Slavnov-Taylor identity in noncommutative geometry

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Abstract

We develop a framework to define quantum Yang-Mills theories on general differential algebras imitating the standard procedure.

1 Field theories on general differential algebras

Let $\mathcal{A}$ be an associative *-algebra over $\mathbb{C}$ and (Ω, d) an $\mathbb{N}$-graded differential *-algebra over $\mathcal{A}$, i.e. $\Omega = \bigoplus_{n=0}^{\infty} \Omega^n$ with $\Omega^0 = \mathcal{A}$ and

$$\Omega^0 = \mathcal{A}\;, \qquad \Omega^k \Omega^l \subset \Omega^{k+l}\;, \qquad d : \Omega^n \to \Omega^{n+1}\;, \qquad {}^* : \Omega^n \to \Omega^n\;, \tag{1}$$
$$(\omega^k \tilde{\omega}^l)^* = (\tilde{\omega}^l)^* (\omega^k)^*\;, \qquad (z\omega^k)^* = \bar{z}\omega^k\;, \tag{2}$$
$$d(\omega^k \tilde{\omega}^l) = d(\omega^k)\,\tilde{\omega}^l + (-1)^l \omega^k\, d(\tilde{\omega}^l)\;, \qquad d(d\omega^k) = 0\;, \tag{3}$$

for $\omega^k \in \Omega^k$, $\tilde{\omega}^l \in \Omega^l$ and $z \in \mathbb{C}$. Moreover, let Ω^n, for each degree n, be equipped with a symmetric non-degenerate positive bilinear form $\langle,\rangle_n : \Omega^n \times \Omega^n \to \mathbb{C}$ which satisfies

$$\begin{aligned} &\langle \omega, \tilde{\omega}\rangle_n = \langle \tilde{\omega}, \omega\rangle_n\;, \qquad \langle \omega a, \tilde{\omega}\rangle_n = \langle \omega, a\tilde{\omega}\rangle_n\;, \qquad \text{for } \omega, \tilde{\omega} \in \Omega^n\;,\; a \in \mathcal{A}\;, \\ &\langle \omega, \omega^*\rangle_n \geq 0 \;\; \forall \omega \in \Omega^n\;, \qquad \langle \omega, \omega^*\rangle_n = 0 \quad \Leftrightarrow \quad \omega = 0\;. \end{aligned} \tag{4}$$

The codifferential $d^* : \Omega^{n+1} \to \Omega^n$ is defined as the adjoint of d via $\langle\;,\;\rangle_n$,

$$\langle da, b\rangle_{n+1} =: \langle a, d^* b\rangle_n\;, \qquad a \in \Omega^n\;, \quad b \in \Omega^{n+1}\;. \tag{5}$$

Our goal is to formulate field theories, i.e. dynamical systems of *amplitudes*. For this purpose it is necessary to assume that there is a basis $\{u_{\mathbf{p}}\}$ in Ω^n labelled by possibly continuous parameters. Instead of taking complex numbers for the amplitudes we allow for Grassmann valued amplitudes $\phi^{\mathbf{p}}_{\mathrm{q},\bar{\mathrm{q}}} \in V_{\mathrm{q},\bar{\mathrm{q}}}$, with

$$\phi^{\mathbf{p}}_{\mathrm{q},\bar{\mathrm{q}}} \phi^{\mathbf{q}}_{\mathrm{r},\bar{\mathrm{r}}} = (-1)^{(\mathrm{q}-\bar{\mathrm{q}})(\mathrm{r}-\bar{\mathrm{r}})} \phi^{\mathbf{q}}_{\mathrm{r},\bar{\mathrm{r}}} \phi^{\mathbf{p}}_{\mathrm{q},\bar{\mathrm{q}}}\;. \tag{6}$$

