \quad & \exp(v_\alpha) := 1_{\mathcal{B}} + \textstyle\sum_{k=1}^\infty \frac{1}{k!}(v_\alpha)^k\,, \\ & v_\alpha \in \mathcal{H}^0 \cap \mathcal{B}\,,\ dv_\alpha - [-\mathrm{i}D, v_\alpha] \in \mathcal{C}^1\ \}\,. \end{aligned} \tag{15}$$

Due to $\exp(v)A\exp(-v) = A + \sum_{k=1}^{\infty}\frac{1}{k!}\underbrace{[v,[v,\dots,[v}_{k},A]\dots]]$ for $A \in \mathcal{L}$, we have a natural degree-preserving representation Ad of $\mathcal{U}$ on $\mathcal{H}$, $\mathrm{Ad}_u(\eta^n) = u\eta^n u^* \in \mathcal{H}^n$, for $\eta^n \in \mathcal{H}^n$ and $u \in \mathcal{U}$.

Definition 5 *The gauge group on an L-cycle is the group $\mathcal{U}$ defined in* (15)*. Gauge transformations of the connection are given by*

$$(\nabla, \nabla_h) \mapsto (\nabla', \nabla'_h) := (\mathrm{Ad}_u \nabla \mathrm{Ad}_{u^*}\,,\, u\nabla_h u^*)\;, \quad u \in \mathcal{U}.$$

Note that the consistency relation iii) in Definition 3 reduces on the infinitesimal level to the condition $dv_\alpha - [-\mathrm{i}D, v_\alpha] \in \mathcal{C}^1$ in (15). The gauge transformation of the curvature form reads $\theta \mapsto \theta' = \mathrm{Ad}_u\theta$.

Definition 6 *Let E_n be the eigenvalues (arranged in decreasing order) of the compact operator $|D|^{-1} = (DD^*)^{-1/2}$ on h_0. The L-cycle is said to be of dimension d^+ if these eigenvalues satisfy $\sum_{n=1}^{N} E_n = O(\sum_{n=1}^{N} n^{-1/\mathrm{d}})$.*

Let $\theta_0 : h_1 \to h_0$ be any representative of the curvature form $\theta \in \hat{\mathcal{H}}^2$. The bosonic action S_B and the fermionic action S_F of the connection (∇, ∇_h) are given by

$$\begin{aligned} S_B(\nabla) &:= \min_{\mathrm{j}^2 \in \mathcal{C}^2 + \pi(\mathcal{J}^2)} \mathrm{Tr}_\omega((\theta_0 + \mathrm{j}^2)(\theta_0 + \mathrm{j}^2)^* |D|^{-\mathrm{d}})\;, \\ S_F(\psi, \nabla_h) &:= \langle \psi, \mathrm{i}\nabla_h \psi\rangle_{h_0}\;, \quad \psi \in h_1\;, \end{aligned} \tag{16}$$

where Tr_ω is the Dixmier trace [1] *and $\langle\;,\;\rangle_{h_0}$ the scalar product on h_0.*

The action (16) is invariant under gauge transformations

$$(\nabla, \nabla_h) \mapsto (\mathrm{Ad}_u \nabla \mathrm{Ad}_{u^*}, u\nabla_h u^*)\;, \quad \psi \mapsto u\psi\;, \quad u \in \mathcal{U}\;. \tag{17}$$

7 Remarks on the standard example

Recall (7) that the general form of an element $\tau^1 \in \pi(\Omega^1)$ is

$$\tau^1 = \sum_{\alpha, z \geq 0} [\pi(a_\alpha^z), [\dots[\pi(a_\alpha^1), [-\mathrm{i}D, \pi(a_\alpha^0)]]\dots]]\;, \qquad a_\alpha^i \in \mathfrak{g}\;. \tag{18}$$

For $a_\alpha^i = f_\alpha^i \otimes \hat{a}_\alpha^i \in C^\infty(M) \otimes \mathfrak{a}$ we get with (2)

$$\begin{aligned} \tau^1 = \textstyle\sum_{\alpha, z\geq 0} \Big(& f_\alpha^z \cdots f_\alpha^1 \not{\partial}(f_\alpha^0) \otimes \hat{\pi}([\hat{a}_\alpha^z, [\dots[\hat{a}_\alpha^1, \hat{a}_\alpha^0]\dots]]) \\ & + f_\alpha^z \cdots f_\alpha^1 f_\alpha^0 \gamma^5 \otimes [\hat{\pi}(\hat{a}_\alpha^z), [\dots[\hat{\pi}(\hat{a}_\alpha^1), [-\mathrm{i}Y, \hat{\pi}(\hat{a}_\alpha^0)]]\dots]]\Big)\;. \end{aligned} \tag{19}$$

