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Graded Lie Algebras with Derivation and Model Building*

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Abstract

Given an algebra, a finite projective right module and a differential algebra over this algebra, a graded Lie algebra with derivation is constructed. It is shown that the algebraic structure used in the Mainz-Marseille approach for the derivation of the standard model can be obtained from this general construction as a special case. Thus, a rigorous mathematical link between Connes' noncommutative geometry and the Mainz-Marseille approach is established.

1 Introduction

Among recent approaches to describe the fundamental interactions within one unified theory the ideas of noncommutative geometry play an important role. In particular, the approach of Connes, cf. [2] and [3], has been the starting point for numerous attempts of this type. The main achievements are, perhaps, the identification of the Higgs field as a part of a unified gauge field and the possibility to include chiral fermions in a natural way, the latter being for instance problematic in theories of Kaluza-Klein type. There is a canonical way to introduce fermions within the K-cycle concept. In some versions one gets predictions for the Weinberg angle and for the masses of physical particles on tree level, see e.g. [4]. In particular, one can get predictions for the mass of the (still not found) Higgs particle. Such predictions should be compared with recent estimates upon renormalization group arguments within the standard model and the (by now) quite exact knowledge of the top quark mass, see e.g. [5]. For the standard model with one Higgs doublet the author of [5] predicts $130\,\mathrm{GeV} < m_{Higgs} < 200\,\mathrm{GeV}$.

An – at first sight – from the mathematical point of view completely different approach is the Mainz-Marseille approach [6]. Here one starts from a certain $\mathbb{Z}_2$-graded Lie algebra with derivation and postulates unified gauge fields and

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their transformation properties rather ad hoc. Also fermions are introduced – comparing with Connes' theory – in a completely different way, namely using the representations theory of graded Lie algebras. There are differences between these two approaches concerning their phenomenological output, for a discussion of this point see [7]. The main difference seems to be the fact that in the Connes-Lott approach one gets predictions of the boson masses in terms of the fermionic mass parameters, whereas in the Mainz-Marseille model there are no relations of this type at all.

The aim of this contribution is to show that, nonetheless, there is a deep structural link between the two above-mentioned theories. The main idea is to use a finite projective module (a notion which was avoided by the Mainz-Marseille group) and a differential algebra over the (underlying) algebra to construct a graded Lie algebra with derivation, which may be mapped by a partial homomorphism onto the graded Lie algebra of the Mainz-Marseille approach. Transporting geometrical objects like connections and curvatures of Connes' theory via this mapping we arrive at the objects postulated in the Mainz-Marseille approach, which leads to a deeper geometrical understanding of the latter. Here we will only sketch the main mathematical ideas and refer to our paper [9] for all details. Moreover, we only mention, that there is a nice physical application of our method: Avoiding the "projection" to the Mainz-Marseille algebra, but nevertheless using the ideas of [6], it is possible to derive the standard model from the simplest two-point K-cycle originally used in [3] to derive the electroweak theory, see the last section of [9] and [10] for further details.

2 Basic notions of noncommutative geomtry

In this section we recall some notions of noncommutative geometry necessary for our construction. A central notion is that of an even K-cycle. This is a tuple (A, h, π, D, Γ), where A is a unital $*$-algebra, h is a Hilbert space, π is a $*$-representation of A in $B(h)$, D is an (in general) unbounded selfadjoint operator on h with the following properties: $[D, \pi(a)] \in B(h)$ for $a \in A$, $(1 + D^2)^{-1}$ is compact. Γ is a grading operator, i.e. $\Gamma = \Gamma^* \in B(h)$, $\Gamma^2 = id_h$, and fulfills $[\Gamma, D]_+ = 0 = [\Gamma, \pi(a)]$.

Any K-cycle gives rise to a canonically related differential algebra Ω_D over A. One starts with the universal differential algebra ΩA (universal differential envelope) defined as follows: $\Omega A = \bigoplus_{k=0}^{\infty} \Omega^k A$, where $\Omega^0 A = A$, $\Omega^1 A = \ker(m : A \otimes A \longrightarrow A)$, $\Omega^k A = \Omega^1 A \otimes_A \cdots \otimes_A \Omega^1 A$. The differential $d : A \longrightarrow \Omega^1 A$ is given by $da = 1 \otimes a - a \otimes 1$, $d : \Omega^1 A \longrightarrow \Omega^2 A$ by $d(\sum_i a_i \otimes db_i) = \sum_i da_i \otimes db_i$, and the differential on higher forms is given inductively by $d(\omega^1 \otimes_A \omega^{k-1}) = d\omega^1 \otimes_A \omega^{k-1} - \omega^1 \otimes_A d\omega^{k-1}$. It follows that $\omega^k \in \Omega^k A$ can be written as $\omega^k = \sum_i a_i^0 da_i^1 \cdots da_i^k$, omitting the $\otimes_A$ between the differentials. In this representation, the differential is given by $d\omega^k = \sum_i da_i^0 da_i^1 \cdots da_i^k$. ΩA becomes a $*$-algebra with $(da)^* = d(a^*)$.

