Noncommutative Quantum Field Theory

Harald GROSSE[1] and Raimar WULKENHAAR[2]

[1] *Fakultät für Physik, Universität Wien*
Boltzmanngasse 5, A-1090 Wien, Austria

[2] *Mathematisches Institut der Westfälischen Wilhelms-Universität*
Einsteinstraße 62, D-48149 Münster, Germany

Abstract

We summarize our recent construction of the ϕ^4-model on four-dimensional Moyal space. This is achieved by solving the quartic matrix model for a general external matrix in terms of the solution of a non-linear equation for the 2-point function and the eigenvalues of that matrix. The β-function vanishes identically. For the Moyal model, the theory of Carleman type singular integral equations reduces the construction to a fixed point problem. The resulting Schwinger functions in position space are symmetric and invariant under the full Euclidean group. The Schwinger 2-point function is reflection positive iff the diagonal matrix 2-point function is a Stieltjes function.

1 Introduction

Perturbatively renormalised quantum field theory is an enormous phenomenological success, a success which lacks a mathematical understanding. The perturbation series is at best an asymptotic expansion which cannot converge at physical coupling constants. Some physical effects such as confinement are out of reach for perturbation theory. In two and partly three dimensions, methods of constructive physics [GJ87, Riv91], often combined with the Euclidean approach [Sch59, OS73, OS75], were used to rigorously establish quantum field theory models.

In four dimensions there was little success so far. It is generally believed that due to asymptotic freedom, non-Abelian gauge theory (i.e. Yang-Mills theory) has the chance of a rigorous construction. But this is a hard problem [JW00]. What makes it so difficult is the fact that any simpler model such as quantum electrodynamics or the $\lambda\phi^4$-model cannot be constructed in four dimensions (Landau ghost problem [LAK54] or triviality [Aiz81, Frö82]).

One of the main difficulties is the *non-linearity* of the models under consideration. Fixed point methods provide a standard approach to non-linear problems,

[1]harald.grosse@univie.ac.at

[2]raimar@math.uni-muenster.de

but they are rarely used in quantum field theory. In this contribution we review a sequence of papers [GW14a, GW13b, GW14b] in which we successfully used symmetry and fixed point methods to exactly solve a toy model for a quantum field theory in four dimensions.

1. Following [GW14a], we show in sec. 2 that a Ward identity for the $U(\infty)$ group action leads to an exact solution of the quartic matrix model $\mathcal{Z} = \int \mathcal{D}[\Phi]\ \exp(\mathrm{trace}(J\Phi - E\Phi^2 - \frac{\lambda}{4}\Phi^4))$ in terms of the solution of a non-linear equation. As by-product we find that any renormalisable quartic matrix model has vanishing β-function. All these steps are completely elementary.
2. Self-dual ϕ^4_4-theory on Moyal space [GW05b, GW05c] is of that type. For extreme noncommutativity $\theta \to \infty$, and after careful discussion of thermodynamic and continuum limit, the non-linear equation is reduced to a fixed-point problem [GW14a] which has a unique non-perturbative and non-trivial solution for $\lambda < 0$ [GW14b]. The key step is the observation that a certain difference function satisfies a linear singular integral equation of Carleman type [Car22, Tri57].
3. Following [GW13b], we identify in sec. 4 a limit to Schwinger functions for a scalar field on $\mathbb{R}^4$. Surprisingly for a highly noncommutative model, these Schwinger functions show full Euclidean symmetry. Otherwise they have unusual properties such as absent momentum transfer in interaction processes. This seems to suggest triviality, but the numerical investigation [GW14b] of the 2-point function shows scattering remnants from a noncommutative geometrical substructure. Most surprisingly, the Schwinger 2-point function seems to be reflection positive in one of its phases.

2 Exact solution of the quartic matrix model

To define a *Euclidean quantum field theory* for a matrix $\Phi \in \mathcal{L}^2(H)$ we give ourselves an action functional

$$S[\Phi] = V\,\mathrm{tr}(E\Phi^2 + P[\Phi])\ . \tag{1}$$

Here, $P[\Phi]$ is a polynomial in Φ with scalar coefficients, and this alone would be a familiar action in the theory of matrix models [DGZ95]. To be closer to field theory on a (compact) manifold $\mathcal{M}$ we add the analogue of the kinetic term $\int_{\mathcal{M}} dx\ (-\Delta\phi)\phi$, that is, we require the external matrix E to be an unbounded selfadjoint positive operator on H with compact resolvent. The volume V will play a crucial rôle. The construction involves several regularisation and limiting procedures. One such regularisation consists in a finite size $\mathcal{N}$ for the matrices, and V will be a certain function of $\mathcal{N}$ which together with $\mathcal{N}$ is sent to ∞.

Adding a source term to the action, we define the *partition function* as

$$\mathcal{Z}[J] = \int \mathcal{D}[\Phi]\ \exp(-S[\Phi] + V\,\mathrm{tr}(\Phi J))\ , \tag{2}$$

where $\mathcal{D}[\Phi]$ is the extension of the Lebesgue measure from finite-rank operators to $\mathcal{L}^2(H)$ and J a test function matrix. What we want, and what we achieve, is to construct $\frac{\mathcal{D}[\Phi]}{\mathcal{Z}[0]}$ for $P[\Phi] = \frac{\lambda}{4}\Phi^4$ in the limit $V \to \infty$. Such a limit cannot be expected for $\mathcal{Z}$. Instead, we pass to the generating functional $\log \mathcal{Z}[J]$ of *connected correlation functions.*

2.1 Ward identity and topological expansion

Unitary operators U belonging to an appropriate unitisation of the compact operators on H give rise to a transformation $\Phi \mapsto \tilde{\Phi} = U\Phi U^*$. Since the space of selfadjoint compact operators is invariant under the adjoint action, we have

$$\int \mathcal{D}[\Phi]\; \exp(-S[\Phi] + V\,\mathrm{tr}(\Phi J)) = \int \mathcal{D}[\tilde{\Phi}]\; \exp(-S[\tilde{\Phi}] + V\,\mathrm{tr}(\tilde{\Phi} J))\;.$$

Unitary invariance $\mathcal{D}[\tilde{\Phi}] = \mathcal{D}[\Phi]$ of the Lebesgue measure implies

$$0 = \int \mathcal{D}[\Phi\; \Big\{ \exp(-S[\Phi] + V\,\mathrm{tr}(\Phi J)) - \exp(-S[\tilde{\Phi}] + V\,\mathrm{tr}(\tilde{\Phi} J)) \Big\}\;.$$

Note that the integrand $\{\dots\}$ itself does not vanish because $\mathrm{tr}(E\Phi^2)$ and $\mathrm{tr}(\Phi J)$ are not unitarily invariant; we only have $\mathrm{tr}(P[\Phi]) = \mathrm{tr}(P[\tilde{\Phi}])$ due to $UU^* = U^*U = \mathrm{id}$ together with the trace property. Linearisation of U about the identity operator leads to the *Ward identity*

$$0 = \int \mathcal{D}[\Phi]\; \Big\{ E\Phi\Phi - \Phi\Phi E - J\Phi + \Phi J \Big\} \exp(-S[\Phi] + V\,\mathrm{tr}(\Phi J))\;. \qquad (3)$$

We can always choose an orthonormal basis of H where E is diagonal (but J is not). Since E is of compact resolvent, E has eigenvalues $E_a > 0$ of finite multiplicity μ_a. We thus label the matrices by an enumeration of the (necessarily discrete) eigenvalues of E and an enumeration of the basis vectors of the finite-dimensional eigenspaces. Writing Φ in $\{\dots\}$ of (3) as functional derivative $\Phi_{ab} = \frac{\partial}{V\partial J_{ba}}$, we have proved (first obtained in [DGMR07]):

Proposition 1 *The partition function $\mathcal{Z}[J]$ of the matrix model defined by the external matrix E satisfies the $|I| \times |I|$ Ward identities*

$$0 = \sum_{n \in I} \Big(\frac{(E_a - E_p)}{V} \frac{\partial^2 \mathcal{Z}}{\partial J_{an} \partial J_{np}} + J_{pn} \frac{\partial \mathcal{Z}}{\partial J_{an}} - J_{na} \frac{\partial \mathcal{Z}}{\partial J_{np}} \Big)\;. \qquad (4)$$

Compactness of the resolvent of E implies that at the expense of adding a measure $\mu_{[m]} = \dim\ker(E - E_m \mathrm{id})$, we can assume that $m \mapsto E_m$ is injective.

In a perturbative expansion, Feynman graphs in matrix models are *ribbon graphs.* Viewed as simplicial complexes, they encode the topology (B, g) of a genus-g Riemann surface with B boundary components. The k^{th} boundary face is characterised by $N_k \geq 1$ external double lines to which we attach the source matrices J. Since E is diagonal, the matrix index is conserved along each strand of the ribbon graph. Therefore, the right index of J_{ab} coincides with the left index of another J_{bc}, or of the same J_{bb}. Accordingly, the k^{th} boundary component carries a cycle $J^{N_k}_{p_1\dots p_{N_k}} := \prod_{j=1}^{N_k} J_{p_j p_{j+1}}$ of N_k external sources, with $N_k + 1 \equiv 1$. This implies the following expansion of $\log \mathcal{Z}[J]$ according to the cycle structure:

$$\log \frac{\mathcal{Z}[J]}{\mathcal{Z}[0]} = \sum_{B=1}^{\infty} \sum_{1\leq N_1 \leq \dots \leq N_B}^{\infty} \sum_{p_1^\beta,\dots,p^\beta_{N_\beta} \in I} \frac{V^{2-B}}{S_{N_1\dots N_B}} G_{|p_1^1\dots p^1_{N_1}|\dots|p_1^B\dots p^B_{N_B}|} \prod_{\beta=1}^{B} \Big(\frac{J^{N_\beta}_{p_1^\beta\dots p^\beta_{N_\beta}}}{N_\beta} \Big) . \tag{5}$$

The symmetry factor $S_{N_1\dots N_B}$ is obtained as follows: If ν_i of the B numbers N_β in a given tuple $(N_1,\dots,N_B)$ are equal to i, then $S_{N_1\dots N_B} = \prod_{i=1}^{N_B} \nu_i!$.

