Slavnov-Taylor identity in noncommutative geometry

RAIMAR WULKENHAAR
Institute for Theoretical Physics, University of Vienna
Boltzmannasse 5, 1090 Wien, Austria

Abstract

We develop a framework to define quantum Yang-Mills theories on general differential algebras imitating the standard procedure.

1 Field theories on general differential algebras

Let $\mathcal{A}$ be an associative $^*$-algebra over $\mathbb{C}$ and $(\Omega, d)$ an $\mathbb{N}$-graded differential $^*$-algebra over $\mathcal{A}$, i.e. $\Omega = \bigoplus_{n=0}^\infty \Omega^n$ with $\Omega^0 = \mathcal{A}$ and

$$
\Omega^0 = \mathcal{A}, \quad \Omega^k \subseteq \mathcal{A}^{k+l}, \quad d : \Omega^n \to \Omega^{n+1}, \quad * : \Omega^n \to \Omega^n,
$$

$$
(\omega^k \tilde{\omega})^* = (\tilde{\omega}^l)^* (\omega^k)^*, \quad (\tilde{\omega}^l)^* = \tilde{\omega}^k,
$$

$$
d(\omega^k \tilde{\omega}^l) = d(\omega^k) \tilde{\omega}^l + (-1)^l \omega^k d(\tilde{\omega}^l), \quad d(\tilde{\omega}^k) = 0,
$$

for $\omega^k \in \Omega^k$, $\tilde{\omega}^l \in \Omega^l$ and $z \in \mathbb{C}$. Moreover, let $\Omega^n$, for each degree $n$, be equipped with a symmetric non-degenerate positive bilinear form $\langle , \rangle_n : \Omega^n \times \Omega^n \to \mathbb{C}$ which satisfies

$$
\langle \omega, \tilde{\omega} \rangle_n = \langle \tilde{\omega}, \omega \rangle_n, \quad \langle \omega a, \tilde{\omega} \rangle_n = \langle \omega, a \tilde{\omega} \rangle_n, \quad \text{for } \omega, \tilde{\omega} \in \Omega^n, \quad a \in \mathcal{A},
$$

$$
\langle \omega, \omega^* \rangle_n \geq 0 \quad \forall \omega \in \Omega^n, \quad \langle \omega, \omega^* \rangle_n = 0 \iff \omega = 0.
$$

The codifferential $d^* : \Omega^{n+1} \to \Omega^n$ is defined as the adjoint of $d$ via $\langle , \rangle_n$,

$$
\langle da, b \rangle_{n+1} =: \langle a, db \rangle_n, \quad a \in \Omega^n, \quad b \in \Omega^{n+1}.
$$

Our goal is to formulate field theories, i.e. dynamical systems of amplitudes. For this purpose it is necessary to assume that there is a basis $\{ u_p \}$ in $\Omega^n$, labelled by possibly continuous parameters. Instead of taking complex numbers for the amplitudes we allow for Grassmann valued amplitudes $\phi^P_{q,q} \in V_{q,q}$, with

$$
\phi^P_{q,q} \phi^Q_{r,r} = (-1)^{(q-r)(r-p)} \phi^Q_{r,r} \phi^P_{q,q}.
$$

Let $\mathbb{G} = \bigoplus_{q,q} \mathbb{G}_{q,q}$ be the bigraded Grassmann algebra generated by the amplitudes $\{ \phi^P_{q,q} \}$ according to (6). Let $\Omega^\ast_{q,q}$ be the space of $\mathbb{G}_{q,q}$-valued $n$-forms which contains in particular elements $\phi_t = \phi_{q,q}^P u_p$ linear in the amplitudes $\phi^P_{q,q} \in V_{q,q}$. Multiplication, differential and bilinear form are extended by linearity from $\Omega^\ast$ to $\Omega^\ast_{q,q}$. Finally, let

$$
\mathcal{C}(\Omega) = \{ z \in \mathcal{A} : dz \equiv 0, \quad [z, \omega] \equiv 0 \quad \forall \omega \in \Omega^\ast_{q,q} \}
$$

(7)
be the set of ‘constant’ central elements of $\Omega_{q,\bar{q}}^n$.