Let $\mathbb{G} = \bigoplus_{\mathrm{q},\bar{\mathrm{q}}} \mathbb{G}_{\mathrm{q},\bar{\mathrm{q}}}$ be the bigraded Grassmann algebra generated by the amplitudes $\{\phi^{\mathbf{p}}_{i,\mathrm{q},\bar{\mathrm{q}}}\}$ according to (6). Let $\Omega^n_{\mathrm{q},\bar{\mathrm{q}}}$ be the space of $\mathbb{G}_{\mathrm{q},\bar{\mathrm{q}}}$-valued n-forms which contains in particular elements $\phi_i = \phi^{\mathbf{p}}_{i,\mathrm{q},\bar{\mathrm{q}}} u_{\mathbf{p}}$ linear in the amplitudes $\phi^{\mathbf{p}}_{i,\mathrm{q},\bar{\mathrm{q}}} \in V_{\mathrm{q},\bar{\mathrm{q}}}$. Multiplication, differential and bilinear form are extended by linearity from Ω^* to $\Omega^*_{*,*}$. Finally, let

$$\mathcal{C}(\Omega) = \{ z \in \mathcal{A} \;:\; \; dz \equiv 0\;, \;\; [z, \omega] \equiv 0 \;\; \forall \omega \in \Omega^*_{*,*}\; \} \tag{7}$$

be the set of 'constant' central elements of $\Omega^*_{*,*}$.

Let $F \in \mathbb{G}_{\mathrm{r},\bar{\mathrm{r}}}$ be some power series of $\{\phi^{\mathbf{P}}_{i,\mathrm{s},\bar{\mathrm{s}}}\}$. We define functional derivatives

$$\frac{\partial}{\partial\phi_i} : \mathbb{G}^{\mathrm{r},\bar{\mathrm{r}}} \ni F \mapsto \frac{\partial F}{\partial\phi_i} \in \Omega^n_{\mathrm{r-q},\bar{\mathrm{r}}-\bar{\mathrm{q}}} \qquad \text{by} \tag{8}$$
$$\lim_{\epsilon\to 0} \frac{1}{\epsilon}\Big(F\Big|_{\phi_i^{\mathbf{P}}\mapsto\phi_i^{\mathbf{P}}+\epsilon x^{\mathbf{P}}} - F - \Big\langle \epsilon x, \frac{\partial F}{\partial\phi_i}\Big\rangle_n\Big) = 0 \;, \qquad \forall\, x = u_{\mathbf{P}}x^{\mathbf{P}} \in \Omega^n_{\mathrm{q},\bar{\mathrm{q}}} \;.$$

Definition 1 *A* <u>*field*</u> *is a homogeneous element of* $\Omega^n_{\mathrm{q},\bar{\mathrm{q}}}$ *linear in the amplitudes spanning* $V_{\mathrm{q},\bar{\mathrm{q}}}$ *with a* <u>*dimension*</u>

$$\dim = n + \mathrm{q} + \bar{\mathrm{q}} \tag{9}$$

assigned to the amplitudes of the field. The numbers $(n;\mathrm{q},\bar{\mathrm{q}})$ *are called* (degree; ghost number, antighost number) *of the field. The differential* $d : \Omega^n_{\mathrm{q},\bar{\mathrm{q}}} \to \Omega^{n+1}_{\mathrm{q},\bar{\mathrm{q}}}$ *is counted as an element of type* $(1;0,0)$.

A <u>*local field monomial*</u> *is the product (in* $\Omega^*_{*,*}$*) of fields and differentials of fields. The dimension of the field monomial is the sum of the dimensions of fields and differentials in it. (By construction, degree and ghost-antighost numbers the field monomial is the sum of the degrees and ghost-antighost numbers, respectively, of the fields and differentials in it.)*

An <u>*integrated local field monomial*</u> *is the contraction of two local field monomials of the same degree* n *via the weighted bilinear form* $\langle z\,.\,,\,.\,\rangle_n$*, with invertible* $z \in \mathcal{C}(\Omega)$.

A <u>*classical action*</u> *is a linear combination of integrated local field monomials (with different weights* z*) of dimension* $\leq D$ *and* <u>*balanced*</u> *ghost-antighost numbers* $\mathrm{q} = \bar{\mathrm{q}}$.