The $*$-representation $\pi : A \longrightarrow B(h)$ can be extended to a $*$-representation $\pi : \Omega A \longrightarrow B(h)$ of algebras by

$$\pi(a^0 da^1 \cdots da^k) = (-\mathrm{i})^k \pi(a^0)[D, \pi(a^1)] \cdots [D, \pi(a^k)] \, .$$

$J_0 = \ker \pi \subset \Omega A$ is an ideal, and $J = J_0 + dJ_0$ is a differential ideal of ΩA. Therefore, $\Omega_D A = \bigoplus_{k=0}^{\infty} \Omega_D^k A$, where $\Omega_D^k A = \Omega^k A / J^k$, $J^k = J \cap \Omega^k A$, is again a differential algebra over A (assuming $\pi|_A$ faithful gives $\Omega_D^0 A = A$). Since $(\ker \pi)^k \subset J^k$, there is a vector space isomorphism

$$\Omega_D^k A \cong \pi(\Omega^k A)/\pi(J^k) = \pi(\Omega^k A)/\pi(dJ_0^{k-1}) \, ,$$

in particular (because of $J_0^0 = \ker(\pi|_A) = 0$)

$$\Omega_D^1 A \cong \pi(\Omega^1 A) = \{ \sum_i \pi(a_i)[D, \pi(b_i)] \mid \ a_i, b_i \in A \} \, .$$

Examples:

(i) Consider the Dirac K-cycle $(C^\infty(X), L^2(X,S), D^{cl}, \gamma^{N+1})$, where X is an $N = 2n$-dimensional compact Riemannian spin manifold, $L^2(X,S)$ the Hilbert space of square integrable sections of the spinor bundle, D^{cl} is the classical Dirac operator, and γ^{N+1} is the product of N orthonormal sections of the linear part of the clifford bundle. Then one has

Theorem 1 $\Omega_{D^{cl}} C^\infty(X) \cong \Lambda(X)$.

Here, $\Lambda(X)$ is the classical algebra of differential forms over X.

(ii) Consider the K-cycle $(\mathbb{C}^2, \mathbb{C}^n \oplus \mathbb{C}^n, \mathcal{M}, \tilde{\Gamma})$, where

$$\mathcal{M} = \begin{pmatrix} 0 & M \\ M^* & 0 \end{pmatrix}, \qquad \tilde{\Gamma} = \begin{pmatrix} 1_{n\times n} & 0 \\ 0 & 1_{n\times n} \end{pmatrix}, \quad M \in M_n\mathbb{C} \, ,$$

$\mathbb{C}$ acts by multiplication in each component of $\mathbb{C}^n \oplus \mathbb{C}^n$ (faithfully). One can show $\pi(dJ_0^k) = 0$ in this case, therefore,

$$\Omega_D^k \mathbb{C}^2 = \pi(\Omega^k \mathbb{C}^2) = \begin{cases} \begin{pmatrix} \mathbb{C}(MM^*)^{k/2} & 0 \\ 0 & \mathbb{C}(M^*M)^{k/2} \end{pmatrix}, & k \text{ even} \\ \begin{pmatrix} 0 & \mathbb{C}(MM^*)^{(k-1)/2}M \\ \mathbb{C}(M^*M)^{(k-1)/2}M^* & 0 \end{pmatrix}, & k \text{ odd} \end{cases}$$

We denote $M_1^t = (MM^*)^t$, $M_2^t = M_1^t M$, $M_4^t = (M^*M)^t$, $M_3^t = M^* M_4^t$ and notice that there is a positive integer m such that in each series $(M_q^t)_{t=0,1,\ldots}$ $(q = 1,2,3,4)$ just the first m terms are linearly independent in $M_n\mathbb{C}$. Therefore,

$$\Omega_D^l \mathbb{C}^2 \subset \sum_{k=0}^{2m+1} \Omega_D^k \mathbb{C}^2 \qquad \text{for } l > 2m+1,$$

where on the right hand side there stands an inner direct sum.