Let $F \in \mathbb{G}_{r,\bar{r}}$ be some power series of $\{\phi_{i,\bar{i}}^P\}$. We define functional derivatives

$$
\frac{\partial}{\partial \phi_i}: \mathbb{G}^r \ni F \mapsto \frac{\partial F}{\partial \phi_i} \in \Omega_{r-q,\bar{r}-\bar{q}}^n \quad \text{by}
$$

$$
\lim_{\epsilon \to 0} \frac{1}{\epsilon} \left( F \bigg|_{\phi_i^P \mapsto \phi_i^P + \epsilon \phi^P} - F - \left\langle \epsilon \phi, \frac{\partial F}{\partial \phi_i} \right\rangle \right) = 0, \quad \forall \epsilon = u_p x^p \in \Omega_{q,\bar{q}}^n.
$$

**Definition 1** A field is a homogeneous element of $\Omega_{q,\bar{q}}^n$ linear in the amplitudes spanning $V_{q,\bar{q}}$ with a dimension

$$
\text{dim} = n + q + \bar{q}
$$

assigned to the amplitudes of the field. The numbers $(n; q, \bar{q})$ are called (degree; ghost number, antighost number) of the field. The differential $d : \Omega_{q,\bar{q}}^n \to \Omega_{q,\bar{q}}^{n+1}$ is counted as an element of type $(1; 0, 0)$.

A local field monomial is the product (in $\Omega_{q,\bar{q}}^n$) of fields and differentials of fields. The dimension of the field monomial is the sum of the dimensions of fields and differentials in it. (By construction, degree and ghost-antighost numbers the field monomial is the sum of the degrees and ghost-antighost numbers, respectively, of the fields and differentials in it.)

An integrated local field monomial is the contraction of two local field monomials of the same degree $n$ via the weighted bilinear form $\langle z, \cdot \rangle_n$, with invertible $z \in \mathcal{C}(\Omega)$.

A classical action is a linear combination of integrated local field monomials (with different weights $z$) of dimension $\leq D$ and balanced ghost-antighost numbers $q = \bar{q}$.

**Definition 2** A Yang-Mills theory on $\Omega_{q,\bar{q}}^n$, in dimension $D = 4$ is a theory of the fields $A, \rho, c, \sigma, \bar{c}, \bar{B}$ whose degrees $n$, ghost-antighost numbers $q - \bar{q}$ and dimensions $\text{dim}$ are given in the following table:

<table>
<thead>
<tr>
<th></th>
<th>$A$</th>
<th>$\rho$</th>
<th>$c$</th>
<th>$\sigma$</th>
<th>$\bar{c}$</th>
<th>$\bar{B}$</th>
<th>$d$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n$</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>$q$</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>$\bar{q}$</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>$\text{dim}$</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

We assume that there exist configurations $A, c$ such that

$$
\alpha A cc + \beta cc A = 0, \quad \gamma d(cc) = 0,
$$

for $\alpha, \ldots, \zeta \in \mathcal{C}(\Omega)$ not necessarily positive, have only the solution $\alpha = \ldots = \zeta = 0$. Then, the theory is governed by the Slavnov-Taylor operator $S$ defined on functionals $\Gamma \in \mathbb{G}$ of the amplitudes of $A, \rho, c, \sigma, \bar{c}, \bar{B}$ by

$$
S(\Gamma) = \left\langle \frac{\partial \Gamma}{\partial \rho}, \frac{\partial \Gamma}{\partial A} \right\rangle + \left\langle \frac{\partial \Gamma}{\partial \sigma}, \frac{\partial \Gamma}{\partial c} \right\rangle + \left\langle B, \frac{\partial \Gamma}{\partial \bar{c}} \right\rangle.
$$
We are looking for the most general solution $\Gamma$ of the Slavnov-Taylor identity $S(\Gamma) = 0$, where $\Gamma$ is a classical action, i.e. an integrated local field polynomial in $A, \rho, c, \sigma, \bar{c}, B$ with dimension $\leq 4$ and balanced ghost-antighost number $q = \bar{q}$. The answer is