Definition 2 *A* <u>*Yang-Mills theory*</u> *on* $\Omega^*_{*,*}$ *in dimension* $D = 4$ *is a theory of the fields* $A, \rho, c, \sigma, \bar{c}, B$ *whose degrees* n*, ghost-antighost numbers* $\mathrm{q}-\bar{\mathrm{q}}$ *and dimensions* dim *are given in the following table:*

	A	ρ	c	σ	$\bar{c}$	B	d
n	1	1	0	0	0	0	1
q	0	0	1	0	0	1	0
$\bar{\mathrm{q}}$	0	1	0	2	1	1	0
dim	1	2	1	2	1	2	1

We assume that there exist configurations A, c *such that*

$$\alpha\, Acc + \beta\, ccA = 0 \;, \qquad \gamma\, d(cc) = 0 \;, \tag{10}$$

for $\alpha, \dots, \zeta \in \mathcal{C}(\Omega)$ *not necessarily positive, have only the solution* $\alpha{=}\dots{=}\zeta{=}0$*. Then, the theory is governed by the* <u>*Slavnov-Taylor operator*</u> $\mathcal{S}$ *defined on functionals* $\Gamma \in \mathbb{G}$ *of the amplitudes of* $A, \rho, c, \sigma, \bar{c}, B$ *by*

$$\mathcal{S}(\Gamma) = \Big\langle \frac{\partial\Gamma}{\partial\rho}, \frac{\partial\Gamma}{\partial A}\Big\rangle_1 + \Big\langle \frac{\partial\Gamma}{\partial\sigma}, \frac{\partial\Gamma}{\partial c}\Big\rangle_0 + \Big\langle B, \frac{\partial\Gamma}{\partial\bar{c}}\Big\rangle_0 \;. \tag{11}$$

We are looking for the most general solution Γ of the *Slavnov-Taylor identity* $\mathcal{S}(\Gamma)=0$, where Γ is a classical action, i.e. an *integrated local field polynomial* in $A, \rho, c, \sigma, \bar{c}, B$ with dimension ≤ 4 and balanced ghost-antighost number $\mathrm{q}=\bar{\mathrm{q}}$. The answer is

Proposition 3 *If (10) is true and if $\frac{\partial\Gamma}{\partial\rho}\not\equiv 0$ and $\frac{\partial\Gamma}{\partial\sigma}\not\equiv 0$, the most general classical non-abelian Yang-Mills action satisfying the Slavnov-Taylor identity is*

$$\begin{aligned}\Gamma = & -\langle \tfrac{1}{4g^2}\, F, F\rangle_2 + \langle \tfrac{\alpha}{2}\, B, B\rangle_0 + \langle d\bar{c}{+}\rho,\ dc{+}[A,c]\rangle_1 - \langle dB, A\rangle_1 - \langle \sigma, cc\rangle_0 \\ & + \langle \beta\, \bar{c}\bar{c}, cc\rangle_0 - \langle \beta\, B, \{\bar{c},c\}\rangle_0 + \langle \gamma\, BA, A\rangle_1 + \langle \gamma\, dc, \{A,\bar{c}\}\rangle_1 + \langle \gamma\, A\{\bar{c},c\}, A\rangle_1 \ ,\end{aligned} \tag{12}$$

with $F=dA{+}AA$, up to a rescaling $A_\mu\mapsto\xi_1 A_\mu$, $\rho^\mu\mapsto\rho^\mu/\xi_1$, $c\mapsto\xi_2 c$, $\sigma\mapsto\sigma/\xi_2$, $\bar{c}\mapsto\xi_3\bar{c}$ and $B\mapsto\xi_3 B$, which leave the Slavnov-Taylor identity unchanged. Here, $g,\alpha,\beta,\gamma,\xi_i$ are positive central elements of $\mathcal{A}$.

There are three degenerate (static) solutions where some parts whose coefficients in (12) are normalized to 1 are missing, given as combinations of

1) $dA=0$ and $dc=0$, $g\mapsto 1$,   additional term $-\langle \frac{m^2}{2} A, A\rangle_1$,

2) $dB=0$ and $d\bar{c}=0$, $\beta\mapsto 1$,

(although these differentials may actually be non-zero).

For the proof one writes down the most general local field polynomial of dimension ≤ 4 and balanced ghost-antighost number and applies the Slavnov-Taylor operator.