(iii) Consider the K-cycle consisting of

$$A = C^\infty(X) \otimes \mathbb{C}^2,$$
$$h = L^2(X,S) \otimes (\mathbb{C}^n \oplus \mathbb{C}^n) = L^2(X,S) \otimes \mathbb{C}^n \otimes \mathbb{C}^2,$$
$$D = D^{cl} \otimes id + \gamma^{N+1} \otimes \mathcal{M} = \begin{pmatrix} D^{cl} \otimes id_{\mathbb{C}^n} & \gamma^{N+1} \otimes M \\ \gamma^{N+1} \otimes M^* & D^{cl} \otimes id_{\mathbb{C}^n} \end{pmatrix},$$
$$\pi \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} f_1 \otimes id_{\mathbb{C}^n} & 0 \\ 0 & f_2 \otimes id_{\mathbb{C}^n} \end{pmatrix},$$
$$\Gamma = \gamma^{N+1} \otimes \tilde{\Gamma},$$

which is the skew tensor product of the K-cycles of the two foregoing examples.

This K-cycle was used in [3] for a construction of the Weinberg-Salam theory. An explicit description of $\Omega_D A$ for this K-cycle was given in [8]. The results obtained there may be summarized as follows: Let us denote $M_1^t = (MM^*)^t$, $M_2^t = M_1^t M$, $M_4^t = (M^*M)^t$, $M_3^t = M^* M_4^t$ and notice that there is a positive integer m such that in each series $(M_q^t)_{t=0,1,\dots}$ $(q = 1,2,3,4)$ just the first m terms are linearly independent in $M_n\mathbb{C}$. Then we have

$$\Omega_D^k A = \begin{pmatrix} \bigoplus_{t=0}^m \Lambda^{k-2t} \otimes \mathbb{C}M_1^t & \bigoplus_{t=0}^m \Lambda^{k-2t-1}\gamma^{N+1} \otimes \mathbb{C}M_2^t \\ \bigoplus_{t=0}^m \Lambda^{k-2t-1}\gamma^{N+1} \otimes \mathbb{C}M_3^t & \bigoplus_{t=0}^m \Lambda^{k-2t} \otimes \mathbb{C}M_4^t \end{pmatrix},$$

where Λ^k denotes the space of differential k-forms on X and the right multiplication with γ^{N+1} is nothing but (a certain variant of) the Hodge star. The product $\bullet$ in $\Omega_D A$ is given by

$$\begin{pmatrix} \alpha_1 \otimes M_1^{t_1} & \alpha_2\gamma^{N+1} \otimes M_2^{t_2} \\ \alpha_3\gamma^{N+1} \otimes M_3^{t_3} & \alpha_4 \otimes M_4^{t_4} \end{pmatrix} \bullet \begin{pmatrix} \beta_1^{l_1} \otimes M_1^{s_1} & \beta_2^{l_2}\gamma^{N+1} \otimes M_2^{s_2} \\ \beta_3^{l_3}\gamma^{N+1} \otimes M_3^{s_3} & \beta_4^{l_4} \otimes M_4^{s_4} \end{pmatrix}$$
$$= \begin{pmatrix} \begin{array}{l} \alpha_1 \wedge \beta_1^{l_1} \otimes M_1^{t_1+s_2} \\ +(-1)^{l_3}\alpha_2 \wedge \beta_3^{l_3} \otimes M_1^{t_2+s_3+1} \end{array} & \begin{array}{l} \alpha_1 \wedge \beta_2^{l_2}\gamma^{N+1} \otimes M_2^{t_1+s_2} \\ +(-1)^{l_4}\alpha_2 \wedge \beta_4^{l_4} \otimes M_2^{t_2+s_4} \end{array} \\ \\ \begin{array}{l} (-1)^{l_1}\alpha_3 \wedge \beta_1^{l_1}\gamma^{N+1} \otimes M_3^{t_3+s_1} \\ +\alpha_4 \wedge \beta_3^{l_3}\gamma^{N+1} \otimes M_3^{t_4+s_3} \end{array} & \begin{array}{l} (-1)^{l_2}\alpha_3 \wedge \beta_2^{l_2} \otimes M_4^{t_3+s_2+1} \\ +\alpha_4 \wedge \beta_4^{l_4} \otimes M_4^{t_4+s_4} \end{array} \end{pmatrix}, \qquad (1)$$

where we put $M_i^t = 0$ for $t > m$, and the upper index of the β's denotes the form degree. This is just multiplication of 2×2-matrices combined with exterior product of differential forms plus suitable signs arising from the exchange of a differential form with γ^{N+1}, and the following rules for the multiplication of the M_i^t (coming also from matrix multiplication): $M_2^t M_3^s = M_1^{t+s+1}$, $M_3^t M_2^s = M_4^{t+s+1}$, $M_i^t M_j^s = M_{k(i,j)}^{t+s}$ for the other values of (i,j) $(k(1,1) = 1$, $k(1,2) = 2$, $k(2,4) = 2$, $k(3,1) = 3$, $k(4,3) = 3$, $k(4,4) = 4)$.