Theorem 2

$$\begin{aligned}\sum_{n\in I} \frac{\partial^2 \mathcal{Z}[J]}{\partial J_{an}\partial J_{np}} &= \delta_{ap}\Big\{ V^2 \sum_{(K)} \frac{J_{P_1}\cdots J_{P_K}}{S_{(K)}} \Big(\sum_{n\in I} \frac{G_{|an|P_1|\dots|P_K|}}{V^{|K|+1}} + \frac{G_{|a|a|P_1|\dots|P_K|}}{V^{|K|+2}} \\ &\qquad + \sum_{r\geq 1} \sum_{q_1,\dots,q_r\in I} \frac{G_{|q_1 a q_1\dots q_r|P_1|\dots|P_K|} J^r_{q_1\dots q_r}}{V^{|K|+1}} \Big) \\ &\qquad + V^4 \sum_{(K),(K')} \frac{J_{P_1}\cdots J_{P_K} J_{Q_1}\cdots J_{Q_{K'}}}{S_{(K)}S_{(K')}} \frac{G_{|a|P_1|\dots|P_K|}}{V^{|K|+1}} \frac{G_{|a|Q_1|\dots|Q_{K'}|}}{V^{|K'|+1}} \Big\} \mathcal{Z}[J] \\ &\quad + \frac{V}{E_p - E_a} \sum_{n\in I} \Big(J_{pn} \frac{\partial \mathcal{Z}[J]}{\partial J_{an}} - J_{na} \frac{\partial \mathcal{Z}[J]}{\partial J_{np}} \Big) .\end{aligned} \tag{6}$$

Proof. We identify four sources of a singular contribution $\sim \delta_{ap}$. The other types of derivatives, collected into $\big(\sum_{n\in I} \frac{\partial^2 \mathcal{Z}[J]}{\partial J_{an}\partial J_{np}} \big)_{\text{reg}}$, persist for $a \neq p$.

2.2 Schwinger-Dyson equations

We can write the action as $S = \frac{V}{2}\sum_{a,b}(E_a + E_b)\Phi_{ab}\Phi_{ba} + V S_{int}[\Phi]$, where E_a are the eigenvalues of E. Functional integration yields, up to an irrelevant constant,

$$\mathcal{Z}[J] = e^{-V S_{int}[\frac{\partial}{V\partial J}]} e^{\frac{V}{2}\langle J,J\rangle_E} , \qquad \langle J, J\rangle_E := \sum_{m,n\in I} \frac{J_{mn}J_{nm}}{E_m + E_n} . \tag{7}$$

Instead of a perturbative expansion of $e^{-V S_{int}[\frac{\partial}{V\partial J}]}$, we apply those J-derivatives to (7) which give rise to a correlation function $G_{\dots}$ on the lhs. On the rhs of (7), these external derivatives combine with internal derivatives from $S_{int}[\frac{\partial}{V\partial J}]$

to certain identities for $G_{\dots}$. These Schwinger-Dyson equations are often of little use because they express an N-point function in terms of $(N{+}2)$-point functions.

In the field-theoretical matrix models under consideration, the Ward identity (6) lets this tower of Schwinger-Dyson equation collapse. Computing $G_{|ab|} = \frac{1}{V\mathcal{Z}[0]}\frac{\partial^2\mathcal{Z}[J]}{\partial J_{ba}\partial J_{ab}}\Big|_{J=0}$ we find with (6) the following result [GW14a]:

Proposition 3 *The 2-point function of a quartic matrix model with action $S = V\,\mathrm{tr}(E\Phi^2 + \frac{\lambda}{4}\Phi^4)$ satisfies for injective $m \mapsto E_m$ the Schwinger-Dyson equation*

$$
\begin{aligned}
G_{|ab|} &= \frac{1}{E_a+E_b} - \frac{\lambda}{E_a+E_b}\frac{1}{V}\sum_{p\in I}\Big(G_{|ab|}G_{|ap|} - \frac{G_{|pb|}-G_{|ab|}}{E_p-E_a}\Big) && \text{(8a)}\\
&\quad - \frac{\lambda}{V^2(E_a+E_b)}\Big(G_{|a|a|}G_{|ab|} + \frac{1}{V}\sum_{n\in I}G_{|an|ab|} \\
&\qquad\qquad + G_{|aaab|} + G_{|baba|} - \frac{G_{|b|b|}-G_{|a|b|}}{E_b-E_a}\Big) && \text{(8b)}\\
&\quad - \frac{\lambda}{V^4(E_a+E_b)}G_{|a|a|ab|}\;. && \text{(8c)}
\end{aligned}
$$

It can be checked [GW14a] that in a genus expansion $G_{\dots} = \sum_{g=0}^{\infty} V^{-2g}G^{(g)}_{\dots}$ (which is probably not convergent but Borel summable), precisely the line (8a) preserves the genus, the lines (8b) increase $g \mapsto g+1$ and the line (8c) increases $g \mapsto g+2$. In particular, in a scaling limit $V \to \infty$ with $\frac{1}{V}\sum_{p\in I}$ finite, the exact Schwinger-Dyson equation for $G_{|ab|}$ coincides with its restriction (8a) to the planar sector $g = 0$, a closed non-linear equation for $G^{(0)}_{|ab|}$ alone:

$$
G^{(0)}_{|ab|} = \frac{1}{E_a+E_b} - \frac{\lambda}{E_a+E_b}\frac{1}{V}\sum_{p\in I}\Big(G^{(0)}_{|ab|}G^{(0)}_{|ap|} - \frac{G^{(0)}_{|pb|}-G^{(0)}_{|ab|}}{E_p-E_a}\Big)\,. \qquad (9)
$$

We have derived in 2007/08 this self-consistency equation for the Moyal model by the graphical method proposed by [DGMR07]. In this form, (9) is meaningless because $\sum_{p\in I}$ diverges. In 2009 we solved the renormalisation problem, namely the renormalisation of infinitely many Feynman graphs at once [GW09]. This renormalisation increases the non-linearity. In [GW09] we have solved (9) perturbatively to $\mathcal{O}(\lambda^3)$. After several years of setbacks with the non-perturbative solution, a breakthrough came in 2012: The equation (9) can be turned into an equation which is linear in the difference $G^{(0)}_{|ab|} - G^{(0)}_{|a0|}$ to the boundary and non-linear only in $G^{(0)}_{|a0|}$!

Calculation gives the Schwinger-Dyson equation for higher N in the form:

$G_{|ab_1\dots b_{N-1}|}$

$$
\left.\begin{aligned}
&= -\frac{\lambda}{E_a+E_{b_1}}\Big(\frac{1}{V}\sum_{p\in I}\Big(G_{|ap|}G_{|ab_1\dots b_{N-1}|} - \frac{G_{|pb_1\dots b_{N-1}|}-G_{|ab_1\dots b_{N-1}|}}{E_p-E_a}\Big) \\
&\qquad\qquad - \sum_{l=1}^{\frac{N-2}{2}} G_{|b_1\dots b_{2l}|}\frac{G_{|b_{2l+1}\dots b_{N-1}a|}-G_{|b_{2l+1}\dots b_{N-1}b_{2l}|}}{E_{b_{2l}}-E_a}\Big)
\end{aligned}\right\} \qquad (10a)
$$

$$
\left.\begin{aligned}
&-\frac{\lambda}{V^2(E_a+E_{b_1})}\Big(G_{|a|a|}G_{|ab_1\dots b_{N-1}|} + \sum_{k=1}^{N-1} G_{|b_1\dots b_k a b_k \dots b_{N-1}a|} \\
&\qquad\qquad + G_{|aaab_1\dots b_{N-1}|} + \frac{1}{V}\sum_{n\in I} G_{|an|ab_1\dots b_{N-1}|} \\
&\qquad\qquad - \sum_{k=1}^{N-1} \frac{G_{|b_1\dots b_k|b_{k+1}\dots b_{N-1}b_k|}-G_{|b_1\dots b_k|b_{k+1}\dots b_{N-1}a|}}{E_{b_k}-E_a}\Big)
\end{aligned}\right\} \qquad (10b)
$$

$$
\left. -\frac{\lambda}{V^4(E_a+E_{b_1})}G_{|a|a|ab_1\dots b_{N-1}|}\;. \right\} \qquad (10c)
$$

Again, the first lines (10a) preserve the genus, whereas $g \mapsto g+1$ in (10b) and $g \mapsto g+2$ in (10c). The planar sector $G^{(0)}_{|ab_1\dots b_{N-1}|}$, exact for $V \to \infty$ with $\frac{1}{V}\sum_{p\in I}$ finite, is a linear inhomogeneous equation with inductively known parameters.

