Yang–Mills–Higgs Models Arising from L-Cycles*

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Abstract

Starting with a Hilbert space endowed with a representation of a unitary Lie algebra and an action of a generalized Dirac operator, we sketch a mathematical concept towards gauge field theories. This concept shares common features with the non-commutative geometry à la Connes/Lott, differs from that, however, by the implementation of unitary Lie algebras instead of associative $*$-algebras.

1 Introduction

We want to derive the classical Lagrangian of a gauge field theory specified by the following set of data:

1. The (Lie) group of local gauge transformations $\mathcal{G}$.
2. Chiral fermions ψ transforming under a representation $\tilde{\pi}$ of $\mathcal{G}$.
3. The fermionic mass matrix $\widetilde{\mathcal{M}}$, i.e. fermion masses plus generalized Kobayashi–Maskawa matrices.
4. Possibly information on the spontaneous symmetry breaking pattern of $\mathcal{G}$.

At first sight, this setting seems to be perfectly adapted to the Connes–Lott prescription of non-commutative geometry (NCG). Namely, one of the most important applications of NCG [1] to physics is a unified description of the standard model. The most elegant version rests upon a K-cycle (nowadays called spectral triple) with real structure [1], see [2] for details of the construction. A K-cycle $(\mathcal{A}, h, D)$ is a collection of a unital associative $*$-algebra $\mathcal{A}$ acting on a Hilbert space h and a generalized Dirac operator D on h, subject to some technical conditions. To relate K-cycles with the physical setting one chooses $\mathcal{A}$ in such a way that it contains the gauge group $\mathcal{G}$ as the set of unitary elements. Moreover, the fermionic multiplets ψ determine the Hilbert space h and the fermionic mass matrix $\widetilde{\mathcal{M}}$ the generalized Dirac operator D.

One could think that the extension of this method to Grand Unified Theories (GUT's) is not difficult. However, it was shown in [3] that only the standard model can be constructed within the above-mentioned understanding of NCG. Yet, there exist possibilities to circumvent this result and to construct GUT's within NCG. Such approaches require additional structures, see e.g. treatments by Chamseddine,

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Felder and Fröhlich. However, only a slight modification of the fundamentals of the Connes–Lott prescription enables the formulation of a large class of physical models.

2 The Idea — or: What is an L-Cycle?

Why is the most elegant NCG-prescription so restrictive to admissible models? The obstruction is the extension of representations $\tilde{\pi}$ of the gauge group $\mathcal{G}$ to representations of the algebra $\mathcal{A}$ containing $\mathcal{G}$ as the set of unitary elements. That extension must be compatible with linear operations, but the representation $\tilde{\pi}$ does not care whether it is linear or not. In general, the only group representations which are extendable are the identity and the complex conjugation. Thus, the most elegant NCG-prescription is compatible only with the fundamental representation of the gauge group and its complex conjugate. Most of the Grand Unified Theories are not of that type.

How can we overcome the restriction? We must linearize the gauge group! But linearization of a Lie group yields its Lie algebra. Thus, the Lie algebra $\mathfrak{g}$ of the gauge group $\mathcal{G}$ is the right object to use, not an algebra covering $\mathcal{G}$.

Therefore, we simply replace in the Definition of a K-cycle [1] the unital associative $*$-algebra $\mathcal{A}$ by a unitary Lie algebra $\mathfrak{g}$. The outcome is called "L-cycle", where the letter L stands for Lie: *An L-cycle* $(\mathfrak{g}, h, D, \pi, \Gamma)$ *consists of a* $*$*-representation* π *of a unitary Lie algebra* $\mathfrak{g}$ *in bounded operators on a Hilbert space* h*, together with a selfadjoint operator* D *on* h *with compact resolvent and a selfadjoint operator* Γ *on* h*,* $\Gamma^2 = \mathrm{id}_h$*, which commutes with* $\pi(\mathfrak{g})$ *and anticommutes with* D*. The operator* D *may be unbounded, but such that* $[D, \pi(\mathfrak{g})]$ *is bounded.*