The differential $\hat{d}$ can be written as

$$\hat{d} = d + [\omega^{N+1}, .]_g,$$

where d is the componentwise ordinary exterior differential,

$$\omega^{N+1} = -\mathrm{i} \begin{pmatrix} 0 & \gamma^{N+1} \otimes M \\ \gamma^{N+1} \otimes M^* & 0 \end{pmatrix} \in \Omega^1_D A\,,$$

and $[.,.]_g$ denotes the graded commutator with respect to the product $\bullet$. Notice that $\Omega_D A$ is also a differential algebra with differential d.

Now, let us recall the notions of a Hermitian finite projective module and of a connection on such a module. Every finite projective right module over an algebra A may be defined as a submodule of some free module A^p of the form $\mathcal{E} = eA^p$, where $e \in End_A(A^p)$, $e^2 = e$. A Hermitian metric on $\mathcal{E}$ is a nondegenerate A-sesquilinear map $(\,.\,,\,.\,)_{\mathcal{E}} : \mathcal{E} \times \mathcal{E} \longrightarrow A$. Such a metric always exists, it can be obtained by restricting the canonical metric on A^p to $\mathcal{E}$. We will always take this metric, and assume $e = e^*$. Given a differential algebra (Λ_A, d) over A, a connection in $\mathcal{E}$ is a $\mathbb{C}$-linear map $\nabla : \mathcal{E} \longrightarrow \mathcal{A} \otimes_A \Lambda^1_A$ with $\nabla(\xi a) = (\nabla\xi)a + \xi \otimes_A da$. ∇ is said to be compatible with $(\,.\,,\,.\,)_{\mathcal{E}}$ if $d((\xi,\xi')_{\mathcal{E}}) = (\nabla\xi,\xi')^{1,0}_{\mathcal{E}} + (\xi,\nabla\xi')^{0,1}_{\mathcal{E}}$ (where $(\,.\,,\,.\,)^{1,0}_{\mathcal{E}}$ and $(\,.\,,\,.\,)^{0,1}_{\mathcal{E}}$ are natural extensions of $(\,.\,,\,.\,)_{\mathcal{E}}$). There is a canonical connection given by $\nabla_0 \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_p \end{pmatrix} = e \begin{pmatrix} d\xi_1 \\ \vdots \\ d\xi_p \end{pmatrix}$, where we have identified e with an element of $M_p(A)$ and $\mathcal{E} \otimes_A \Lambda_A \cong e(\Lambda_A)^p$. The connection form of a connection ∇ is defined to be $\rho = \nabla - \nabla_0 \in Hom_A(\mathcal{E}, \mathcal{E} \otimes_A \Lambda^1_A)$. ∇ is compatible iff $\rho = \rho^*$. Extending ∇ in a natural manner to a map $\mathcal{E} \otimes_A \Lambda^1_A \longrightarrow \mathcal{E} \otimes_A \Lambda^2_A$, one obtains the curvature $\Theta = \nabla^2 \in Hom_A(\mathcal{E}, \mathcal{E} \otimes_A \Lambda^2_A)$. The group of gauge transformations is the unitary group of the module, $U(\mathcal{E}) = \{u \in End_A(\mathcal{E}) \mid uu^* = u^*u = id_{\mathcal{E}}\}$. Connection, connection form, and curvature are gauge transformed as follows: $\nabla \longrightarrow u\nabla u^*$, $\rho \longrightarrow u\rho u^* + udu^*$, $\Theta \longrightarrow u\Theta u^*$.