The first line belongs to $\Lambda^1 \otimes \hat{\pi}(\mathfrak{a})$, because the gamma matrices occurring in $\not{\partial}$ provide a 1-form basis. In physical terminology, these Lie algebra-valued 1-forms are Yang–Mills fields acting via the representation $\mathrm{id} \otimes \hat{\pi}$ on the fermions. In the second line of (19) we split Y into generators of irreducible representations of $\mathfrak{a}$. Obviously, these irreducible representations are spanned after taking the commutators with $\hat{\pi}(\hat{a}^i_\alpha)$. Thus, the second line of (19) contains sums of function-valued [γ^5 can be ignored for this argumentation] representations of the matrix Lie algebra, which are physically interpreted as Higgs fields. In other words, the prototype τ^1 of a connection form (=gauge potential) describes representations of both Yang–Mills and Higgs fields on the fermionic Hilbert space.

This is a more satisfactory picture than the usual noncommutative geometric construction of Yang–Mills–Higgs models [2, 3]. Namely, descending from Connes' noncommutative geometry [1, 4] one obtains gauge potentials composed of *matrix*-valued 1-forms and *matrix*-valued 0-forms, but for general physical models we need these forms to be *representation*-valued. As we have shown, such objects arise naturally from our framework.

However, this is not the full story, as the gauge potential ρ belongs to $\mathcal{H}^1 \supset \pi(\Omega^1)$. Fortunately, it turns out [7] that after imposing a locality condition for the connection (which is equivalent to saying that ρ commutes with functions), possible additional $\mathcal{H}^1$-degrees of freedom are either of Yang–Mills type or Higgs type. In particular, this allows for u(1)-gauge fields, which are not covered by (19).

The formula for the curvature form θ in proposition 4 can be evaluated and the result is

$$\theta = \boldsymbol{d}\rho + \rho^2 - \mathrm{i}\{\gamma^5 Y, \rho\} + \hat{\sigma}_{\mathfrak{g}}(\rho)\gamma^5 + \mathcal{C}^2 + \pi(\mathcal{J}^2) \;, \tag{20}$$

where $\boldsymbol{d}$ is the exterior differential. The linear map $\hat{\sigma}_{\mathfrak{g}}$ is for $\rho = \tau^1 \in \pi(\Omega^1)$ as in (18) given by

$$\hat{\sigma}_{\mathfrak{g}}(\tau^1) = \sum_{\alpha, z \geq 0} [\pi(a^z_\alpha), [\dots [\pi(a^1_\alpha), [Y^2, \pi(a^0_\alpha)]] \dots]] \;.$$

It can be extended to $\mathcal{H}^1$ if $\mathcal{H}^1 \neq \pi(\Omega^1)$. The dimension d of the L-cycle coincides with the dimension of the manifold M and the Dixmier trace reduces to the usual matrix trace and integration over M:

$$\mathrm{Tr}_\omega(B\,|D|^{-\mathrm{d}}) = \int_M dx\; \mathrm{tr}(B) \;, \qquad B \in \mathcal{B}(L^2(\mathcal{S})) \otimes M_F\mathbb{C} \;.$$

8 Application to the standard model

In this section we review the application of our framework to the standard model. The L-cycle of the standard model and the derivation of its action was first presented in [9]. We will recall here the initial data and the tree-level predictions. At the end, we review recent results [10] on the metric structure of the standard model derived from definition 1. The contents of this section was not presented at the workshop.

The complete L-cycle splits according to (2) into spacetime and matrix part so that we can restrict our presentation to matrices. The matrix Hilbert space is $\mathbb{C}^{48}$ if we include right neutrinos. The matrix Lie algebra is

$$\begin{aligned}\mathfrak{a} &= \mathrm{su}(3)\oplus\mathrm{su}(2)\oplus\mathrm{u}(1)\\ &\ni \{\boldsymbol{g},\boldsymbol{a},e\} \equiv \{\mathrm{i}(\textstyle\sum_{j=1}^{8} g^j\lambda^j)\,,\ \mathrm{i}(a\sigma^3{+}b\sigma^1{+}c\sigma^2)\,,\ e\}\ ,\end{aligned} \tag{21}$$

where λ^j are the Gell-Mann matrices, σ^k the Pauli matrices and $g^j, a, b, c, e \in \mathbb{R}$. This Lie algebra acts on $\mathbb{C}^{48}$ via the representation