It is convenient to impose the gauge fixing condition

$$\frac{\partial\Gamma}{\partial B}=\alpha B - d^*A\ . \tag{13}$$

2 Generating functionals

The classical action $\Gamma[A,\rho,c,\sigma,\bar{c},B]$ is regarded as a special example of a generating functional of 1PI (one-particle irreducible) Green's functions. In general, deriving such a functional with respect to the fields $\phi_i=\{A,\rho,c,\sigma,\bar{c},B\}$ (considered as test functions), $\Gamma_{1\ldots n}:=\frac{\partial^n\Gamma}{\partial\phi_1\ldots\partial\phi_n}\Big|_{\phi_i=0}$, one can associate to $\Gamma_{1\ldots n}$ a graph which remains connected after cutting an arbitrary line. In particular, external lines of $\Gamma_{1\ldots n}$ are amputated.

In the general case on can pass from Γ to a generating functional Z^c of connected Green's functions by Legendre transformation

$$Z^c[J,\mathcal{J},\bar{\jmath},\jmath,\rho,\sigma]:=\Gamma[A,B,c,\bar{c},\rho,\sigma]+\langle A,J\rangle_1+\langle B,\mathcal{J}\rangle_0+\langle\bar{\jmath},c\rangle_0+\langle\jmath,\bar{c}\rangle_0\ , \tag{14}$$

where the fields $A,B,c,\bar{c}$ have to be replaced by the (inverse) solution of

$$J=-\frac{\partial\Gamma}{\partial A}\in\Omega^1_0\,,\quad \mathcal{J}=-\frac{\partial\Gamma}{\partial B}\in\Omega^0_0\,,\quad \jmath=\frac{\partial\Gamma}{\partial\bar{c}}\in\Omega^0_1\,,\quad \bar{\jmath}=\frac{\partial\Gamma}{\partial c}\in\Omega^0_{-1}\ . \tag{15}$$

with $\Omega^n_Q = \bigoplus_{\mathrm{q=max}(-Q,0)}^{\infty} \Omega^n_{\mathrm{q+Q,q}}$. The generating functional of general (not necessarily connected) Green's functions is defined as

$$Z := \mathrm{e}^{-\frac{1}{\hbar}Z^c} \ . \tag{16}$$

In particular, we can take for Γ the bilinear part Γ_{bil} of the gauge fixed classical action Γ_{cl}:

$$\Gamma_{\mathrm{bil}} = -\langle \tfrac{1}{4g^2}\, dA, dA\rangle_2 + \langle \tfrac{\alpha}{2}\, B, B\rangle_0 - \langle dB, A\rangle_1 + \langle d\bar{c}, dc\rangle_1 \ . \tag{17}$$

$$-(s{-}1)^2M^2\langle \tfrac{1}{2g^2}\, A, A\rangle_1 + (s{-}1)^2M^2\langle \tfrac{\alpha}{g^2}\, \bar{c}, c\rangle_0 \ . \tag{18}$$

The mass terms proportional to $(s{-}1)^2M^2$ are auxiliary ones to deal with possible infrared divergences. It is convenient not to include $\langle\rho, dc\rangle_1$ in Γ_{bil}.

Restricted for the moment to the bilinear part we obtain

$$\begin{aligned} J &= \tfrac{1}{g^2}(\tfrac{1}{2}d^*d + (s{-}1)^2M^2)A + dB \ , & j &= (d^*d + \tfrac{\alpha}{g^2}(s{-}1)^2M^2)c \ , \\ \mathcal{J} &= -\alpha B + d^*A \ , & \bar{\jmath} &= -(d^*d + \tfrac{\alpha}{g^2}(s{-}1)^2M^2)\bar{c} \ . \end{aligned} \tag{19}$$