**Proposition 3** If (10) is true and if $\frac{\partial \Gamma}{\partial \rho} \neq 0$ and $\frac{\partial \Gamma}{\partial c} \neq 0$, the most general classical non-abelian Yang-Mills action satisfying the Slavnov-Taylor identity is

$$
\Gamma = -\frac{1}{16\pi^2} F_i F_i + \frac{1}{2} B_i B_i + e d\tilde{c} + \rho_i + dc + \{A_i, c_i\} - \{d B_i, A_i\} - \langle \sigma, \epsilon \rangle_0 + \langle \beta \bar{c} c_i, \epsilon \rangle_0 - \langle \beta_i B_i, \{\bar{c}, c_i\}\rangle_0 + \langle \gamma B_i A_i, A_i\rangle_1 + \langle \gamma d c_i, \{A_i, \bar{c}\}\rangle_1 + \langle \gamma A_i \{\bar{c}, c_i\}, A_i\rangle_1,
$$

with $F = dA + AA$, up to a rescaling $A_i \mapsto \xi_i A_i, \rho_i \mapsto \rho_i / \xi_i, c \mapsto \sigma / \xi_i$, $\bar{c} \mapsto \xi_i \bar{c}$ and $B \mapsto \xi_2 B$, which leave the Slavnov-Taylor identity unchanged. Here, $g, \alpha, \beta, \gamma, \xi_i$ are positive central elements of $\mathcal{A}$.

There are three degenerate (static) solutions where some parts whose coefficients in (12) are normalized to 1 are missing, given as combinations of

1) $dA = 0$ and $dc = 0$, $g \mapsto 1$, additional term $-(\frac{m^2}{2} A, A)$,

2) $dB = 0$ and $d\bar{c} = 0$, $\beta \mapsto 1$, (although these differentials may actually be non-zero).

For the proof one writes down the most general local field polynomial of dimension $\leq 4$ and balanced ghost-antighost number and applies the Slavnov-Taylor operator.

It is convenient to impose the gauge fixing condition

$$
\frac{\partial \Gamma}{\partial B} = \alpha B - d^* A.
$$

2 Generating functionals

The classical action $\Gamma[A, \rho, c, \sigma, \bar{c}, B]$ is regarded as a special example of a generating functional of 1PI (one-particle irreducible) Green’s functions. In general, deriving such a functional with respect to the fields $\phi_i = \{A, \rho, c, \sigma, \bar{c}, B\}$ (considered as test functions), $\Gamma_{1...n} := \frac{\partial^2 \Gamma}{\partial \phi_i \partial \phi_j} |_{\phi_i = 0}$, one can associate to $\Gamma_{1...n}$ a graph which remains connected after cutting an arbitrary line. In particular, external lines of $\Gamma_{1...n}$ are amputated.

In the general case one can pass from $\Gamma$ to a generating functional $Z^e$ of connected Green’s functions by Legendre transformation

$$
Z^e[J, \bar{J}, j, \bar{j}, \rho, \sigma] := \Gamma[A, B, c, \bar{c}, \rho, \sigma] + \langle A_i J_i \rangle_1 + \langle B_i \bar{J}_i \rangle_0 + \langle j_i c_i \rangle_0 + \langle \bar{j}_i \bar{c}_i \rangle_0,
$$

where the fields $A, B, c, \bar{c}$ have to be replaced by the (inverse) solution of

$$
J = -\frac{\partial \Gamma}{\partial A} \in \Omega^1, \quad \bar{J} = -\frac{\partial \Gamma}{\partial \bar{B}} \in \Omega^0, \quad j = \frac{\partial \Gamma}{\partial c} \in \Omega^1, \quad \bar{j} = \frac{\partial \Gamma}{\partial \bar{c}} \in \Omega^0.
$$
with \( \Omega_Q^0 = \bigoplus_{n=\max(-Q,0)}^{\infty} \Omega^0_{Q+nQ} \). The generating functional of general (not necessarily connected) Green’s functions is defined as

\[
Z := e^{-\frac{i}{\hbar} \mathcal{L}}.
\]