It turns out that a real theory with $\Phi = \Phi^*$ admits a short-cut which directly gives the higher N-point functions without any index summation. Since the equations for $G_{\dots}$ are real and $\overline{J_{ab}} = J_{ba}$, the reality $\mathcal{Z} = \overline{\mathcal{Z}}$ implies (in addition to invariance under cyclic permutations) invariance under orientation reversal

$$
G_{|p_0^1 p_1^1 \dots p_{N_1-1}^1|\dots|p_0^B p_1^B \dots p_{N_B-1}^B|} = G_{|p_0^1 p_{N_1-1}^1 \dots p_1^1|\dots|p_0^B p_{N_B-1}^B \dots p_1^B|}\;. \qquad (11)
$$

Whereas empty for $G_{|ab|}$, in $(E_a+E_{b_1})G_{ab_1b_2\dots b_{N-1}} - (E_a+E_{b_{N-1}})G_{ab_{N-1}\dots b_2 b_1}$ the identities (11) lead to many cancellations which result in a universal algebraic recursion formula:

Proposition 4

$$
\begin{aligned}
G_{|b_0b_1\dots b_{N-1}|} &= (-\lambda)\sum_{l=1}^{\frac{N-2}{2}} \frac{G_{|b_0b_1\dots b_{2l-1}|}G_{|b_{2l}b_{2l+1}\dots b_{N-1}|} - G_{|b_{2l}b_1\dots b_{2l-1}|}G_{|b_0b_{2l+1}\dots b_{N-1}|}}{(E_{b_0}-E_{b_{2l}})(E_{b_1}-E_{b_{N-1}})} \\
&\quad + \frac{(-\lambda)}{V^2}\sum_{k=1}^{N-1} \frac{G_{|b_0b_1\dots b_{k-1}|b_kb_{k+1}\dots b_{N-1}|} - G_{|b_kb_1\dots b_{k-1}|b_0b_{k+1}\dots b_{N-1}|}}{(E_{b_0}-E_{b_k})(E_{b_1}-E_{b_{N-1}})}\;. \qquad (12)
\end{aligned}
$$

The last line of (12) increases the genus and is absent in $G^{(0)}_{|b_0b_1\dots b_{N-1}|}$.

We make the following key observation: An affine transformation $E \mapsto ZE+C$ together with a corresponding rescaling $\lambda \mapsto Z^2\lambda$ leaves the algebraic equations invariant:

Theorem 5 *Given a real quartic matrix model with* $S = V\,\mathrm{tr}(E\Phi^2 + \frac{\lambda}{4}\Phi^4)$ *and* $m \mapsto E_m$ *injective, which determines the set* $G_{|p_1^1 \dots p_{N_1}^1|\dots|p_1^B \dots p_{N_B}^B|}$ *of* $(N_1 + \dots + N_B)$*-point functions. Assume that the basic functions with all* $N_i \leq 2$ *are turned finite by* $E_a \mapsto Z(E_a + \frac{\mu^2}{2} - \frac{\mu_{bare}^2}{2})$ *and* $\lambda \mapsto Z^2\lambda$. *Then all functions with one* $N_i \geq 3$

1. *are finite without further need of a renormalisation of* λ, *i.e. all renormalisable quartic matrix models have vanishing* β*-function,*
2. *are given by universal algebraic recursion formulae in terms of renormalised basic functions with* $N_i \leq 2$. □

The theorem tells us that vanishing of the β-function for the self-dual Φ^4_4-model on Moyal space (proved in [DGMR07] to all orders in perturbation theory) is generic to all quartic matrix models, and the result even holds non-perturbatively!

The universal recursion formula (12) computes the planar N-point function $G_{|b_0 \dots b_{N-1}|}$ at $B = 1$ as a sum of fractions with products of 2-point functions in the numerator and products of differences of eigenvalues of E in the denominator. This structure admits an interesting graphical interpretation. We draw the indices $b_0, \dots b_{N-1}$ in cyclic order on the circle S^1 and represent a factor $G_{b_i b_j}$ as a chord connecting b_i with b_j and a factor $\frac{1}{E_{b_i} - E_{b_j}}$ as an arrow from b_i to b_j

The chords form the non-crossing chord diagrams counted by the Catalan number $C_{\frac{N}{2}} = \frac{N!}{(\frac{N}{2}+1)!\frac{N}{2}!}$. The arrows form two disjoint trees, one connecting the even vertices and one connecting the odd vertices. By rational fraction expansion it is possible to achieve that each tree intersects the chord only in the vertices. The assignment of trees to a given chord diagram is, in general, not unique. A canonical choice is not known to us.

3 Φ^4_4-theory on Moyal space as a fixed point problem

3.1 Preliminaries

A large class of examples of noncommutative geometries comes from deformations of the algebra of functions on manifolds. Schwartz functions on Euclidean space $\mathbb{R}^4$ admit an $\mathbb{R}^4$-group action by translation. As shown by Rieffel [Rie93], this group action induces a noncommutative associative product on the space of Schwartz functions, the Moyal product:

$$(f \star g)(x) = \int_{\mathbb{R}^4 \times \mathbb{R}^4} \frac{dy\, dk}{(2\pi)^4} f(x + \tfrac{1}{2}\Theta k)\; g(x+y)\; e^{\mathrm{i}\langle k,y\rangle}\;, \quad \Theta = -\Theta^t \in M_4(\mathbb{R})\;. \tag{13}$$

Whether or not the Moyal space $(\mathbb{R}^4, \star)$ is relevant for Planck scale physics is pure speculation (although a refinement can be justified by uncertainty relations for position operators [DFR95]). In any case the Moyal space is a nice toy model on which it is easy to formulate and to study (quantum) field theories. To formulate a Euclidean quantum field theory on Moyal space it is, at first sight, enough to replace in the action of a usual field theory the pointwise product of functions by the $\star$-product. The simplest example is the $\phi_4^{\star 4}$-model with action

$$S[\phi] = \int_{\mathbb{R}^4} dx \,\Big(\frac{1}{2}\phi \star (-\Delta + \mu^2)\phi + \frac{\lambda}{4}\phi \star \phi \star \phi \star \phi\Big)(x)\;. \tag{14}$$

The resulting Feynman rules [Fil96] lead to situations where a multiple insertion of non-planar subgraphs gives rise to divergences of arbitrarily high degree (ultraviolet/infrared mixing [MVS00]). See [CR00] for a thorough investigation of this problem. Relativistic quantum field theories on noncommutative Minkowski space are much more difficult [BDFP02]. Here the UV/IR-mixing problem occurs in different types of graphs [Bah10].

The Moyal algebra $(\mathcal{S}(\mathbb{R}^4), \star)$ has a matrix basis [GV88, GGISV03], in which the $\phi_4^{\star 4}$-interaction (14) becomes a matrix product (we write ϕ for a function and Φ for a matrix):

$$S[\phi] = (2\pi\theta)^2 \sum_{\underline{k},\underline{l},\underline{m},\underline{n}\in\mathbb{N}^2} \Big(\tfrac{1}{2}\Phi_{\underline{k}\underline{l}}(\Delta_{\underline{k}\underline{l};\underline{m}\underline{n}} + \mu^2\delta_{\underline{k}\underline{n}}\delta_{\underline{l}\underline{m}})\Phi_{\underline{m}\underline{n}} + \frac{\lambda}{4}\Phi_{\underline{k}\underline{l}}\Phi_{\underline{l}\underline{m}}\Phi_{\underline{m}\underline{n}}\Phi_{\underline{n}\underline{k}}\Big)\;. \tag{15}$$

The matrix kernel $\Delta_{\underline{k}\underline{l};\underline{m}\underline{n}}$ of the Laplacian $(-\Delta)$, viewed as map from $\mathbb{N}^4$ to $\mathbb{N}^4$, consists of a local interaction plus nearest neighbour interaction.

In [GW05b] we studied the renormalisation group flow of the $\phi_4^{\star 4}$-model in matrix representation (using a power-counting theorem [GW05a] for matrix models with kernel $\Delta_{\underline{k}\underline{l};\underline{m}\underline{n}}$). We noticed that the marginal parts of the local term and of the nearest neighbour term in $\Delta_{\underline{k}\underline{l};\underline{m}\underline{n}}$ have different flows. To absorb these different flows a 4^{th} relevant/marginal operator in the action functional is necessary. This operator corresponds to a harmonic oscillator potential:

$$S[\phi] = 64\pi^2\!\int d^4x \Big(\frac{Z}{2}\phi\star\big(-\Delta+\Omega^2(2\Theta^{-1}x)^2 + \mu^2_{bare}\big)\phi + \frac{\lambda Z^2}{4}\phi\star\phi\star\phi\star\phi\Big)(x)\;. \tag{16}$$

We proved in [GW05b] that the corresponding Euclidean quantum field theory is renormalisable to all orders in perturbation theory. This result was reestablished by various methods, see [Riv07a] for a review.