This gives

$$\begin{aligned} A &= -g^2\tilde{\Delta}(J + \tfrac{1}{\alpha}d\mathcal{J}) \ , & c &= -\Delta j \ , \\ B &= -\tfrac{g^2}{\alpha}d^*\tilde{\Delta}J + \tfrac{1}{g^2}(s{-}1)^2M^2\Delta\mathcal{J} \ , & \bar{c} &= \Delta\bar{\jmath} \ , \end{aligned} \tag{20}$$

with the propagators $\tilde{\Delta}, \Delta$ defined by

$$\tilde{\Delta}\big(\tfrac{1}{2}d^*d + \tfrac{g^2}{\alpha}dd^* + (s{-}1)^2M^2\big) = -\mathrm{id}_{\Omega^1} \ , \quad \Delta\big(d^*d + \tfrac{\alpha}{g^2}(s{-}1)^2M^2\big) = -\mathrm{id}_{\mathcal{A}} \ .$$

We have used the identity $\frac{g^2}{\alpha}\tilde{\Delta}d = d\Delta$.

A lengthy but straightforward computation leads to

$$Z^c_{\mathrm{bil}} = -\langle \tfrac{g^2}{2}\, J, \tilde{\Delta}J\rangle_1 - \langle \tfrac{g^2}{\alpha}\, d\mathcal{J}, \tilde{\Delta}J\rangle_1 + (s{-}1)^2M^2\langle \tfrac{1}{2g^2}\, \mathcal{J}, \Delta\mathcal{J}\rangle_0 - \langle \bar{\jmath}, \Delta j\rangle_0 \tag{21}$$

and consequently to

$$Z_{\mathrm{bil}}[J,\mathcal{J},j,\bar{\jmath}] = \mathrm{e}^{\left(\left\langle \frac{g^2}{2\hbar}\, J,\tilde{\Delta}J\right\rangle_1 + \left\langle \frac{g^2}{\alpha\hbar}\, d\mathcal{J},\tilde{\Delta}J\right\rangle_1 - (s-1)^2M^2\left\langle \frac{1}{2g^2\hbar}\, \mathcal{J},\Delta\mathcal{J}\right\rangle_0 + \frac{1}{\hbar}\left\langle \bar{\jmath},\Delta j\right\rangle_0\right)} \ . \tag{22}$$

We quantize our theory axiomatically by the principle that the full generating functional is given by

$$\begin{aligned} &Z[\rho,\sigma,J,J_B,j,\bar{\jmath}] \\ &\quad := \mathcal{N}\mathrm{e}^{-\frac{1}{\hbar}\Gamma_{\mathrm{int}}[A,c,\bar{c},B,\rho,\sigma]}\Big|_{A\mapsto -\hbar\frac{\partial}{\partial J}\ ,\ c\mapsto -\hbar\frac{\partial}{\partial\bar{\jmath}}\ ,\ \bar{c}\mapsto -\hbar\frac{\partial}{\partial j}\ ,\ B\mapsto -\hbar\frac{\partial}{\partial\mathcal{J}}} Z_{\mathrm{bil}}[J,\mathcal{J},j,\bar{\jmath}] \ , \end{aligned} \tag{23}$$

where $\Gamma_{\mathrm{int}} = \Gamma_{\mathrm{cl}} - \Gamma_{\mathrm{bil}}|_{s=1}$ and $\mathcal{N}$ is an (ill-defined) normalization factor determined by $Z[0] = 1$. In many cases the expansion of (23) leads to infinities even if the possible problem with $\mathcal{N}$ is ignored. We have to fix a regularization scheme so that (23) becomes a formal power series in $\hbar$ consisting of finite terms. It is not important for us whether the series converges or not.

Due to $\dfrac{\partial Z^c}{\partial \rho} = \dfrac{\partial \Gamma}{\partial \rho}$ and $\dfrac{\partial Z^c}{\partial \sigma} = \dfrac{\partial \Gamma}{\partial \sigma}$ we have

$$\begin{aligned} \mathcal{S}\Gamma &= \Big\langle \frac{\partial \Gamma}{\partial \rho}, \frac{\partial \Gamma}{\partial A} \Big\rangle_1 + \Big\langle \frac{\partial \Gamma}{\partial \sigma}, \frac{\partial \Gamma}{\partial c} \Big\rangle_0 + \Big\langle B, \frac{\partial \Gamma}{\partial \bar{c}} \Big\rangle_0 \\ &= \Big\langle -J, \frac{\partial Z^c}{\partial \rho} \Big\rangle_1 + \Big\langle \bar{\jmath}, \frac{\partial Z^c}{\partial \sigma} \Big\rangle_0 + \Big\langle j, \frac{\partial Z^c}{\partial \mathcal{J}} \Big\rangle_0 \equiv \mathcal{S}Z^c \ , \end{aligned} \tag{24}$$