Let us sketch how L-cycles are related to our physical input data: First, one constructs a gauge field theory on a compact Euclidian manifold X. The completion of the space of fermions ψ yields the Hilbert space h of the L-cycle. In some cases, it may be necessary to work with several copies of the fermions. Given the (Lie) group of local gauge transformations $\mathcal{G}$, we take $\mathfrak{g}$ as the Lie algebra of $\mathcal{G}$. The representation π of $\mathfrak{g}$ on the Hilbert space is just the differential of the group representation $\tilde{\pi}$. We take $D = \mathsf{D} + \gamma^5 \mathcal{M}$, where D is the usual Dirac operator on X. The matrix $\mathcal{M}$ contains the fermionic mass parameters and possibly contributions required by the desired symmetry breaking scheme, where $\gamma^5 \mathcal{M}$ coincides with the fermionic mass matrix $\widetilde{\mathcal{M}}$ on chiral fermions. The grading operator Γ represents the chirality properties of the fermions. After the Wick rotation to Minkowski space we use Γ to impose a chirality condition on h.

3 The General Scheme

Now we have to transcribe the Connes–Lott prescription to our case. The first step is to enlarge our Lie algebra $\mathfrak{g}$ to a universal graded differential Lie algebra $\Omega^* \mathfrak{g}$ given as the set of repeated graded commutators of $\mathfrak{g}$ and $d\mathfrak{g}$. The graded Lie algebra $\Omega^* \mathfrak{g}$ is universal in the following sense: Each graded differential Lie algebra generated by

$\pi(\mathfrak{g})$ and $\mathrm{d}\pi(\mathfrak{g})$ can be obtained by factorization of $\Omega^*\mathfrak{g}$ with respect to a differential ideal. For instance, the information contained in an L-cycle determines uniquely such a differential ideal. Thus, there is a canonical graded differential Lie algebra $\Omega^*_D\mathfrak{g}$ associated to an L-cycle.

To find this differential Lie algebra we represent $\Omega^*\mathfrak{g}$ on the Hilbert space h, using the data specified in the L-cycle. This representation extends $\pi(\mathfrak{g})$ and is defined by

$$\pi(da) = [-\mathrm{i}D, \pi(a)] \;, \qquad \pi([\omega^k, \tilde{\omega}^l]) := \pi(\omega^k)\pi(\tilde{\omega}^l) - (-1)^{kl}\pi(\tilde{\omega}^l)\pi(\omega^k) \;, \tag{1}$$

for $a \in \mathfrak{g}$, $\omega^k \in \Omega^k\mathfrak{g}$ and $\tilde{\omega}^l \in \Omega^l\mathfrak{g}$. Here, it is essential to have the grading operator Γ, which detects the correct sign for $(-1)^{kl}$. Now we take the differential ideal $\mathcal{J}^*\mathfrak{g} := \ker\pi + d\ker\pi \subset \Omega^*\mathfrak{g}$ and obtain the graded differential Lie algebra

$$\Omega^*_D\mathfrak{g} = \bigoplus_{n=0}^{\infty} \Omega^n_D\mathfrak{g} \;, \qquad \Omega^n_D\mathfrak{g} := \frac{\Omega^n\mathfrak{g}}{\mathcal{J}^n\mathfrak{g}} \cong \frac{\pi(\Omega^n\mathfrak{g})}{\pi(\mathcal{J}^n\mathfrak{g})} \;. \tag{2}$$

In non-commutative geometry, the connection form is simply an element of $\Omega^1_D\mathcal{A}$ and the curvature an element of $\Omega^2_D\mathcal{A}$. In our approach, the situation is different. If one tries to find a reasonable definition for the connection, one encounters more freedom than one expects. Moreover, it is not possible to describe gauge field theories containing U(1)-groups if one takes $\Omega^1_D\mathfrak{g}$-valued connection forms. Therefore, an additional structure is necessary: Not the graded differential Lie algebra $\Omega^*_D\mathfrak{g}$ is the correct space where the connection form and the curvature live, but the space of certain graded Lie endomorphisms $\hat{\mathcal{H}}^*\mathfrak{g}$ of $\Omega^*_D\mathfrak{g}$. Then, the connection form is an element of $\hat{\mathcal{H}}^1\mathfrak{g}$ and the curvature an element of $\hat{\mathcal{H}}^2\mathfrak{g}$. The gauge group is obtained via the exponential mapping. Bosonic and fermionic actions are defined in complete analogy to NCG. The mathematical details of our method can be found in [4].