Presence of the harmonic oscillator term $\Omega \neq 0$ breaks translation invariance. Conversely, this term achieves covariance under Langmann-Szabo duality transformation [LS02] which consists in exchanging $x \leftrightarrow p$ and $\phi(x) \leftrightarrow \hat{\phi}(p)$ followed by Fourier transform back to the original variables. Remarkably, this transformation leaves $\displaystyle\int dx\;\phi\star\phi\star\phi\star\phi$ invariant, and it exchanges $\int dx\;\phi(-\Delta)\phi$ with $\int dx\;\phi|2\Theta^{-1}x|^2\phi$. Presence of the oscillator term gives rise to an interesting spectral noncommutative geometry [GW13a] (see also [GW12]) which is conceptually simpler than the isospectral deformation [GGISV03] of $\mathbb{R}^4$. Most importantly, the oscillator term cures the Landau ghost problem [LAK54] of usual ϕ_4^4-theory:

We have discovered in [GW04, GW05c] that the one-loop renormalisation group flows of Ω and λ influence each other in such a way that the running coupling constant $\lambda(\Lambda)$ remains finite at any scale Λ. Even more, at the self-duality point $\Omega = 1$ the β-function of the $\lambda\Phi_4^4$-coupling vanishes to all orders in perturbation theory [DGMR07]. This result was obtained by an ingenious combination of Ward identities and Schwinger-Dyson equations (see [DR07] for an explicit three-loop calculation). In [GW14a] we have generalised the method of Disertori-Gurau-Magnen-Rivasseau [DGMR07] to the whole class of quartic matrix models (reviewed in sec. 2). Vanishing of the β-function is often connected with integrability, and together with the absent Landau ghost problem a non-perturbatively constructed ϕ_4^4-model on Moyal space came into reach. The first milestone was the derivation of the self-consistency equation (9) and the understanding of its renormalisation in [GW09]. It took us several years to fully understand this equation, and it is only recently that we finished the solution/construction of the Moyal space ϕ_4^4-model [GW14a]. In the sequel we review this construction.

3.2 Renormalisation and integral representation

At the self-duality point $\Omega = 1$, the matrix kernel $\Delta^{\Omega=1}_{\underline{kl};\underline{mn}}$ of the Schrödinger operator $H = -\Delta + \|2\Theta^{-1}x\|^2$ becomes purely local and turns the action (16) in matrix basis into a (field-theoretical) quartic matrix model with action

$$S[\Phi] = V\Big(\sum_{\underline{m},\underline{n}\in\mathbb{N}^2_{\mathcal{N}}} E_{\underline{m}}\,\Phi_{\underline{mn}}\Phi_{\underline{nm}} + \frac{Z^2\lambda}{4} \sum_{\underline{m},\underline{n},\underline{k},\underline{l}\in\mathbb{N}^2_{\mathcal{N}}} \Phi_{\underline{mn}}\Phi_{\underline{nk}}\Phi_{\underline{kl}}\Phi_{\underline{lm}}\Big)\,, \tag{17}$$
$$E_{\underline{m}} = Z\Big(\frac{|\underline{m}|}{\sqrt{V}} + \frac{\mu^2_{bare}}{2}\Big)\,, \qquad |\underline{m}| := m_1 + m_2 \leq \mathcal{N}\,,\quad V = \Big(\frac{\theta}{4}\Big)^2 .$$

Our general results on quartic matrix models imply that the planar 2-point function $G^{(0)}_{|\underline{ab}|}$ satisfies the self-consistency equation (9),

$$G^{(0)}_{|\underline{ab}|} = \frac{1}{E_{\underline{a}}+E_{\underline{b}}} - \frac{Z^2\lambda}{E_{\underline{a}}+E_{\underline{b}}}\frac{1}{V}\sum_{\underline{p}\in\mathbb{N}^2_{\mathcal{N}}} \Big(G^{(0)}_{|\underline{ab}|}G^{(0)}_{|\underline{ap}|} - \frac{G^{(0)}_{|\underline{pb}|} - G^{(0)}_{|\underline{ab}|}}{E_{\underline{p}}-E_{\underline{a}}}\Big)\,. \tag{18}$$

We have introduced a cut-off $\mathbb{N}^2_{\mathcal{N}}$ in the matrix size; the index sum diverges for $\mathbb{N}^2_{\mathcal{N}} \mapsto \mathbb{N}^2$. As usual, the renormalisation strategy consists in adjusting Z, μ_{bare} in such a way that the limit $\mathbb{N}^2_{\mathcal{N}} \mapsto \mathbb{N}^2$ exists. This will be achieved by normalisation conditions for the 1PI function $\Gamma_{\underline{ab}}$ defined by $G^{(0)}_{|\underline{ab}|} =: (H_{\underline{ab}} - \Gamma_{\underline{ab}})^{-1}$, where $H_{\underline{ab}} := E_{\underline{a}} + E_{\underline{b}}$. We express (18) in terms of $\Gamma_{\underline{ab}}$,

$$\Gamma_{\underline{ab}} = -\frac{\lambda Z^2}{V}\sum_{\underline{p}\in\mathbb{N}^2_{\mathcal{N}}} \Big(\frac{1}{H_{\underline{ap}}-\Gamma_{\underline{ap}}} + \frac{1}{H_{\underline{pb}}-\Gamma_{\underline{pb}}} - \frac{1}{(H_{\underline{pb}}-\Gamma_{\underline{pb}})}\frac{\Gamma_{\underline{pb}}-\Gamma_{\underline{ab}}}{\frac{Z}{\sqrt{V}}(|\underline{p}|-|\underline{a}|)}\Big)\,, \tag{19}$$

and write $\Gamma_{\underline{a}\underline{b}}$ as first-order Taylor formula with remainder $\Gamma^{ren}_{\underline{a}\underline{b}}$,

$$\Gamma_{\underline{a}\underline{b}} = Z\mu^2_{bare} - \mu^2 + \tfrac{(Z-1)}{\sqrt{V}}(|\underline{a}| + |\underline{b}|) + \Gamma^{ren}_{\underline{a}\underline{b}}\;, \quad \Gamma^{ren}_{\underline{0}\underline{0}} = 0\;, \quad (\partial\Gamma^{ren})_{\underline{0}\underline{0}} = 0\;.$$

Equation (19) for $\Gamma_{\underline{a}\underline{b}}\big[\Gamma^{ren}_{\underline{a}\underline{b}}, \mu^2_{bare}, Z\big]$ together with $\Gamma^{ren}_{\underline{0}\underline{0}} = 0$ and $(\partial\Gamma^{ren})_{\underline{0}\underline{0}}$ constitute three equations to determine the three functions $\Gamma^{ren}_{\underline{a}\underline{b}}, \mu^2_{bare}, Z$. Eliminating μ^2_{bare}, Z thus gives rise to a *closed equation for the renormalised function* $\Gamma^{ren}_{\underline{a}\underline{b}}$ *alone.* For this elimination it is important to note that the equations for $\Gamma^{ren}_{\underline{a}\underline{b}}, \mu^2_{bare}, Z$ depend on $\underline{a}, \underline{b}$ only via the norms $|\underline{a}|, |\underline{b}|$ which parametrise the spectrum of E. Therefore, $\Gamma_{\underline{a}\underline{b}}$ is actually a function only of $|\underline{a}|, |\underline{b}|$, and consequently the index sum reduces to $\sum_{\underline{p}\in\mathbb{N}^2_{\mathcal{N}}} f(|\underline{p}|) = \sum_{|\underline{p}|=0}^{\mathcal{N}}(|\underline{p}|{+}1)f(|\underline{p}|)$.

We study a particular scaling limit in which matrix size $\mathcal{N}$ and volume V are simultaneously sent to ∞ such that the ratio $\frac{\mathcal{N}}{\sqrt{V\mu^4}} = \Lambda^2(1{+}\mathcal{Y})$ is kept fixed. Note that $V = \big(\frac{\theta}{4}\big)^2 \to \infty$ is a limit of extreme noncommutativity! The new parameter $(1{+}\mathcal{Y})$ corresponds to a finite wavefunction renormalisation, identified later to decouple our equations. The parameter Λ^2 represents an ultraviolet cut-off which is sent to $\Lambda \to \infty$ in the very end (continuum limit). In the scaling limit, functions of $\frac{|\underline{p}|}{\sqrt{V}} =: \mu^2(1+\mathcal{Y})p$ converge to functions of 'continuous matrix indices' $p \in [0, \Lambda^2]$. In the same way, $\Gamma^{ren}_{\underline{a}\underline{b}}$ converges to a function $\mu^2\Gamma_{ab}$ with $a, b \in [0, \Lambda^2]$, and the discrete sum converges to a Riemann integral

$$\frac{1}{V}\sum_{|\underline{p}|=0}^{\mathcal{N}}(|\underline{p}|+1)f\big(\tfrac{|\underline{p}|}{\sqrt{V}}\big) \longrightarrow \mu^4(1+\mathcal{Y})^2\int_0^{\Lambda^2} p\,dp\; f\big(\mu^2(1+\mathcal{Y})p\big)\;.$$

This limit makes the restriction to the planar sector (9) of (8) exact.