when expressing both lines of the equation in terms of the same variables. Since we have a more explicit formula for Z than for Z^c, the identity (24) suggests to study the problem

$$\mathcal{S}Z := \Big\langle -J, \frac{\partial Z}{\partial \rho} \Big\rangle_1 + \Big\langle \bar{\jmath}, \frac{\partial Z}{\partial \sigma} \Big\rangle_0 + \Big\langle j, \frac{\partial Z}{\partial \mathcal{J}} \Big\rangle_0 \ . \tag{25}$$

For a given model, i.e. given $(\Omega, d, \langle\ ,\ \rangle)$, it is possible to compute $\mathcal{S}Z$. On a formal level one always obtains $\mathcal{S}Z = 0$, however, the counterterms introduced to remove the infinities will lead in general to corrections breaking the Slavnov-Taylor identity. These corrections have to be characterized by the *quantum action principle* concerning dimension, ghost-antighost numbers and structure of the field monomials. The model is called *perturbatively renormalizable* if $\mathcal{S}Z = 0$ can be achieved when replacing the classical action $\Gamma_{\rm cl}$ defining Z by a power series in $\hbar$ of the same form as $\Gamma_{\rm cl}$.

Out of Z given by (23) we obtain the generating functional of the connected Green's functions Z^c via (16) and the generating functional Γ of the 1PI Green's functions by inverting (14):

$$\Gamma[\rho, \sigma, A, c, \bar{c}, B] = Z^c[\rho, \sigma, J, \mathcal{J}, j, \bar{\jmath}] - \langle A, J\rangle_1 - \langle B, \mathcal{J}\rangle_0 - \langle \bar{\jmath}, c\rangle_0 - \langle j, \bar{c}\rangle_0 \ ,$$

where the sources $J, \mathcal{J}, j, \bar{\jmath}$ are replaced by the solution of

$$A = \frac{\partial Z^c}{\partial J} \ , \qquad c = \frac{\partial Z^c}{\partial \bar{\jmath}} \ , \qquad \bar{c} = \frac{\partial Z^c}{\partial j} \ , \qquad B = \frac{\partial Z^c}{\partial \mathcal{J}} \ .$$

The functional Γ is a formal power series in $\hbar$, $\Gamma = \sum_{n=0}^{\infty} \hbar^n \Gamma_{(n)}$, and one can check that $\Gamma_{(0)} = \Gamma_{{\rm cl},s} = \Gamma_{\rm int} + \Gamma_{\rm bil}$. Moreover, the contributions to $\Gamma_{(n)}$ correspond to Feynman graphs with n loops. One easily convinces oneself that $\Gamma_{(n)}$ for $n > 0$ cannot be written as contractions via $\langle\ ,\ \rangle_k$. The only way to evaluate it is in components with respect to a basis for a concrete model.

The three Slavnov-Taylor identities $\mathcal{S}Z = 0$, $\mathcal{S}Z^c = 0$ and $\mathcal{S}\Gamma = 0$ for generating functionals of renormalized Green's functions are equivalent.

3 Example: The noncommutative $\mathbb{R}^4$

An example of an algebra fitting into our setting is the noncommutative $\mathbb{R}^4$. We consider four hermitian 'coordinates' x_μ, $\mu = 1, \dots, 4$, satisfying

$$[x_\mu, x_\mu] = -2\mathrm{i}\pi\theta_{\mu\nu} \;, \qquad \theta_{\mu\nu} = -\theta_{\nu\mu} \in \mathbb{R} \;.$$