4 Physical Models

4.1 The Standard Model

The L-cycle for the standard model is the direct transcription of the physical situation. We take the Lie algebra $C^\infty(X) \otimes (\mathrm{su}(3) \oplus \mathrm{su}(2) \oplus \mathrm{u}(1))$, together with its usual representation on the fermionic Hilbert space. The formalism generates one complex Higgs doublet and the well-known quartic potential for it. The bosonic action contains in a unified form the Yang–Mills Lagrangian, the covariant derivatives of the Higgs fields and the Higgs potential. The fermionic action unifies the gauge field couplings with the Yukawa couplings. In the same way as in NCG we obtain tree-level predictions for all bosonic masses. For the simplest scalar product we find in the case that right neutrinos are included

$$m_W = \tfrac{1}{2}m_t \;, \qquad m_Z = m_W/\cos\theta_W \;, \qquad \sin^2\theta_W = \tfrac{3}{8} \;, \qquad m_H = \tfrac{3}{2}m_t \;. \tag{3}$$

Without right neutrinos, the only modification is $m_H = \sqrt{\frac{43}{20}}m_t$. Here, m_t, m_W, m_Z and m_H are the masses of the top quark, the W bosons, the Z boson and the Higgs boson. The details of the construction are given in [5].

4.2 The Flipped $\mathrm{SU}(5)\times\mathrm{U}(1)$-Grand Unification Model

The Lie algebra of the L-cycle is $\mathfrak{g} = C^\infty(X)\otimes \mathrm{su}(5)$. Nevertheless, we do obtain an additional u(1)-gauge field and U(1)-gauge transformations due to the completion of $\Omega^1_D\mathfrak{g}$ to $\hat{\mathcal{H}}^1\mathfrak{g}$. Remarkably, the representation of that u(1)-gauge field on the fermionic Hilbert space is unique and realized in nature. The 3 generations of 16 fermions occurring in nature are assigned to the su(5)-representation $(\underline{10}\oplus\underline{5}^*\oplus\underline{1})\otimes\mathbb{C}^3$. For technical reasons we need 4 copies of the fermions. Thus, the representation π of $a\in\mathfrak{g}$ on the fermionic Hilbert space is defined by

$$\pi(a) := \mathrm{diag}\big(\hat{A}\ ,\ \hat{A}\ ,\ \overline{\hat{A}}\ ,\ \overline{\hat{A}}\big)\ ,\qquad \hat{A} := \mathrm{diag}\big(\pi_{10}(a)\otimes\mathbf{1}_3\ ,\ \overline{a}\otimes\mathbf{1}_3\ ,\ 0_3\big)\ . \tag{4}$$

Here, $\pi_{10}(a)$ is the embedding of $a\in\mathrm{su}(5)\equiv\underline{24}$ into $\mathrm{End}(\underline{10}) = \underline{1}\oplus\underline{24}\oplus\underline{75}$.

The mass matrix $\mathcal{M}$ of the L-cycle consists of two different contributions,

$$\mathcal{M} = \left(\begin{array}{cc|cc} 0 & \mathcal{M}_i & \mathcal{M}_f & 0 \\ \mathcal{M}_i^* & 0 & 0 & \mathcal{M}_f \\ \hline \mathcal{M}_f^* & 0 & 0 & \overline{\mathcal{M}_i} \\ 0 & \mathcal{M}_f^* & \mathcal{M}_i^T & 0 \end{array}\right)\ . \tag{5}$$

The 48×48-matrix $\mathcal{M}_f = \mathcal{M}_f^T$ is the fermionic mass matrix. Let M_u, M_d, M_e be the mass matrices for the (u,c,t), (d,s,b) and (e,μ,τ) fermion sectors. Let M_n and M_N be Dirac and Majorana mass matrices for the (ν_e,ν_μ,ν_τ) neutrino sector. Let $M_{\tilde{u}} = \frac{1}{4}(3M_u+M_n)$ and $M_{\tilde{n}} = \frac{1}{4}(M_u-M_n)$. The sites where the generation matrices occur in $\mathcal{M}_f$ coincide with a combination of the representations $\underline{5}$, $\underline{45}$ and $\underline{50}$ of su(5). Let n, n', m' be appropriate elements of $\underline{5}, \underline{45}^*, \underline{50}$, in this order. Then one has