After elimination of μ^2_{bare}, but before elimination of Z, our equation for Γ_{ab} becomes

$$\begin{aligned}
&(Z-1)(1+\mathcal{Y})(a+b) + \Gamma_{ab} \\
&= -\lambda(1{+}\mathcal{Y})^2\int_0^{\Lambda^2} p\,dp\,\Big(\frac{Z^2}{(a+p)(1{+}\mathcal{Y}) + 1 - \Gamma_{ap}} - \frac{Z^2}{p(1{+}\mathcal{Y}) + 1 - \Gamma_{0p}}\Big) \\
&- \lambda(1{+}\mathcal{Y})^2\int_0^{\Lambda^2} p\,dp\,\Big(\frac{Z}{(b+p)(1{+}\mathcal{Y}) + 1 - \Gamma_{pb}} - \frac{Z}{p(1{+}\mathcal{Y}) + 1 - \Gamma_{p0}} \\
&\qquad - \frac{Z}{(b+p)(1{+}\mathcal{Y}) + 1 - \Gamma_{pb}}\,\frac{\Gamma_{pb} - \Gamma_{ab}}{(1+\mathcal{Y})(p-a)} \\
&\qquad + \frac{Z}{p(1+\mathcal{Y}) + 1 - \Gamma_{p0}}\,\frac{\Gamma_{p0}}{p(1+\mathcal{Y})}\Big)\;.
\end{aligned} \tag{20}$$

Applying $\frac{d}{db}\big|_{a=b=0}$ we get Z in terms of Γ_{ab} (and its derivative). Inserted back one gets a highly non-linear integro-differential equation. Fortunately we can reduce the non-linearity by subtracting from (20) the same equation taken at $b = 0$.

This subtraction eliminates the second line of (20) containing Z^2. In terms of $G_{ab} := \big((a+b)(1+\mathcal{Y}) + 1 - \Gamma_{ab}\big)^{-1}$, this difference equation reads

$$\frac{Z^{-1}}{(1+\mathcal{Y})}\Big(\frac{1}{G_{ab}} - \frac{1}{G_{a0}}\Big) = b - \lambda \int_0^{\Lambda^2} p\,dp\; \frac{\frac{G_{pb}}{G_{ab}} - \frac{G_{p0}}{G_{a0}}}{p-a}\;. \tag{21}$$

Differentiation $\frac{d}{db}\big|_{a=b=0}$ of (21) yields Z in terms of G_{ab} and its derivative. The resulting derivative G' can be avoided by adjusting

$$\mathcal{Y} := -\lambda \lim_{b\to 0} \int_0^{\Lambda^2} dp\; \frac{G_{pb} - G_{p0}}{b}\;.$$

This choice leads to $\dfrac{Z^{-1}}{(1+\mathcal{Y})} = 1 - \lambda \displaystyle\int_0^{\Lambda^2} dp\; G_{p0}$, which is a perturbatively divergent integral for $\Lambda \to \infty$. Inserting Z^{-1} and $\mathcal{Y}$ back into (21) we end up in a *linear* integral equation for the difference function $D_{ab} := \frac{a}{b}(G_{ab} - G_{a0})$ to the boundary:

$$\Big(\frac{b}{a} + \frac{1}{aG_{a0}}\Big) D_{ab} + G_{a0} = \lambda \int_0^{\Lambda^2} dp\; \Big(\frac{D_{pb} - D_{ab}\frac{G_{p0}}{G_{a0}}}{p-a}\Big)\;. \tag{22}$$

The non-linearity restricts to the boundary function G_{a0} where the second index is put to zero. Assuming $a \mapsto G_{ab}$ Hölder-continuous, we can pass to Cauchy principal values. In terms of the *finite Hilbert transform*

$$\mathcal{H}^\Lambda_a[f(\bullet)] := \frac{1}{\pi} \lim_{\epsilon\to 0} \Big(\int_0^{a-\epsilon} + \int_{a+\epsilon}^{\Lambda^2}\Big) \frac{f(q)\,dq}{q-a}\;, \tag{23}$$

the integral equation (22) becomes

$$\Big(\frac{b}{a} + \frac{1 + \lambda\pi a \mathcal{H}^\Lambda_a\big[G_{\bullet 0}\big]}{aG_{a0}}\Big) D_{ab} - \lambda\pi \mathcal{H}^\Lambda_a\big[D_{\bullet b}\big] = -G_{a0}\;. \tag{24}$$

3.3 The Carleman solution

Equation (24) is a well-known singular integral equation of Carleman type [Car22, Tri57]:

Theorem 6 ([Tri57], transformed from $[-1,1]$ to $[0,\Lambda^2]$) *The singular linear integral equation*

$$h(a)y(a) - \lambda\pi \mathcal{H}^\Lambda_a[y] = f(a)\;, \qquad a \in\,]0,\Lambda^2[\;,$$

is for $h(a)$ continuous on $]0,\Lambda^2[$, Hölder-continuous near $0,\Lambda^2$, and $f \in L^p$ for some $p>1$ (determined by $\vartheta(0)$ and $\vartheta(\Lambda^2)$) solved by

$$\begin{aligned} y(a) = \frac{\sin(\vartheta(a)) e^{-\mathcal{H}^\Lambda_a[\pi-\vartheta]}}{\lambda\pi a} \Big(& a\, f(a) e^{\mathcal{H}^\Lambda_a[\pi-\vartheta]} \cos(\vartheta(a)) \\ & + \mathcal{H}^\Lambda_a\Big[e^{\mathcal{H}^\Lambda_\bullet[\pi-\vartheta]} \bullet f(\bullet) \sin(\vartheta(\bullet))\Big] + C\Big) \end{aligned} \tag{25a}$$

$$\stackrel{*}{=} \frac{\sin(\vartheta(a))e^{\mathcal{H}^\Lambda_a[\vartheta]}}{\lambda\pi}\Big(f(a)e^{-\mathcal{H}^\Lambda_a[\vartheta]}\cos(\vartheta(a)) + \mathcal{H}^\Lambda_a\Big[e^{-\mathcal{H}^\Lambda_\bullet[\vartheta]}f(\bullet)\sin(\vartheta(\bullet))\Big] + \frac{C'}{\Lambda^2-a}\Big)\;, \tag{25b}$$

where $\vartheta(a) = \operatorname*{arctan}_{[0,\pi]}\Big(\frac{\lambda\pi}{h(a)}\Big)$, $\sin(\vartheta(a)) = \frac{|\lambda\pi|}{\sqrt{(h(a))^2+(\lambda\pi)^2}} \geq 0$ *and* C, C' *are arbitrary constants.*

The possibility of $C, C' \neq 0$ is due to the fact that the finite Hilbert transform has a kernel, in contrast to the infinite Hilbert transform with integration over $\mathbb{R}$. The two formulae (25a) and (25b) are formally equivalent, but the solutions belong to different function classes and normalisation conditions may (and will) make a choice.

In principle, (25) provides the solution G_{ab} of (24), where the angle function

$$\vartheta_b(a) := \operatorname*{arctan}_{[0,\pi]}\left(\frac{\lambda\pi a}{b + \frac{1+\lambda\pi a\mathcal{H}^\Lambda_a[G_{\bullet 0}]}{G_{a0}}}\right) \tag{26}$$

plays a key rôle. This solution involves multiple Hilbert transforms which are difficult to control. A better strategy starts from the observation that the angle (26) satisfies, for $b = 0$, again a Carleman type singular integral equation

$$\lambda\pi\cot\vartheta_0(a)G_{a0} - \lambda\pi\mathcal{H}^\Lambda_a[G_{\bullet 0}] = \tfrac{1}{a}$$

with solution

$$G_{a0} = \frac{e^{-\mathcal{H}^\Lambda_a[\pi-\vartheta_0]}\sin(\vartheta_0(a))}{\lambda\pi a}\big(e^{\mathcal{H}^\Lambda_a[\pi-\vartheta_0]}\cos(\vartheta_0(a)) + \mathcal{H}^\Lambda_a\big[e^{\mathcal{H}^\Lambda_\bullet[\pi-\vartheta_0]}\sin(\vartheta_0(\bullet))\big] + C\big) \tag{27a}$$

$$\stackrel{*}{=} \frac{e^{\mathcal{H}^\Lambda_a[\vartheta_0]}\sin(\vartheta_0(a))}{\lambda\pi}\Big(\frac{e^{-\mathcal{H}^\Lambda_a[\vartheta_0]}\cos(\vartheta_0(a))}{a} + \mathcal{H}^\Lambda_a\Big[\frac{e^{-\mathcal{H}^\Lambda_\bullet[\vartheta_0]}\sin(\vartheta_0(\bullet))}{\bullet}\Big] + \frac{C'}{\Lambda^2-a}\Big)\;. \tag{27b}$$

Tricomi's identities [Tri57, §4.4(28+18)], which can be arranged as

$$e^{\pm\mathcal{H}^\Lambda_a[\vartheta_b]}\cos(\vartheta_b(a)) \mp \mathcal{H}^\Lambda_a\big[e^{\pm\mathcal{H}^\Lambda_\bullet[\vartheta_b]}\sin(\vartheta_b(\bullet))\big] = 1\;,$$

and rational fraction expansion $\mathcal{H}^\Lambda_a\big[\frac{f(\bullet)}{\bullet}\big] = \frac{1}{a}\big(\mathcal{H}^\Lambda_a\big[f(\bullet)\big] - \mathcal{H}^\Lambda_0\big[f(\bullet)\big]\big)$ simplify (27) to

$$G_{a0} = \frac{e^{-\mathcal{H}^\Lambda_a[\pi-\vartheta_0]}\sin(\vartheta_0(a))}{\lambda\pi a}\big(C-1\big) \tag{28a}$$

$$\stackrel{*}{=} \frac{e^{\mathcal{H}^\Lambda_a[\vartheta_0]}\sin(\vartheta_0(a))}{\lambda\pi a}\Big(e^{-\mathcal{H}^\Lambda_0[\vartheta_0]}\cos(\vartheta_0(0)) + \frac{C'a}{\Lambda^2-a}\Big)\;. \tag{28b}$$

Both lines are formally equivalent, but we have to guarantee the normalisation $\lim_{a\to 0} G_{a0} = 1$. From (26) one concludes $\lim_{p\to 0} \vartheta_0(p) = \left\{ \begin{array}{ll} 0 & \text{for } \lambda \geq 0 \\ \pi & \text{for } \lambda < 0 \end{array} \right\}$.