$$\begin{aligned}
\hat{\pi}(\boldsymbol{g},\boldsymbol{a},e) &= \begin{pmatrix} \hat{\pi}_\ell(\boldsymbol{g},\boldsymbol{a},e) & 0 \\ 0 & \hat{\pi}_q(\boldsymbol{g},\boldsymbol{a},e)\end{pmatrix}\ ,\\
\hat{\pi}_\ell(\boldsymbol{g},\boldsymbol{a},e) &= \begin{pmatrix} \mathrm{i}(a-e)\otimes 1_3 & \mathrm{i}(b-\mathrm{i}c)\otimes 1_3 & 0 & 0\\ \mathrm{i}(b+\mathrm{i}c)\otimes 1_3 & \mathrm{i}(-a-e)\otimes 1_3 & 0 & 0\\ 0 & 0 & 0_3 & 0\\ 0 & 0 & 0 & -2\mathrm{i}e\otimes 1_3\end{pmatrix},\\
\hat{\pi}_q(\boldsymbol{g},\boldsymbol{a},e) &= \begin{pmatrix} (\mathrm{i}(a{+}\frac{1}{3}e)1_3{+}\boldsymbol{g})\otimes 1_3 & \mathrm{i}(b{-}\mathrm{i}c)1_3\otimes 1_3 & 0 & 0\\ \mathrm{i}(b{+}\mathrm{i}c)1_3\otimes 1_3 & (\mathrm{i}(-a{+}\frac{1}{3}e)1_3{+}\boldsymbol{g})\otimes 1_3 & 0 & 0\\ 0 & 0 & (\frac{4}{3}\mathrm{i}e1_3{+}\boldsymbol{g})\otimes 1_3 & 0\\ 0 & 0 & 0 & (-\frac{2}{3}\mathrm{i}e1_3{+}\boldsymbol{g})\otimes 1_3\end{pmatrix}.
\end{aligned} \tag{22}$$

The generalized Dirac operator is the Yukawa operator

$$\begin{aligned}
Y &= \begin{pmatrix} Y_\ell & 0\\ 0 & Y_q\end{pmatrix}\ ,\\
Y_\ell &= \begin{pmatrix} 0 & 0 & \mathcal{M}_\nu & 0\\ 0 & 0 & 0 & \mathcal{M}_e\\ \mathcal{M}_\nu^* & 0 & 0 & 0\\ 0 & \mathcal{M}_e^* & 0 & 0\end{pmatrix},\quad Y_q = \begin{pmatrix} 0 & 0 & 1_3\otimes\mathcal{M}_u & 0\\ 0 & 0 & 0 & 1_3\otimes\mathcal{M}_d\\ 1_3\otimes\mathcal{M}_u^* & 0 & 0 & 0\\ 0 & 1_3\otimes\mathcal{M}_d^* & 0 & 0\end{pmatrix},
\end{aligned} \tag{23}$$

where $\mathcal{M}_{e,\nu,u,d}$ are 3×3-mass matrices of the fermions.

The next step is to write down the general form of an element $\tau^1 \in \pi(\Omega^1)$, which splits according to (19) into Yang–Mills part and Higgs part. Then one computes the anti-commutator $\{\tau^1,\tau^1\} \in \pi(\Omega^2)$ and the ideal $\pi(\mathcal{J}^2)$. The analysis of (12) shows that the relevant degrees of $\hat{\mathcal{H}}$ differ not very much from $\pi(\Omega)$, in particular, there are no additional Yang–Mills or Higgs fields. This simplifies the computation of the curvature form (20). Finally, one evaluates the action (16), which coincides with the usual standard model action, subject to the following tree-level predictions:

$$m_W = \tfrac{1}{2}m_t\ ,\qquad m_H = \tfrac{3}{2}m_t\ ,\qquad m_Z = m_W/\cos\theta_W\ ,\qquad \sin^2\theta_W = \tfrac{3}{8}\ . \tag{24}$$

Here, m_W, m_Z, m_H are the masses of the $W^\pm$, Z and Higgs bosons, m_t is the mass of the top quark and θ_W the Weinberg angle. Details are given in [9].