$$\mathcal{M}_f := \begin{pmatrix} \mathrm{i}\pi_{10,10}(n)\otimes M_d + \mathrm{i}m'\otimes M_N & \mathrm{i}\pi_{10,5}(n)\otimes M_{\tilde{u}} + \mathrm{i}n'\otimes M_{\tilde{n}} & 0 \\ \mathrm{i}\pi_{10,5}(n)^T\otimes M_{\tilde{u}}^T + \mathrm{i}n'^T\otimes M_{\tilde{n}}^T & 0 & \mathrm{i}\pi_{5,1}(n)\otimes M_e \\ 0 & \mathrm{i}\pi_{5,1}(n)^T\otimes M_e^T & 0 \end{pmatrix}, \tag{6}$$

where $\pi_{i,j}(n)$ denote embeddings of $n\in\underline{5}$.

The block diagonal part $\mathcal{M}_i$ is responsible for the desired symmetry breaking pattern of su(5). Let $m\in\underline{24}$ be the appropriate generator of su(5). Then we have

$$\mathcal{M}_i := \mathrm{diag}(\mathrm{i}\pi_{10}(m)\otimes M_{10}\ ,\ \overline{-\mathrm{i}m\otimes M_5}\ ,\ 0_3)\ , \tag{7}$$

where M_{10} and M_5 are arbitrary 3×3-matrices with sufficiently large eigenvalues.

To this L-cycle we apply our formalism, which performs the following job: It extends the matrix $a\in\mathrm{su}(5)$ to a su(5)–gauge field and generates a u(1)-gauge field. The representation of $\mathrm{i}f\in C^\infty(X)\otimes\mathrm{u}(1)$ on the fermionic Hilbert space is

$$\pi(\mathrm{i}f) := \mathrm{diag}\big(\tilde{F},\tilde{F},\overline{\tilde{F}},\overline{\tilde{F}}\big)\,,\quad \tilde{F} := \mathrm{diag}\big(-\tfrac{1}{2}\mathrm{i}f\mathbf{1}_{10}\otimes\mathbf{1}_3\,,-\tfrac{3}{2}\overline{\mathrm{i}f}\mathbf{1}_5\otimes\mathbf{1}_3\,,-\tfrac{5}{2}\mathrm{i}f\otimes\mathbf{1}_3\big)\,. \tag{8}$$

Moreover, the formalism extends the matrices m, n, n', m' to $\underline{24}, \underline{5}, \underline{45}^*, \underline{50}$-Higgs multiplets, in that order. Totally, there are 224 Higgs fields and 25 gauge bosons. The connection form contains both the Yang–Mills gauge fields and the Higgs fields. The bosonic Lagrangian contains the usual Yang–Mills Lagrangian, the covariant derivatives of the Higgs fields and the (very complicated) Higgs potential, whose minimum is given by the configuration $\{m, n, n', m'\}$. The coefficients of products of Higgs fields occurring in the Higgs potential are certain traces of combinations of the mass matrices $M_{u,d,e,n,N}$ and $M_{10,5}$. Thus, if we fix the mass matrix $\mathcal{M}$ then all masses of Higgs and gauge bosons are determined.

The spontaneous symmetry breaking from $C^\infty(X) \otimes (\mathrm{su}(5) \oplus \mathrm{u}(1))$ to $C^\infty(X) \otimes (\mathrm{su}(3)_C \oplus \mathrm{u}(1)_{EM})$ removes 16 Higgs components and gives mass to 16 gauge bosons. There occurs precisely one light Higgs field H, whose upper bound $m_H = 1.45\, m_t$ for the mass is independent of the GUT-matrices M_{10} and M_5. The reason that only an upper bound can be given is the incomplete knowledge of the input parameters. The Higgs field H is a certain linear combination of neutral Higgs fields of the $\underline{5}$-representation and the $\underline{45}^*$-representation. It has precisely the same properties as the standard model Higgs field. We get the same predictions (3) for the Weinberg angle and for the masses of the $W^\pm$ and Z bosons as in the standard model.

The remaining 207 Higgs fields and 13 gauge fields are too heavy to observe. All particles with fractional-valued electric charge, which therefore lead to proton decay, receive a mass of the order of the GUT-scale. Particles with integer-valued charge receive a mass either of the order of that GUT-scale or few orders inferior. The details of the construction can be found in [6].

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