Consequently, $e^{-\mathcal{H}_0^\Lambda[\vartheta_0]} = \exp\big(-\frac{1}{\pi}\int_0^{\Lambda^2} \frac{dp}{p}\vartheta_0(p)\big) \stackrel{\lambda<0}{\longrightarrow} 0$, which means that (28b) reduces for $\lambda < 0$ to (28a), with $C' \mapsto C-1$. Similarly, $\lim_{a\to 0} e^{-\mathcal{H}_a^\Lambda[\pi-\vartheta_0]} \stackrel{\lambda\geq 0}{=} 0$, so that (28a) is only consistent with $\lambda < 0$. The normalisation $\lim_{a\to 0} G_{a0} = 1$ leads with $\lim_{a\to 0} \frac{\sin\vartheta_0(a)}{|\lambda|\pi a} = 1$ to $1-C = e^{-\mathcal{H}_0^\Lambda[\pi-\vartheta_0]}$ in (28a), whereas (28b) stays as it is for $\lambda > 0$. These results can be summarised as follows:

Lemma 7 *The angle function* $\tau_b(a) := \underset{[0,\pi]}{\arctan}\left(\dfrac{|\lambda|\pi a}{b + \frac{1+\lambda\pi a\mathcal{H}_a^\Lambda[G_{\bullet 0}]}{G_{a0}}}\right)$ *is for* $b = 0$ *reverted to*

$$G_{a0} = \frac{\sin(\tau_0(a))}{|\lambda|\pi a} \mathrm{e}^{\mathrm{sign}(\lambda)(\mathcal{H}_0^\Lambda[\tau_0(\bullet)]-\mathcal{H}_a^\Lambda[\tau_0(\bullet)])} \begin{cases} 1 & \text{for } \lambda < 0\,, \\ \big(1+\frac{Ca}{\Lambda^2-a}\big) & \text{for } \lambda > 0\,, \end{cases} \tag{29}$$

where C is an arbitrary constant.

Recall that G_{a0} forms the inhomogeneity in the Carleman equation (24). We insert (29) into the Carleman solution (25) for (24) and obtain with the addition theorem $|\lambda|\pi a \sin\big(\tau_d(a) - \tau_b(a)\big) = (b-d)\sin\tau_b(a)\sin\tau_d(a)$ after essentially the same steps as in the proof of (29):

Theorem 8 ([GW14b]) *The full matrix 2-point function G_{ab} of self-dual ϕ^4_4-theory on Moyal space is in the limit $\theta \to \infty$ given in terms of the boundary 2-point function G_{a0} by the equation*

$$G_{ab} = \frac{\sin(\tau_b(a))}{|\lambda|\pi a} \mathrm{e}^{\mathrm{sign}(\lambda)(\mathcal{H}_0^\Lambda[\tau_0(\bullet)]-\mathcal{H}_a^\Lambda[\tau_b(\bullet)])} \begin{cases} 1 & \text{for } \lambda < 0\,, \\ \big(1+\frac{Ca+bF(b)}{\Lambda^2-a}\big) & \text{for } \lambda > 0\,, \end{cases} \tag{30}$$

where C is a undetermined constant and $b\,F(b)$ an undetermined function of b vanishing at $b = 0$.

Some remarks:

- We have proved this theorem in 2012 for $\lambda > 0$ under the assumption $C' = 0$ in (25b), but knew that non-trivial solutions of the homogeneous Carleman equation parametrised by $C' \neq 0$ are possible. That no such term arises for $\lambda < 0$ (if angles are redefined $\vartheta \mapsto \tau$) is a recent result [GW14b].
- We expect C, F to be Λ-dependent so that $\big(1+\frac{Ca+bF(b)}{\Lambda^2-a}\big) \stackrel{\Lambda\to\infty}{\longrightarrow} 1+\tilde{C}a+b\tilde{F}(b)$.
- An important observation is $G_{ab} \geq 0$, at least for $\lambda < 0$. This is a truly non-perturbative result; individual Feynman graphs show no positivity at all!

- As in [GW09], the equation for G_{ab} can be solved perturbatively. Matching at $\lambda = 0$ requires C, F to be flat functions of λ (all derivatives vanish at zero). Because of $\mathcal{H}^\Lambda_a[G_{\bullet 0}] \stackrel{a\to\Lambda^2}{\longrightarrow} -\infty$, the naïve arctan series is dangerous for $\lambda > 0$. Unless there are cancellations, we expect zero radius of convergence!
- From (30) we deduce the finite wavefunction renormalisation

$$\mathcal{Y} := -1 - \frac{dG_{ab}}{db}\Big|_{a=b=0} = \int_0^{\Lambda^2} \frac{dp}{(\lambda\pi p)^2 + \big(\frac{1+\lambda\pi p\mathcal{H}^\Lambda_p[G_{\bullet 0}]}{G_{p0}}\big)^2} - \begin{cases} 0 & \text{for } \lambda < 0\,, \\ F(0) & \text{for } \lambda > 0\,. \end{cases} \tag{31}$$

- The partition function $\mathcal{Z}$ is undefined for $\lambda < 0$. But the Schwinger-Dyson equations for G_{ab} and for higher functions, and with them $\log \mathcal{Z}$, extend to $\lambda < 0$. These extensions are unique but probably not analytic in a neighbourhood of $\lambda = 0$.

It remains to identify the boundary function G_{a0}. The Carleman equation (24) for G_{ab} was obtained from the difference (20)$-$(20)$_{b=0}$. Consequently, (20)$_{b=0}$ gives the second relation between G_{ab} and G_{a0} from which both are determined. But the resulting equation turns out to be of little use: The integrals are individually divergent for $\Lambda\to\infty$ so that we have to rely on cancellations on which we have no control.

We compensate this lack by a symmetry argument. Given the boundary function G_{a0}, the Carleman theory computes the full 2-point function G_{ab} via (30). In particular, we get G_{0b} as function of G_{a0}. But the 2-point function is symmetric, $G_{ab} = G_{ba}$, and the special case $b = 0$ leads to the following self-consistency equation:

Proposition 9 *The limit $\theta \to \infty$ of ϕ^4_4-theory on Moyal space is determined by the solution of the fixed point equation $G = TG$,*

$$G_{b0} = \frac{\left\{\begin{matrix} 1 & \textit{for } \lambda<0 \\ 1+bF(b) & \textit{for } \lambda>0 \end{matrix}\right\}}{1+b} \exp\left(-\lambda \int_0^b dt \int_0^{\Lambda^2} \frac{dp}{(\lambda\pi p)^2 + \big(t + \frac{1+\lambda\pi p\mathcal{H}^\Lambda_p[G_{\bullet 0}]}{G_{p0}}\big)^2}\right)\,. \tag{32}$$

At this point we can eventually send $\Lambda \to \infty$. Any solution of (32) is automatically smooth and (for $\lambda > 0$ but $F = 0$) monotonously decreasing. Any solution of the true equation (20) (without the difference to $b = 0$) also solves the master equation (32), but not necessarily conversely. In case of a unique solution of (32), it is enough to check one candidate.

Existence of a solution of (32) is established (for $\lambda > 0$ but $F(b) = 0$) by the Schauder fixed point theorem. This solution provides G_{ab} via (30) and all higher correlation functions via the universal algebraic recursion formulae (12), etc, or via the linear equations for the basic $(N_1 + \ldots + N_B)$-point functions. The recursion formula (12) becomes after transition to continuous matrix indices

$$G_{b_0 \ldots b_{N-1}} = \frac{(-\lambda)}{(1+\mathcal{Y})^2} \sum_{l=1}^{\frac{N-2}{2}} \frac{G_{b_0 b_1 \ldots b_{2l-1}} G_{b_{2l} b_{2l+1} \ldots b_{N-1}} - G_{b_{2l} b_1 \ldots b_{2l-1}} G_{b_0 b_{2l+1} \ldots b_{N-1}}}{(b_0 - b_{2l})(b_1 - b_{N-1})} \,. \tag{33}$$

It involves the finite wavefunction renormalisation $1 + \mathcal{Y} = -\frac{dG_{ab}}{db}\big|_{a=b=0}$ given by (31). Of particular interest is the effective coupling constant $\lambda_{\mathit{eff}} = -G_{0000}$. This limit of coinciding indices is not so easy; therefore we directly solve the integral equation for G_{a000} before using the reality condition. We find [GW14a]

$$\lambda_{\mathit{eff}} = \lambda \Big\{ 1 + \frac{\lambda}{(1{+}\mathcal{Y})} \int_0^\infty dp \, \frac{\Big(\dfrac{1 - G_{p0}}{(1+\mathcal{Y})p} - G_{p0} \Big) G_{p0}}{\big(\lambda \pi p G_{p0}\big)^2 + \big(1 + \lambda \pi p \mathcal{H}_p^\infty[G_{\bullet 0}]\big)^2} \Big\} \,. \tag{34}$$

3.4 Computer simulations [GW14b]

A numerical investigation of (32), for $F(b) \equiv 0$, reveals interesting properties of the ϕ^4_4-theory on Moyal space. We approximate G_{a0} as piecewise linear function on $[0, \Lambda^2]$ sampled according to a geometric progression and view (32) as iteration $G^{n+1}_{a0} = (TG^n)_{a0}$ for some initial function G^0. In this way we find numerically that T satisfies, for any $\lambda \in \mathbb{R}$, the assumptions of the Banach fixed point theorem for Lipschitz functions on $[0, \Lambda^2]$, i.e. T is contractive and (G^n) converges to a fixed point which approximates G_{a0}. Whereas (G^n) converges for any sign of λ (without discontinuity at $\lambda = 0$), the necessary consistency condition $G_{ab} = G_{ba}$ for (30) turns out to be maximally violated for $\lambda > 0$ if $C = 0 = F(b)$ is assumed, and satisfied (within numerical error bounds) for $\lambda \leq 0$. Taking $C, F(b) \neq 0$ for $\lambda > 0$ into account is not feasible at the moment so that our numerical results are reliable only for $\lambda \leq 0$.