In particular, we can take for \( \Gamma \) the bilinear part \( \Gamma_{\text{bil}} \) of the gauge-fixed classical action \( \Gamma_{\text{cl}} \):

\[
\Gamma_{\text{bil}} = -\langle \frac{1}{2g^2} dA, dA \rangle_2 + \langle \frac{g}{2} B, B \rangle_0 - \langle dB, A \rangle_1 + \langle d\bar{c}, dc \rangle_1 \quad \text{(17)}
\]

\[
- (s-1)^2 M^2 \langle \frac{1}{2g^2} A, A \rangle_1 + (s-1)^2 M^2 \langle \frac{g}{2} \bar{c}, \bar{c} \rangle_0 \quad \text{(18)}
\]

The mass terms proportional to \((s-1)^2 M^2\) are auxiliary ones to deal with possible infrared divergences. It is convenient not to include \( \langle \rho, dc \rangle_1 \) in \( \Gamma_{\text{bil}} \).

Restricted for the moment to the bilinear part we obtain

\[
J = \frac{1}{g^2} (\frac{1}{2} d^* d + (s-1)^2 M^2) A + d B, \quad j = (d^* d + \frac{g}{2} (s-1)^2 M^2) c, \quad \tilde{j} = -(d^* d + \frac{g}{2} (s-1)^2 M^2) \tilde{c}.
\]

This gives

\[
A = -g^2 \Delta (J + \frac{1}{\alpha} d \tilde{J}) \quad \text{and} \quad c = -\Delta j, \quad \bar{c} = \Delta \tilde{j},
\]

with the propagators \( \Delta, \Delta \) defined by

\[
\tilde{\Delta} (\frac{1}{2} d^* d + \frac{g^2}{\alpha} d^* d + (s-1)^2 M^2) = -\text{id}_{\Omega^1}, \quad \Delta (d^* d + \frac{g}{g} (s-1)^2 M^2) = -\text{id}_{A}.
\]

We have used the identity \( \frac{g^2}{\alpha} \tilde{\Delta} d = d \Delta \).

A lengthy but straightforward computation leads to

\[
Z_{\text{bil}}^0 = -\langle \frac{g^2}{2} J, \tilde{\Delta} J \rangle_1 - \langle \frac{g^2}{2} d \tilde{J}, \tilde{\Delta} J \rangle_1 + (s-1)^2 M^2 \langle \frac{1}{2g^2} J, \Delta J \rangle_0 - \langle \tilde{j}, \Delta j \rangle_0 \quad \text{(21)}
\]

and consequently to

\[
Z_{\text{bil}}[J, J, j, \tilde{j}] = e^{\left( \left( \frac{g^2}{2} J, \tilde{\Delta} J \right)_1 + \langle \frac{g^2}{2} d \tilde{J}, \tilde{\Delta} J \rangle_1 - (s-1)^2 M^2 \langle \frac{1}{2g^2} J, \Delta J \rangle_0 + \frac{1}{\hbar} \langle \tilde{j}, \Delta j \rangle_0 \right)}. \quad \text{(22)}
\]

We quantize our theory axiomatically by the principle that the full generating functional is given by

\[
Z[\rho, \sigma, J, J_B, j, \tilde{j}] := \mathcal{N} \sqrt[\alpha]{\Gamma_{\text{int}}[A, \bar{c}, \bar{c}, B, \rho, \sigma]} \left[ A_{\rho+\hbar\rho}, \bar{c}_{\sigma+\hbar\sigma} \right] \cdot \left[ A_{\rho-\hbar\rho}, \bar{c}_{\sigma-\hbar\sigma} \right], \quad Z_{\text{bil}}[J, J, j, \tilde{j}] \quad \text{with} \quad \Gamma_{\text{int}} = \Gamma_{\text{cl}} - \Gamma_{\text{bil}} \big|_{\alpha=1} \quad \text{and} \quad \mathcal{N} \text{ is an (ill-defined) normalization factor determined by } Z[0] = 1. \quad \text{(23)}
\]

In many cases the expansion of (23) leads to infinities even if
the possible problem with $\mathcal{N}$ is ignored. We have to fix a regularization scheme
so that (23) becomes a formal power series in $\hbar$ consisting of finite terms. It is
not important for us whether the series converges or not.