3 The algebraic structure of the Mainz-Marseille approach

The central mathematical object of this approach is a certain $\mathbb{Z}_2$-graded Lie algebra with derivation contained in the $\mathbb{Z}_2$-graded differential algebra $\Lambda(X) \otimes M_4\mathbb{C}$, the latter one being considered as the $\mathbb{Z}_2$-graded tensor product of $\Lambda(X)$ and $M_4\mathbb{C}$. Even and odd parts of $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in M_4\mathbb{C}$ are defined to be $M_0 = \frac{1}{2}(M + \Gamma_0 M \Gamma_0) = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}$ and $M_1 = \frac{1}{2}(M - \Gamma_0 M \Gamma_0) = \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}$. With the usual matrix multiplication, $M_4\mathbb{C}$ becomes a $\mathbb{Z}_2$-graded algebra, and with the corresponding graded commutator a $\mathbb{Z}_2$-graded Lie algebra, which we denote by $pl(2,2)$. The graded differential is introduced as the graded commutator with the odd element $\mathfrak{m} = -\mathrm{i}\begin{pmatrix} 0 & 1_{2\times 2} \\ 1_{2\times 2} & 0 \end{pmatrix}$. It is also a graded differential of the graded Lie subalgebra $sl(2,2) = \{M \in M_4\mathbb{C} \,|\, tr(\Gamma_0 M) = 0\}$ of $pl(2,2)$, where $\Gamma_0 = \begin{pmatrix} 1_{2\times 2} & 0 \\ 0 & -1_{2\times 2} \end{pmatrix} \in M_4\mathbb{C}$. Now, it is standard to define the graded tensor product $\Lambda(X) \otimes M_4\mathbb{C}$ of differential algebras. Notice, in particular, that the differential $\mathfrak{d}$ can be written in the form $\mathfrak{d}(b = \beta \otimes M) = d\beta \otimes M + (-1)^{\partial\beta}\beta \otimes [\mathfrak{m}, M] = db + [1 \otimes \mathfrak{m}, b]_g$. It turns out that $\Lambda(X) \otimes spl(2,2) \subset \Lambda(X) \otimes M_4\mathbb{C}$ as a graded differential Lie subalgebra. Now, define a graded Lie subalgebra of $\Lambda(X) \otimes spl(2,2)$ by

$$\Lambda(X) \otimes spl(2,1) = \{b \in \Lambda(X) \otimes spl(2,2) \,|\, b = \mathfrak{e} b \mathfrak{e}\},$$

where $\mathfrak{e} = \begin{pmatrix} 1_{2\times 2} & 0 \\ 0 & \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \end{pmatrix}$. Elements of $\Lambda(X) \otimes spl(2,1)$ just have zeroes in the last row and column. The differential $\mathfrak{d}$ descends to a derivation (not a differential!) of $\Lambda(X) \otimes spl(2,1)$ given by

$$\mathcal{D}b = \mathfrak{e}\mathfrak{d}b = db + [1 \otimes \mathfrak{e}\mathfrak{m}\mathfrak{e}, b]_g \,.$$

A connection in the Mainz-Marseille approach is an operator

$$\nabla = \mathfrak{e}\mathfrak{d} + \mathfrak{a}$$

with

$$\mathfrak{a} = -\mathfrak{a}^* = \begin{pmatrix} A_{11} & A_{12} & -\mathrm{i}\Phi_1 & 0 \\ A_{21} & A_{22} & -\mathrm{i}\Phi_2 & 0 \\ -\mathrm{i}\Phi_1 & -\mathrm{i}\Phi_2 & B & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \in \Lambda(X) \otimes spl(2,1)\,,$$

and $A_{ij} = -\bar{A}_{ji} \in \Lambda^1(X)$, $B = -\bar{B} \in \Lambda^1(X)$, $A_{11} + A_{22} = B$, $\Phi_i \in \Lambda^0(X)$. The curvature of such a connection is defined by $\mathfrak{f} = \nabla^2 = \mathfrak{e}(\mathfrak{d}\mathfrak{e})(\mathfrak{d}\mathfrak{e})\mathfrak{e} + \mathcal{D}\mathfrak{a} + \frac{1}{2}[\mathfrak{a}, \mathfrak{a}]_g$. Gauge transformations are defined on the infinitesimal level: $\gamma_t(\mathfrak{a}) = \mathfrak{a} - \mathcal{D}t + [t, \mathfrak{a}]_g$ with $t = -t^* = \begin{pmatrix} T_{11} & T_{12} & 0 & 0 \\ T_{21} & T_{22} & 0 & 0 \\ 0 & 0 & T_{33} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \in \Lambda(X) \otimes spl(2,1)$, where $T_{ij} = -\bar{T}_{ji} \in \Lambda^0(X)$ and $tr(\Gamma_0 t) = 0$.