We find clear evidence for a second-order phase transition: $\mathcal{Y}'$ is discontinuous at $\lambda_c = -0.396$, and we have in reasonable approximation a critical behaviour

$$1 + \mathcal{Y} = \begin{cases} A(\lambda - \lambda_c)^\alpha & \text{for } \lambda \geq \lambda_c \,, \\ 0 & \text{for } \lambda < \lambda_c \,, \end{cases} \tag{35}$$

for some $A, \alpha > 0$. Of course, there cannot be a discontinuity in $\mathcal{Y}'$ for finite Λ, but we have numerical evidence for a critical behaviour in the limit $\Lambda^2 \to \infty$.

4 Schwinger functions and reflection positivity

Under conditions identified by Osterwalder-Schrader [OS73, OS75], Schwinger functions [Sch59] of a Eulidean quantum field theory permit an analytical continuation to Wightman functions [Wig56, SW64] of a true relativistic quantum field theory. In simplified terms, the reconstruction theorem of Osterwalder-Schrader says:

Theorem 10 ([OS73, OS75]) *If the Schwinger functions $S(x_1,\dots,x_N)$ satisfy growth conditions, Euclidean covariance, reflection positivity*[1] *and permutation symmetry, then the $S(\xi_1,\dots\xi_{N-1})\big|_{\xi_i^0>0}$, with $\xi_i = x_i - x_{i+1}$, are Laplace-Fourier transforms of Wightman functions in a relativistic quantum field theory. If in addition the $S(x_1,\dots,x_N)$ satisfy clustering then the Wightman functions satisfy clustering, too.*

Representation as Laplace transform in ξ^0 requires analyticity in $\mathrm{Re}(\xi^0)>0$. For the Schwinger 2-point function such analyticity in ξ^0 is a corollary of analyticity of the function $a \mapsto G_{aa}$ in $\mathbb{C}\setminus\,]{-\infty},0]$. We will show that analyticity and reflection positivity boil down to *Stieltjes functions*, i.e. functions $f:\mathbb{R}_+\to\mathbb{R}$ which have a representation as a Stieltjes transform (see [Wid38])

$$f(x) = c + \int_0^\infty \frac{d(\rho(t))}{x+t}\;, \qquad c = f(\infty) \geq 0\;, \tag{36}$$

where ρ is non-negative and non-decreasing. We prove:

Proposition 11 *The Schwinger function $S_c(\mu\xi) = \displaystyle\int_{\mathbb{R}^4} \frac{dp}{(2\pi\mu)^4} e^{\mathrm{i}p\xi} G_{\frac{\|p\|^2}{2\mu^2(1+\mathcal{Y})}\frac{\|p\|^2}{2\mu^2(1+\mathcal{Y})}}$ is the analytic continuation of a Wightman 2-point function if and only if $a\mapsto G_{aa}$ is Stieltjes.*

Proof. This is verified by explicit calculation. If $a\mapsto G_{aa}$ is Stieltjes, we have in terms of $\omega_{\vec{p}}(t) := \sqrt{\vec{p}^2 + 2\mu^2(1+\mathcal{Y})t}$ after using the residue theorem

$$\begin{aligned}
S_c(\mu\xi)\big|_{\xi^0>0} &= \int_{\mathbb{R}^3} \frac{d\vec{p}}{(2\pi\mu)^3} \int_{-\infty}^\infty \frac{dp^0}{2\pi\mu} e^{\mathrm{i}p^0\xi^0 + \mathrm{i}\vec{p}\cdot\vec{\xi}} \int_0^\infty \frac{d\rho(t)}{t + \frac{(p^0)^2+\vec{p}^2}{2\mu^2(1+\mathcal{Y})}} \\
&= \int_0^\infty \frac{2(1+\mathcal{Y})\,d\rho(t)}{\mu^4} \int_0^\infty dq^0 \int_{\mathbb{R}^3} d\vec{q}\;\hat{W}_t(q) e^{-q^0\xi^0 + \mathrm{i}\vec{q}\cdot\vec{x}}\;, \qquad \text{(37a)} \\
\hat{W}_t(q) &:= \frac{\theta(q^0)}{(2\pi)^3}\delta\Big(\frac{(q^0)^2 - \vec{q}^2 - 2\mu^2(1+\mathcal{Y})t}{\mu^2}\Big)\;. \qquad \text{(37b)}
\end{aligned}$$

We observe that $\hat{W}_t(q)$ is precisely the Källén-Lehmann spectral representation [Käl52, Leh54] of a Wightman 2-point function. $\square$

Remarkably, the Stieltjes property can be tested by purely real conditions:

[1] For each assignment $N \mapsto f_N \in \mathcal{S}^N$ of test functions,

$$\sum_{M,N} \int dx\,dy\; S(x_1,\dots,x_N,y_1,\dots,y_M)\overline{f_N(x_1^r,\dots,x_N^r)} f_M(y_1,\dots,y_M) \geq 0\;,$$

where $(x^0,x^1,\dots x^{d-1})^r := (-x^0,x^1,\dots x^{d-1})$

Theorem 12 (Widder [Wid38]) *A function $f:\mathbb{R}_+\to\mathbb{R}$ is Stieltjes iff it is smooth, non-negative and satisfies $L_{k,t}[f(\bullet)]\geq 0$, where*

$$L_{k,t}[f(\bullet)]:=\frac{(-t)^{k-1}}{c_k}\frac{d^{2k-1}}{dt^{2k-1}}\big(t^k f(t)\big)\;,\qquad c_1=1,\;c_{k>1}=k!(k-2)!\;.$$

In that case, the measure is recovered by $\rho'(t)=\lim_{k\to\infty}L_{k,t}[f(\bullet)]$ (weakly and almost everywhere).

Already a perturbative calculation shows that the anomalous dimension $\eta=-2\lambda+\mathcal{O}(\lambda^2)$ is negative for $\lambda>0$, which implies that $a\mapsto G_{aa}$ cannot be Stieltjes for $\lambda>0$. The restriction to negative coupling constant is reminiscent of the planar wrong-sign $\lambda\phi^4_4$-model [tHo82, Riv83]. Recall that our matrix model also reduces to the planar sector, but as result of the infinite volume limit and not by hand. We nonetheless keep a non-trivial topology in form of $B\geq 1$ boundary components. Moreover, we have an exact solution for $S(x_1,\dots,x_N)$, not only an existence proof.

Whether or not $a\mapsto G_{aa}$ is a Stieltjes function for $\lambda<0$ is a highly interesting question. A first idea has been obtained by computer simulations. We find clear evidence that $a\mapsto G_{aa}$ is not a Stieltjes function for $\lambda<\lambda_c$, where $\lambda_c\approx -0.396$ locates the discontinuity of $\mathcal{Y}'(\lambda)$. For $\lambda\in[\lambda_c,0]$ the results are not yet conclusive. Since G_{aa} and G_{a0} show a very similar behaviour, the functions $L_{k,t}[G_{\bullet 0}]$ (which are easy to compute) give some indication about $L_{k,t}[G_{\bullet\bullet}]$ (which we are interested in). From (32) one can prove the following identity [GW14b]:

$$\frac{(\log G_{a0})^{(\ell)}}{(\ell-1)!}=\frac{(-1)^\ell}{(1+a)^\ell}+(-1)^\ell\,\mathrm{sign}(\lambda)\,\mathcal{H}^\Lambda_0\Big[\sin\big(\ell\tau_a(\bullet)\big)\Big(\frac{\sin\tau_a(\bullet)}{|\lambda|\pi\bullet}\Big)^\ell\Big]\;.\qquad(38)$$

We have first results for the integrated 'substitute mass densities' $\tilde{\rho}_k(m^2)=\int_0^{m^2}dt\;L_{k,t}[G_{\bullet 0}]$: There is clear evidence for a mass gap, $\lim_{k\to\infty}\tilde{\rho}_k(\mu^2)=0$ for $0\leq\mu^2\leq m^2$. For $\lambda\nearrow 0$ the integrated mass density approaches (as expected) a step function, whereas for $\lambda\searrow\lambda_c$ we notice a power-law behaviour typical for critical phenomena. In particular, for $\lambda_c<\lambda<0$ there is no further gap in the support of $\tilde{\rho}'$, which signals scattering right away from m^2 (not only from the two-particle threshold on). We interpret this as scattering of a massive particle with an infrared cloud. This scattering would be a remnant of the underlying non-trivial matrix model before the projection to diagonal matrices.