It is very enlightening to compute the metric structure (in the sense of definition 1) of the standard model [10]. We consider functionals $\chi_{\boldsymbol{\epsilon},\boldsymbol{\alpha},\delta;p}$ on $\mathfrak{g} = C^\infty(M)\otimes\mathfrak{a}$ of the form

$$\chi_{\boldsymbol{\epsilon},\boldsymbol{\alpha},\delta;p}(\boldsymbol{g},\boldsymbol{a},e) := \alpha a(p) + \beta b(p) + \gamma c(p) + (2\delta{-}\alpha)e(p) + \textstyle\sum_{j=1}^{8}\epsilon^j g^j(p)\,, \tag{25}$$

where $\alpha,\beta,\gamma,\delta,\epsilon^j \in \mathbb{R}$ and $p \in M$. Compared with (21) we have changed the notation that e is now a function on M and $e(p)$ its value at the point $p \in M$, and so on. The discussion of the ϵ^j can be avoided by defining $\mathrm{extr}_p \sum_{j=1}^8 \epsilon^j g^j(p) =: \epsilon\|\boldsymbol{g}\|$. The condition $\|\chi_{\boldsymbol{\epsilon},\boldsymbol{\alpha},\delta;p}\| = 1$ of definition 1 has a very interesting solution:

$$
\begin{aligned}
&|\epsilon| \leq 1 \;, \\
&\delta = \tfrac{1}{6}\epsilon + R\cos\phi \;, && 0 \leq \phi \leq \pi \;, \quad R = 1 - \tfrac{1}{2}|\epsilon| \;, \\
&\alpha = R(\cos\phi + \cos\phi') \;, && 0 \leq \phi' \leq \pi \;, \\
\beta^2+\gamma^2 = &\begin{cases} 4R^2(1+\cos\phi)^2 - 4R\alpha(1+\cos\phi) \\ 1-\alpha^2 \\ 4R^2(1-\cos\phi)^2 + 4R\alpha(1-\cos\phi) \end{cases} && \text{for} \;\; \begin{matrix} \cos\phi' - \cos\phi \geq \frac{2-2|\epsilon|}{2-|\epsilon|} \\ |\cos\phi - \cos\phi'| \leq \frac{2-2|\epsilon|}{2-|\epsilon|} \\ \cos\phi - \cos\phi' \geq \frac{2-2|\epsilon|}{2-|\epsilon|} \end{matrix}
\end{aligned} \tag{26}
$$

This means that for fixed $\{\boldsymbol{\epsilon},\delta\}$ the remaining parameters $\boldsymbol{\alpha}$ form a two-dimensional object which can be described as the surface of a unit ball whose polar regions are rotary-grinded to paraboloids. We call such an object a *gyro*.

In order to discuss the metric properties of the space of functionals $\chi_{\boldsymbol{\epsilon},\boldsymbol{\alpha},\delta;p}$ it is convenient to restrict oneself first to constant matrices $\{\boldsymbol{g},\boldsymbol{a},e\} \in \mathfrak{a}$. Then, the condition

$$1 \geq \|[D,\pi(\boldsymbol{g},\boldsymbol{a},e)]\| = \|[Y,\pi(\boldsymbol{g},\boldsymbol{a},e)]\| = \sqrt{(a-e)^2+b^2+c^2}\, m_t \tag{27}$$

(where m_t is the mass of the top quark) implies $\mathrm{dist}(\chi_{\boldsymbol{\epsilon},\boldsymbol{\alpha},\delta;p},\chi_{\boldsymbol{\epsilon}',\boldsymbol{\alpha}',\delta';q}) = \infty$ if $\delta \neq \delta'$ or $\epsilon^j \neq \epsilon^{j\prime}$ for at least one $j = 1,\dots,8$. This means that the geometry of the standard model is a nine-parametric family (the parameters are $\{\delta,\epsilon^j\}$) of infinitely distant worlds. It is therefore only interesting to compute the distance of functionals with fixed $\{\delta,\epsilon^j\}$ (i.e. within one world). For fixed point $p \in M$ the second consequence of (27) is

$$\mathrm{dist}(\chi_{\boldsymbol{\epsilon},\boldsymbol{\alpha},\delta;p},\chi_{\boldsymbol{\epsilon},\boldsymbol{\alpha}',\delta;p}) = \mathrm{dist}(\boldsymbol{\alpha},\boldsymbol{\alpha}')/m_t = \sqrt{(\alpha-\alpha')^2+(\beta-\beta')^2+(\gamma-\gamma')^2}/m_t \;.$$

This means that the distance between points on a gyro is equal to the Euclidean length (in units $1/m_t$) of the string through the interior connecting the points.