Due to $\frac{\partial Z^c}{\partial \rho} = \frac{\partial \Gamma}{\partial \rho}$ and $\frac{\partial Z^c}{\partial \sigma} = \frac{\partial \Gamma}{\partial \sigma}$ we have

$$S\Gamma = \left\langle \frac{\partial \Gamma}{\partial \rho}, \frac{\partial \Gamma}{\partial \sigma} \right\rangle_1 + \left\langle \frac{\partial \Gamma}{\partial \sigma}, \frac{\partial \Gamma}{\partial c} \right\rangle_0 + \left\langle B, \frac{\partial \Gamma}{\partial \bar{c}} \right\rangle_0$$

$$= \left\langle -J, \frac{\partial Z^c}{\partial \rho} \right\rangle_1 + \left\langle \bar{J}, \frac{\partial Z^c}{\partial \sigma} \right\rangle_0 + \left\langle j, \frac{\partial Z^c}{\partial \bar{J}} \right\rangle_0 \equiv SZ^c , \quad (24)$$

down when expressing both lines of the equation in terms of the same variables. Since
we have a more explicit formula for $Z$ than for $Z^c$, the identity (24) suggests to
study the problem

$$SZ := \left\langle -J, \frac{\partial Z^c}{\partial \rho} \right\rangle_1 + \left\langle \bar{J}, \frac{\partial Z^c}{\partial \sigma} \right\rangle_0 + \left\langle j, \frac{\partial Z^c}{\partial \bar{J}} \right\rangle_0 . \quad (25)$$

For a given model, i.e. given $(\Omega, d, \langle \cdot, \cdot \rangle)$, it is possible to compute $SZ$. On a
formal level one always obtains $SZ = 0$, however, the counterterms introduced to
remove the infinities will lead in general to corrections breaking the Slavnov-
Taylor identity. These corrections have to be characterized by the quantum action
principle concerning dimension, ghost-antighost numbers and structure of the
field monomials. The model is called perturbatively renormalizable if $SZ = 0$ can
be achieved when replacing the classical action $\Gamma_0$ defining $Z$ by a power series
in $\hbar$ of the same form as $\Gamma_{cl}$.

Out of $Z$ given by (23) we obtain the generating functional of the connected
Green’s functions $Z^c$ via (16) and the generating functional $\Gamma$ of the 1PI Green’s
functions by inverting (14):

$$\Gamma[\rho, \sigma, A, c, \bar{c}, B] = Z^c[\rho, \sigma, J, j, \bar{j}] - \langle A, J \rangle_1 - \langle B, J \rangle_0 - \langle J, c \rangle_0 - \langle j, \bar{c} \rangle_0 ,$$

where the sources $J, J, j, \bar{j}$ are replaced by the solution of

$$A = \frac{\partial Z^c}{\partial J}, \quad c = \frac{\partial Z^c}{\partial j}, \quad \bar{c} = \frac{\partial Z^c}{\partial \bar{j}}, \quad B = \frac{\partial Z^c}{\partial \bar{J}} .$$

The functional $\Gamma$ is a formal power series in $\hbar$, $\Gamma = \sum_{n=0}^{\infty} \hbar^n \Gamma_{(n)}$, and one can check
that $\Gamma_{(0)} = \Gamma_{cl, s} = \Gamma_{int} + \Gamma_{bil}$. Moreover, the contributions to $\Gamma_{(n)}$ correspond to
Feynman graphs with $n$ loops. One easily convinces oneself that $\Gamma_{(n)}$ for $n > 0$
cannot be written as contractions via $\langle \cdot, \cdot \rangle_k$. The only way to evaluate it is in
components with respect to a basis for a concrete model.