5 Summary

We have shown that the ϕ^4_4-model on noncommutative Moyal space, considered in the limit $\theta\to\infty$ of extreme noncommutativity, is an exactly solvable and non-trivial matrix model. Euclidean symmetry is violated in the beginning, but we identified a limit which projects to diagonal matrices where Euclidean symmetry is restored. One would not expect that such a brutal projection can respect any quantum field theory axioms. Surprisingly, the first consistency checks, positivity of the lowest Widder criteria $L_{k,t}[G_{\bullet\bullet}]$, are passed for the only interesting interval $[\lambda_c, 0]$ of the coupling constant!

If these miracles continue and all Osterwalder-Schrader axioms (except for clustering) hold, we would get a relativistic quantum field theory in four dimensions. This theory is somewhat strange as 'particles' keep their momenta in interaction processes. Nevertheless, the theory is not completely trivial. We find scattering remnants from the noncommutative geometrical (i.e. matricial) substructure. Only the external matrix indices are put 'on-shell', internally all degrees of freedom contribute.

We have seen that clustering is maximally violated. The interaction is insensitive to positions in different boundary components. In particular, 'particles' are never asymptotically free.

Acknowledgements

HG would like to cordially thank George Zoupanos for the invitation to the Corfu Workshop and the enjoyable atmosphere.

References

[Aiz81] M. Aizenman, "Proof of the triviality of ϕ^4_d field theory and some mean field features of Ising models for $d > 4$," Phys. Rev. Lett. **47** (1981) 1–4.

[BDFP02] D. Bahns, S. Doplicher, K. Fredenhagen and G. Piacitelli, "On the unitarity problem in space-time noncommutative theories," Phys. Lett. B **533** (2002) 178–181 [hep-th/0201222].

[Bah10] D. Bahns, "The ultraviolet infrared mixing problem on the noncommutative Moyal space," arXiv:1012.3707 [hep-th].

[Car22] T. Carleman, "Sur la résolution de certaines équations intégrales," Arkiv for Mat., Astron. och Fysik **16** (1922), 19pp.

[CR00] I. Chepelev and R. Roiban, "Convergence theorem for noncommutative Feynman graphs and renormalization," JHEP **0103** (2001) 001 [hep-th/0008090].

[DGZ95] P. Di Francesco, P. H. Ginsparg and J. Zinn-Justin, "2D gravity and random matrices," Phys. Rept. **254** (1995) 1–133 [hep-th/9306153].

[DR07] M. Disertori and V. Rivasseau, "Two and three loops beta function of non commutative ϕ^4_4 theory," Eur. Phys. J. C **50** (2007) 661–671 [hep-th/0610224].

[DGMR07] M. Disertori, R. Gurau, J. Magnen and V. Rivasseau, "Vanishing of beta function of non commutative ϕ^4_4 theory to all orders," Phys. Lett. B **649** (2007) 95–102 [hep-th/0612251].

[DFR95] S. Doplicher, K. Fredenhagen and J. E. Roberts, "The Quantum structure of space-time at the Planck scale and quantum fields," Commun. Math. Phys. **172** (1995) 187–220 [hep-th/0303037].

[Fil96] T. Filk, "Divergencies in a field theory on quantum space," Phys. Lett. B **376** (1996) 53–58.

[Frö82] J. Fröhlich, "On the triviality of $\lambda\phi_d^4$ theories and the approach to the critical point in $d \geq 4$ dimensions," Nucl. Phys. B **200** (1982) 281–296.

[GGISV03] V. Gayral, J. M. Gracia-Bondía, B. Iochum, T. Schücker and J. C. Várilly, "Moyal planes are spectral triples," Commun. Math. Phys. **246** (2004) 569–623 [hep-th/0307241].

[GW13a] V. Gayral and R. Wulkenhaar, "Spectral geometry of the Moyal plane with harmonic propagation," J. Noncommut. Geom. **7** (2013) 939–979 [arXiv:1108.2184 [math.OA]].

[GJ87] J. Glimm and A. M. Jaffe, *Quantum physics. A functional integral point of view*, Springer-Verlag, New York (1987).

[GV88] J. M. Gracia-Bondía and J. C. Várilly, "Algebras of distributions suitable for phase space quantum mechanics. I.," J. Math. Phys. **29** (1988) 869–879.

[GW04] H. Grosse and R. Wulkenhaar, "The β-function in duality-covariant noncommutative ϕ^4-theory," Eur. Phys. J. C **35** (2004) 277–282 [hep-th/0402093].

[GW05a] H. Grosse and R. Wulkenhaar, "Power-counting theorem for non-local matrix models and renormalisation," Commun. Math. Phys. **254** (2005) 91–127 [hep-th/0305066].

[GW05b] H. Grosse and R. Wulkenhaar, "Renormalisation of ϕ^4-theory on noncommutative $\mathbb{R}^4$ in the matrix base," Commun. Math. Phys. **256** (2005) 305–374 [hep-th/0401128].

[GW05c] H. Grosse and R. Wulkenhaar, "Renormalization of ϕ^4-theory on noncommutative $\mathbb{R}^4$ to all orders," Lett. Math. Phys. **71** (2005) 13–26 [hep-th/0403232].

[GW09] H. Grosse and R. Wulkenhaar, "Progress in solving a noncommutative quantum field theory in four dimensions," arXiv:0909.1389 [hep-th].

[GW12] H. Grosse and R. Wulkenhaar, "8D-spectral triple on 4D-Moyal space and the vacuum of noncommutative gauge theory," J. Geom. Phys. **62** (2012) 1583–1599 [arXiv:0709.0095 [hep-th]].

[GW14a] H. Grosse and R. Wulkenhaar, "Self-dual noncommutative ϕ^4-theory in four dimensions is a non-perturbatively solvable and non-trivial quantum field theory," Commun. Math. Phys. (2014) doi:10.1007/s00220-014-1906-3, [arXiv:1205.0465 [math-ph]].

[GW13b] H. Grosse and R. Wulkenhaar, "Solvable limits of a 4D noncommutative QFT," arXiv:1306.2816 [math-ph].

[GW14b] H. Grosse and R. Wulkenhaar, "Phase transitions in exactly solvable 4D noncommutative QFT," in preparation.

[JW00] A. M. Jaffe and E. Witten, "Quantum Yang-Mills theory," in: *The millenium prize problems*, pp. 129–152, eds. J. Carlson et al, Amer. Math. Soc., Providence (2006).

[Käl52] G. Källén, "On the definition of the renormalization constants in quantum electrodynamics," Helv. Phys. Acta **25** (1952) 417–434.

[LAK54] L. D. Landau, A. A. Abrikosov and I. M. Khalatnikov, "On the removal of infinities in quantum electrodynamics," (in russ.) Dokl. Akad. Nauk SSSR **95** (1954) 497–500.

[LS02] E. Langmann and R. J. Szabo, "Duality in scalar field theory on noncommutative phase spaces," Phys. Lett. B **533** (2002) 168–177 [hep-th/0202039].

[Leh54] H. Lehmann, "Über Eigenschaften von Ausbreitungsfunktionen und Renormierungskonstanten quantisierter Felder," Nuovo Cim. **11** (1954) 342–357.

[MVS00] S. Minwalla, M. Van Raamsdonk and N. Seiberg, "Noncommutative perturbative dynamics," JHEP **0002** (2000) 020 [hep-th/9912072].

[OS73] K. Osterwalder and R. Schrader, "Axioms for Euclidean Green's functions," Commun. Math. Phys. **31** (1973) 83–112.

[OS75] K. Osterwalder and R. Schrader, "Axioms for Euclidean Green's functions II," Commun. Math. Phys. **42** (1975) 281–305.

[Rie93] M. A. Rieffel, "Deformation quantization for actions of $\mathbb{R}^d$", Memoirs AMS **506** (1993) 1–96.

[Riv83] V. Rivasseau, "Construction and Borel summability of planar four-dimensional Euclidean field theory," Commun. Math. Phys. **95** (1984) 445–486.

[Riv91] V. Rivasseau, *From perturbative to constructive renormalization*, Princeton Univ. Press (1991).

[Riv07a] V. Rivasseau, "Non-commutative renormalization" In: *Quantum spaces (Séminaire Poincaré X)*, pp 19–109, eds. B. Duplantier and V. Rivasseau, Birkhäuser Verlag Basel (2007) [arXiv:0705.0705 [hep-th]].

[Sch59] J. Schwinger, "Euclidean quantum electrodynamics," Phys. Rev. **115** (1959) 721–731.

[SW64] R. F. Streater and A. S. Wightman, *PCT, spin and statistics, and all that,* Benjamin, New York (1964).

[tHo74] G. 't Hooft, "A planar diagram theory for strong interactions," Nucl. Phys. B **72** (1974) 461–473.

[tHo82] G. 't Hooft, "Rigorous construction of planar diagram field theories in four-dimensional Euclidean space," Commun. Math. Phys. **88** (1983) 1–25.

[Tri57] F. G. Tricomi, *Integral equations*, Interscience, New York (1957).

[Wid38] D. V. Widder, "The Stieltjes transform," Trans. Amer. Math. Soc. **43** (1938) 7–60.

[Wig56] A. S. Wightman, "Quantum field theory in terms of vacuum expectation values," Phys. Rev. **101** (1956) 860–866.