In [6], the gauge and Higgs bosons of the electroweak theory were unified in the "connection form" $\mathfrak{a}$. Notice that the above constructions may be easily generalized using $M_{2p}\mathbb{C}$ instead of $M_4\mathbb{C}$, which leads to $pl(p,p)$, $spl(p,p)$ etc.

4 A general construction of graded Lie algebras with derivation

Let us start with the following data: Let A be a unital $*$-algebra over $\mathbb{C}$, let $(\Lambda_A, \bullet, *, d)$ be an involutive differential algebra over A ($\Lambda_A^0 = A$), and let $\mathcal{E} = eA^p$ be a finite projective right A-module with Hermitian structure $(.,.)_{\mathcal{E}}$. We put

$$\mathcal{E}^* = \bigoplus_{k=0}^{\infty} \mathcal{E}^k, \quad \text{where} \quad \mathcal{E}^k = \mathcal{E} \otimes_A \Lambda_A^k.$$

$\mathcal{E}^*$ is a right Λ_A-module in a natural way, and there are natural extensions of the Hermitian metric to mappings $(.,.)_{\mathcal{E}}^{k,l} : \mathcal{E}^k \times \mathcal{E}^l \longrightarrow \Lambda_A^{k+l}$. Now, we define

$$\mathcal{H} = \bigoplus_{k=0}^{\infty} \mathcal{H}^k, \quad \mathcal{H}^k = Hom_A(\mathcal{E}, \mathcal{E}^k).$$

$\mathcal{H}$ can be given the structure of an associative $\mathbb{N}$-graded involutive algebra over $\mathbb{C}$: The product $\bullet$ is defined by

$$(\rho^k \bullet \rho^l)(\xi) = (id_{\mathcal{E}} \otimes_A \bullet) \circ (\rho^k \otimes_A id_{\Lambda_A^l}) \circ \rho^l(\xi).$$

$id_{\mathcal{E}}$ is the unit for this multiplication, and the involution is defined by

$$(\xi, (\rho^k)^*(\xi'))_{\mathcal{E}}^{0,k} = (\rho^k(\xi), \xi')_{\mathcal{E}}^{k,0}.$$

With the graded commutator, $\mathcal{H}$ becomes also an $\mathbb{N}$-graded Lie algebra, and it acts from the left on $\mathcal{E}^*$: $\rho^k \bullet \xi^l = (id_{\mathcal{E}} \otimes_A \bullet) \circ (\rho^k \otimes_A id_{\Lambda_A^l})(\xi^l)$. Finally, there is a graded derivation $D_{\mathcal{H}} : \mathcal{H}^k \longrightarrow \mathcal{H}^{k+1}$ inherited from the canonical compatible connection ∇_0 on $\mathcal{E}$, which stems from the differential d of Λ_A:

$$(D_{\mathcal{H}}\rho^k)(\xi) = \nabla_0(\rho^k(\xi)) - (-1)^k \rho^k \bullet (\nabla_0(\xi)).$$

$D_{\mathcal{H}}$ fails to be a differential:

$$D_{\mathcal{H}}^2(\rho) = \Theta_0 \bullet \rho - \rho \bullet \Theta_0,$$

where Θ_0 is the curvature of ∇_0. For the curvature of a connection $\nabla = \nabla_0 + \rho$, one obtains

$$\Theta = \Theta_0 + \mathcal{D}_{\mathcal{H}}\rho + \rho \bullet \rho.$$

These definitions have a nice matrix form: Let $(\epsilon_i)_{i=1}^p$ be the canonical basis of A^p (ϵ_i having the unit of A as entry at the i-th place, zeroes at the other places). Then, the projection e is given by $e(\epsilon_i) = \epsilon_j e_{ji}$, $e_{ji} \in A$ with $e_{ij}e_{jk} = e_{ik}$. $\xi \in \mathcal{E}$ is characterized by $(e\xi)_i = e_{ij}\xi_j = \xi_i$. An element $\rho \in \mathcal{H}^k$ is characterized by a matrix $(\rho_{ij})_{i,j=1}^p$, $\rho_{ij} \in \Lambda_A^k$, with $e_{ij}\rho_{jk}e_{kl} = \rho_{il}$ (in short, $e\rho e = \rho$), the multiplication in $\mathcal{H}$ is given by matrix multiplication, $(\rho \bullet \rho')_{ij} = \rho_{ik} \bullet \rho'_{kj}$, and the derivation $D_{\mathcal{H}}$ is given by componentwise action of d: $(D_{\mathcal{H}}\rho)_{ij} = e_{ik} d\rho_{kl} e_{lj}$ $(D_{\mathcal{H}}\rho = ed\rho e)$. Notice that $(\Theta_0)_{ij} = e_{ik} de_{kl} \bullet de_{lm} e_{mj}$.