The three Slavnov-Taylor identities $SZ = 0$, $SZ^c = 0$ and $S\Gamma = 0$ for generating
functionals of renormalized Green’s functions are equivalent.
3 Example: The noncommutative $\mathbb{R}^4$

An example of an algebra fitting into our setting is the noncommutative $\mathbb{R}^4$. We consider four hermitian ‘coordinates’ $x_\mu$, $\mu = 1, \ldots, 4$, satisfying

$$[x_\mu, x_\nu] = -2i\pi \theta_{\mu\nu} , \quad \theta_{\mu\nu} = -\theta_{\nu\mu} \in \mathbb{R} .$$

Introducing $u_p := \exp(2i\pi p^\mu x_\mu) = 1 + 2i\pi p^\mu x_\mu + (1/2!)(2i\pi p^\mu x_\mu)^2 + \ldots$, with $p = (p^1, p^2, p^3, p^4) \in \mathbb{R}^4$, the Baker-Campbell-Hausdorff formula gives

$$u_p u_q = e^{i\theta(p,q)} u_{p+q} , \quad \theta(p,q) = -\theta(q,p) = \theta_{\mu\nu} p^\mu p'^\nu . \quad (26)$$

Moreover, $(u_p)^* = u_{-p}$. The noncommutative $\mathbb{R}^4$ is the algebra spanned by $\{u_p\}$ and will be denoted by $\mathbb{R}^4_\Omega$. The differential algebra $(\Omega, d)$ over $\mathbb{R}^4_\Omega$ is the tensor product of $\mathbb{R}^4_\Omega$ with some Grassmann algebra of $D = 4$ generators $\{\gamma_\mu\}_{\mu=1,\ldots,4}$, satisfying $\gamma_\mu \gamma_\nu = -\gamma_\nu \gamma_\mu$. Then,

$$\Omega_0^p = \text{span}_\mathbb{C}(u_p , \ p \in \mathbb{R}^4) \equiv \mathbb{R}^4_\Omega ,$$

$$\Omega_1^p = \text{span}_\mathbb{C}(u_{p\mu} = \gamma_\mu u_p , \ 1 \leq \mu \leq 4, \ p \in \mathbb{R}^4) ,$$

$$\Omega_2^p = \text{span}_\mathbb{C}(u_{p\mu\nu} = \gamma_\mu \gamma_\nu u_p , \ 1 \leq \mu \leq \nu \leq 4, \ p \in \mathbb{R}^4) ,$$

$$\Omega_3^p = \text{span}_\mathbb{C}(u_{p\mu\nu\rho} = \gamma_\mu \gamma_\nu \gamma_\rho u_p , \ 1 \leq \mu \leq \nu < \rho \leq 4, \ p \in \mathbb{R}^4) ,$$

$$\Omega_4^p = \text{span}_\mathbb{C}(u_{p5} = \gamma_1 \gamma_2 \gamma_3 \gamma_4 u_p , \ p \in \mathbb{R}^4) ,$$

and $\Omega_n^p \equiv 0$ for $n \geq 5$. The product in $\Omega$ is the usual product of tensor products, for instance $u_p u_q (\gamma_\mu u_q) = \gamma_\mu (u_p u_q)$, $(\gamma_\mu u_p) (\gamma_\nu u_q) = \gamma_\nu (u_p u_q)$, etc. The differential is defined as

$$d(\gamma_\mu \cdots \gamma_\nu u_p) := i p^\rho \gamma_\rho \gamma_\mu \cdots \gamma_\nu u_p , \quad (28)$$

with summation over $\rho$ from 1 to 4. The sequence $\gamma_\mu \cdots \gamma_\nu$ might be empty. Developing the differential in a basis we get

$$d u_p = \delta^\mu_p u_{q\mu} , \quad d u_{q\mu} = i p^\rho \delta^\mu_q ,$$

$$d u_{q\mu\nu} = \delta^\mu_{pq} u_{q\nu} , \quad d u_{q\nu\rho} = i (p^\mu \delta^\nu_q - p^\nu \delta^\mu_q) u_{q\nu} . \quad (29)$$

We extend the star to $\Omega$ by $(\gamma_\mu)^* := \gamma_\mu$. The bilinear forms are defined by

$$\langle \gamma_\mu_1 \cdots \gamma_\mu_n, u_p, \gamma_\nu_1 \cdots \gamma_\nu_n, u_q \rangle_n = \delta_{\mu_1 \nu_1} \cdots \delta_{\mu_n \nu_n} \delta_{p,-q} , \quad (30)$$

with $\mu_i < \mu_j , \nu_i < \nu_j$ for $i < j$. The properties (3) and (4) are easy to verify.