Introducing $u_p := \exp(2\mathrm{i}\pi p^\mu x_\mu) = 1 + 2\mathrm{i}\pi p^\mu x_\mu + (1/2!)(2\mathrm{i}\pi p^\mu x_\mu)^2 + \dots$, with $p = (p^1, p^2, p^3, p^4) \in \mathbb{R}^4$, the Baker-Campbell-Hausdorff formula gives

$$u_p u_q = \mathrm{e}^{\mathrm{i}\theta(p,q)} u_{p+q} \;, \qquad \theta(p,q) = -\theta(q,p) = \theta_{\mu\nu} p^\mu q^\nu \;. \tag{26}$$

Moreover, $(u_p)^* = u_{-p}$. The noncommutative $\mathbb{R}^4$ is the algebra spanned by $\{u_p\}$ and will be denoted by $\mathbb{R}^4_\theta$. The differential algebra (Ω_θ, d) over $\mathbb{R}^4_\theta$ is the tensor product of $\mathbb{R}^4_\theta$ with some Grassmann algebra of $D = 4$ generators $\{\gamma_\mu\}_{\mu=1,\dots,4}$, satisfying $\gamma_\mu\gamma_\nu = -\gamma_\nu\gamma_\mu$. Then,

$$\begin{aligned}
\Omega^0_\theta &= \mathrm{span}_{\mathbb{C}}(u_p \;, \;\; p \in \mathbb{R}^4) \equiv \mathbb{R}^4_\theta \;, \\
\Omega^1_\theta &= \mathrm{span}_{\mathbb{C}}(u_{p\mu} = \gamma_\mu u_p \;, \;\; 1 \le \mu \le 4, \; p \in \mathbb{R}^4) \;, \\
\Omega^2_\theta &= \mathrm{span}_{\mathbb{C}}(u_{p\mu\nu} = \gamma_\mu\gamma_\nu u_p \;, \;\; 1 \le \mu < \nu \le 4, \; p \in \mathbb{R}^4) \;, \\
\Omega^3_\theta &= \mathrm{span}_{\mathbb{C}}(u_{p\mu\nu\rho} = \gamma_\mu\gamma_\nu\gamma_\rho u_p \;, \;\; 1 \le \mu < \nu < \rho \le 4, \; p \in \mathbb{R}^4) \;, \\
\Omega^4_\theta &= \mathrm{span}_{\mathbb{C}}(u_{p5} = \gamma_1\gamma_2\gamma_3\gamma_4 u_p \;, \;\; p \in \mathbb{R}^4) \;,
\end{aligned} \tag{27}$$

and $\Omega^n_\theta \equiv 0$ for $n \ge 5$. The product in Ω is the usual product of tensor products, for instance $u_p u_(\gamma_\mu u_q) = \gamma_\mu(u_p u_q)$, $(\gamma_\mu u_p)(\gamma_\nu u_q) = \gamma_\mu\gamma_\nu(u_p u_q)$, etc. The differential is defined as

$$d(\gamma_\mu \cdots \gamma_\nu u_p) := \mathrm{i}p^\rho \gamma_\rho \gamma_\mu \cdots \gamma_\nu u_p \;, \tag{28}$$

with summation over ρ from 1 to 4. The sequence $\gamma_\mu \cdots \gamma_\nu$ might be empty. Developing the differential in a basis we get

$$\begin{aligned}
du_p &= d^{q\mu}_p u_{q\mu} \;, & d^{q\mu}_p &= \mathrm{i}p^\mu \delta^q_p \;, \\
du_{p\rho} &= d^{q\mu\nu}_{p\rho} u_{q\mu\nu} \;, & d^{q\mu\nu}_{p\rho} &= \mathrm{i}(p^\mu \delta^\nu_\rho - p^\nu \delta^\mu_\rho)\delta^q_p \;.
\end{aligned} \tag{29}$$

We extend the star to Ω by $(\gamma_\mu)^* := \gamma_\mu$. The bilinear forms are defined by

$$\langle \gamma_{\mu_1} \cdots \gamma_{\mu_n} u_p, \gamma_{\nu_1} \cdots \gamma_{\nu_n} u_q \rangle_n = \delta_{\mu_1\nu_1} \cdots \delta_{\mu_n\nu_n} \delta_{p,-q} \;, \tag{30}$$

with $\mu_i < \mu_j$, $\nu_i < \nu_j$ for $i < j$. The properties (3) and (4) are easy to verify.