In order to come from these general definition of an algebra with derivation to the algebraic structure of the Mainz-Marseille approach, we have first to specify the data of our definition. For the chosen case, it is possible to introduce a suitable condition of tracefreeness on $\mathcal{H}$ and a certain surjective mapping whose application just leads to the structures of the foregoing section. First, we take the algebra $A = C^\infty(X) \otimes \mathbb{C}^2$ and the differential algebra $\Lambda_A = \Omega_D A$ of the K-cycle described in the example of section 2. For this case, and for any module $\mathcal{E}$, we can construct a certain graded Lie subalgebra $\mathcal{H}_0$ of $\mathcal{H}$ as follows: We define a $\mathbb{C}$-linear map $T_\Lambda : \Lambda_A \longrightarrow \Lambda(X)$ by

$$T_\lambda \left(\begin{pmatrix} \alpha_1 \otimes M_1^{t_1} & \alpha_2 \gamma^{N+1} \otimes M_2^{t_2} \\ \alpha_3 \gamma^{N+1} \otimes M_3^{t_3} & \alpha_4 \otimes M_4^{t_4} \end{pmatrix} \right) = \alpha_1 + \alpha_4.$$

This is a generalized trace in the sense that $T_\Lambda(\Gamma_\Lambda[\lambda, \lambda']_g) = 0$, where $\Gamma_\Lambda = \begin{pmatrix} 1 \otimes 1_{n\times n} & 0 \\ 0 & -1 \otimes 1_{n\times n} \end{pmatrix} \in A$. Now, we define $T_{\mathcal{H}} : \mathcal{H} \longrightarrow \Lambda(X)$ by

$$T_{\mathcal{H}}(\rho) = \sum_{i=0}^{p} T_\Lambda(\Gamma_\Lambda \rho_{ii}),$$

which is also a generalized trace: $T_{\mathcal{H}}([\rho, \rho']_g) = 0$. Therefore, $\mathcal{H}_0 = \bigoplus_{k=0}^\infty \mathcal{H}_0^k$, $\mathcal{H}_0^k = \{\rho \in \mathcal{H}^k | T_{\mathcal{H}}(\rho) = 0\}$, is a graded Lie subalgebra of $\mathcal{H}$.

Recall that there are two differentials $\hat{d}$ and d on $\Omega_D A$. For both we can construct, using the corresponding compatible connections $\hat{\nabla}_0$ and ∇_0, graded derivations $\hat{\mathcal{D}}_{\mathcal{H}}$ and $\mathcal{D}_{\mathcal{H}}$ of $\mathcal{H}$, which turn out to be also graded derivations of $\mathcal{H}_0$. They are related by

$$\hat{\mathcal{D}}_{\mathcal{H}}\rho = \mathcal{D}_{\mathcal{H}}\rho + [\mu, \rho]_g,$$

where $\mu = e(1_{p\times p} \otimes \omega^{N+1})e \in \mathcal{H}^1$.

To come to the Mainz Marseille setting, we now have to perform two steps:

1. In matrix representation, elements of $\mathcal{H}$ are $p \times p$-matrices with entries from $\Lambda_A \subset \Lambda(X) \otimes End(\mathbb{C}^n) \otimes M_2\mathbb{C}$. We treat them now as 2×2-matrices with entries from $\Lambda(X) \otimes End(\mathbb{C}^n) \otimes M_p\mathbb{C}$. This is just going from one standard representation of a Kronecker product of matrices to the other one. Moreover, we can remove the γ^{N+1} without loosing information. Thus, we get an injection

$$i : \Lambda_A \otimes M_p\mathbb{C} \longrightarrow \Lambda(X) \otimes M_p\mathbb{C} \otimes End(\mathbb{C}^n) \otimes M_2\mathbb{C}$$

   of vector spaces. Elements of $i(\Lambda_A^k \otimes M_p\mathbb{C})$ have the form

$$\begin{pmatrix} A_1^{k-2t_1} \otimes M_1^{t_1} & A_2^{k-2t_2-1} \otimes M_2^{t_2} \\ A_3^{k-2t_3-1} \otimes M_3^{t_3} & A_4^{k-2t_4} \otimes M_4^{t_4} \end{pmatrix}$$