It remains to compute the distance between functionals at different points p,q. The exact result would involve the diagonalization of 4×4-matrices because of the complicated interplay between exterior differential $[\mathrm{i}\partial\!\!\!/ ,\,.\,]$ and matrix derivation $[\gamma^5 Y,\,.\,]$ in $[D,\,.\,] \leq 1$. The best one can do is to give a lower bound by considering $\mathfrak{a}$-valued functions with $a = e$ and $b = c = 0$. This removes $[\gamma^5 Y,\,.\,]$ and leads to

$$\mathrm{dist}(\chi_{\boldsymbol{\epsilon},\boldsymbol{\alpha},\delta;p},\chi_{\boldsymbol{\epsilon},\boldsymbol{\alpha}',\delta;q}) \geq \Delta\,\mathrm{dist}(p,q) \;, \qquad \Delta = \max(|\epsilon|, |\delta - \tfrac{1}{6}\epsilon| + \tfrac{1}{2}|\epsilon|) \;.$$

If we now rise $|a-e|,|b|,|c|$, then the distance will also grow at first but very soon this growth is compensated by the necessity to decrease $[\mathrm{i}\partial\!\!\!/ ,\,.\,]$ at expense of the growth of $[\gamma^5 Y,\,.\,]$. A simple estimating yields the final result

$$
\begin{aligned}
&\max\{\Delta\,\mathrm{dist}(p,q)\,,\,\mathrm{dist}(\boldsymbol{\alpha},\boldsymbol{\alpha}')/m_t\} \\
&\qquad\qquad \leq \mathrm{dist}(\chi_{\boldsymbol{\epsilon},\boldsymbol{\alpha},\delta;p},\chi_{\boldsymbol{\epsilon},\boldsymbol{\alpha}',\delta;q}) \leq \Delta\,\mathrm{dist}(p,q) + 2/m_t \;.
\end{aligned} \tag{28}
$$

Note that $1/m_t$ is of the order 10^{-16} cm so that for macroscopic geodesic distances the geometry of the standard model is in accurate agreement with Δ times the Riemannian geometry of the underlying manifold. At scales of the order $1/m_t$ however, spacetime should reveal a completely different structure. That what macroscopically is a point becomes an extended object – a gyro.

There is a nine-parametric family of infinitely distant worlds whose points (on macroscopic scales) are gyros (on scales $1/m_t$). The scale factor Δ is constant on each world and cannot be detected. Thus, there is no problem in saying that the true geometry is $(1/\Delta)$ times the measured geometry. This means that any of the allowed values for $\{\epsilon, \delta\}$ according to (26) could be realized in our world.

In other words, we recover the old Kaluza–Klein idea of additional spacetime dimensions, which are compactified to very small size so that they are not apparent in every day's life. The essential progress of our method lies in its effectiveness and the fact that it implements chiral fermions from the very beginning – an obstacle for traditional Kaluza–Klein theories. For the geometric structure we suggest the following physical interpretation:

The standard model contains three massive Yang-Mills fields which generate the internal space $\mathbb{R}^3$ in a first step. The norm=1 condition selects a certain hypersurface in $\mathbb{R}^3$ – our gyro. The nine-dimensional disconnectedness reflects the nine massless Yang–Mills fields (photon and gluons) of the standard model. Again, the norm=1 condition selects a compact region of $\mathbb{R}^9$ as shown implicitely in (26).

On each world, the size and shape of the gyros are fixed so that there is no geometry other than dimension related to Yang–Mills fields. The geometry of the gyros is rather related to the Higgs field. Following the Chamseddine–Connes approach [11] we replace the Dirac–Yukawa operator by the Dirac–Yukawa–Higgs operator. This means to replace all fermion masses m_i by ϕm_i, where ϕ is the Higgs field whose vacuum expectation value $\langle\phi\rangle_0$ equals 1. Now the distance scale on the gyro becomes $1/(\phi m_t)$ instead of $1/m_t$, and is therefore subject to change if the Higgs field varies.

We handle the gyros on the same footing as Riemannian spaces M. The distance between points of M is not the Euclidean distance but obtained on infinitesimal level by taking the metric tensor $g_{\mu\nu}$ into consideration. Just as the Higgs field on the rigid gyro, the metric tensor determines the scale on the rigid coordinate space.

Both metric tensor and Higgs field have non-vanishing vacuum expectation value: $\langle g_{\mu\nu}\rangle_0 = \eta_{\mu\nu}$ (Minkowski tensor) and $\langle\phi\rangle_0 = 1$. The diameter of the gyro is determined by $1/m_t$ or rather by the inverse mass of the Higgs field, see (24). Our analogy then implies that the diameter of the universe coordinate space should be of the order of the inverse mass of the graviton, and therefore equal to infinity. This explains why four coordinates are expanded whereas some internal coordinates remain compactified.

In conclusion, L-cycles turned out to be a very useful tool to construct gauge field theories and to investigate their metric structure.

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