with $A_q^l \in \Lambda^l(X) \otimes M_p\mathbb{C}$. Moreover, we have $i(e) = \begin{pmatrix} e_1 & 0 \\ 0 & e_4 \end{pmatrix}$ with $e_q = e_q^2 = e_q^* \in C^\infty(X) \otimes M_p\mathbb{C}$. Elements of $i(\mathcal{H})$ are characterized by $A_1 = e_1 A_1 e_1$, $A_2 = e_1 A_2 e_4$, $A_3 = e_4 A_3 e_1$, $A_4 = e_4 A_4 e_4$, those of $i(\mathcal{H}_0)$ in addition by $trA_1 = trA_4$. Transporting the product $\bullet$ of $\mathcal{H}$ leads to a product of the same form as in $\Lambda_A = \Omega_D A$, formula (1). One has to replace there $\alpha \longrightarrow A$, $\beta \longrightarrow B$, one has to omit γ^{N+1} and one has to interprete $\wedge$ as exterior product of forms combined with multiplication of $p \times p$-matrices.

2. We define a surjection $\mathfrak{p} : i(\Lambda_A \otimes M_p\mathbb{C}) \longrightarrow \Lambda(X) \otimes M_{2p}\mathbb{C}$ by

$$\mathfrak{p} \begin{pmatrix} A_1 \otimes M_1^{t_1} & A_2 \otimes M_2^{t_2} \\ A_3 \otimes M_3^{t_3} & A_4 \otimes M_4^{t_4} \end{pmatrix} = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}.$$

Theorem 2

(i) $\mathfrak{p} \circ i(\mathcal{H}_0) = \{b \in \Lambda(X) \otimes spl(p,p) \mid b = \mathfrak{e}b\mathfrak{e}\}$
with $\mathfrak{e} = i(e) = \begin{pmatrix} e_1 & 0 \\ 0 & e_4 \end{pmatrix}$ *(see above).*

(ii) $(\mathfrak{p} \circ i(\rho))^* = \mathfrak{p} \circ i(\rho^*)$, $\rho \in \mathcal{H}$.

(iii) $\mathfrak{p} \circ i([\rho^k, \rho^l]_g) = [\mathfrak{p} \circ i(\rho^k), \mathfrak{p} \circ i(\rho^k)]_g$ *for* $k + l \leq 2m + 1$, $\rho^k \in \mathcal{H}^k$, $\rho^l \in \mathcal{H}^l$.

(iv) $\mathfrak{p} \circ i(\hat{\mathcal{D}}_{\mathcal{H}}(\rho^k)) = \mathcal{D}(\mathfrak{p} \circ i(\rho^k))$ *for* $k \leq 2m$, $\rho^k \in \mathcal{H}^k$.

Notice that also the analogue $\mathfrak{p} \circ i(\mathcal{D}_{\mathcal{H}}(\rho^k)) = d(\mathfrak{p} \circ i(\rho^k))$ of (iv) is true.

The theorem says that $\mathfrak{p} \circ i$ is a partial homomorphism of $\mathbb{Z}_2$-graded involutive Lie algebras with derivation. This mapping is not injective on $\mathcal{H}$ or $\mathcal{H}_0$, its restriction, however, to any sum of subsequent homogeneous components $(\Lambda_A^k \oplus \Lambda_A^{k+1}) \otimes M_p\mathbb{C}$ is injective. Since we assume $MM^* \notin \mathbb{C}1$, we have $m \geq 1$. Therefore, in particular, $\mathfrak{p}$ is a monomorphism on the graded Lie subalgebra $i(\mathcal{H}^0 \oplus \mathcal{H}^1)$, and it commutes with the derivation of elements of $i(\mathcal{H}^0)$ and $i(\mathcal{H}^1)$. However, under the application of $\mathfrak{p}$ the $\mathbb{N}$-grading of $i(\mathcal{H})$ is lost and only a $\mathbb{Z}_2$-grading remains.

It is now easy to see, that we arrive at the algebraic setting of the Mainz-Marseille approach starting with the choice $p = 2$ and $e = \begin{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & 0 \\ 0 & \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \end{pmatrix}$. Using the above theorem, it is almost obvious that under the mapping $\mathfrak{p} \circ i$ the geometric objects living in the projective module $\mathcal{E} = eA^2$ are transformed into corresponding objects of the Mainz-Marseille scheme. In particular, due to the partial injectivity of $\mathfrak{p} \circ i$ discussed above, no information about the objects relevant for gauge theories (connections and curvatures) is lost. Moreover, the scheme is completed by giving a natural definition of the (noninfinitesimal) gauge group and of the module where the connection acts.

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