Renormalisation group approach to noncommutative quantum field theory

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Summary. We present the main ideas of our proof that the real ϕ^4-models on noncommutative $\mathbb{R}^2$ and noncommutative $\mathbb{R}^4$ are renormalisable to all orders when working in the matrix base. The proof uses renormalisation group techniques, power-counting theorems for ribbon graphs as well as orthogonal polynomials.

1 Introduction

Quantum field theory on Euclidean or Minkowski space is extremely successful. For suitably chosen action functionals one achieves a remarkable agreement of up to 10^{-11} between theoretical predictions and experimental data. However, combining the fundamental principles of both general relativity and quantum mechanics one concludes that space(-time) cannot be a differentiable manifold [1]. To the best of our knowledge, such a possibility was first discussed in [2].

Since geometric concepts are indispensable in physics, we need a replacement for the space-time manifold which still has a geometric interpretation. Quantum physics tells us that whenever there are measurement limits we have to describe the situation by non-commuting operators on a Hilbert space. Fortunately for physics, mathematicians have developed a generalisation of geometry, baptised noncommutative geometry [3], which is perfectly designed for our purpose. However, in physics we need more than just a better geometry: We need renormalisable quantum field theories modelled on such a noncommutative geometry.

Remarkably, it turned out to be very difficult to renormalise quantum field theories even on the simplest noncommutative spaces [4]. It would be a wrong conclusion, however, that this problem singles out the standard commutative geometry as the only one compatible with quantum field theory. The problem tells us that we are still at the very beginning of *understanding* quantum field theory. Thus, apart from curing the contradiction between gravity and quantum physics, in doing quantum field theory on noncommutative geometries we learn a lot about quantum field theory itself.

2 Field theory on noncommutative $\mathbb{R}^D$ in momentum space

The simplest noncommutative generalisation of Euclidean space is the so-called noncommutative $\mathbb{R}^D$. Although this space arises naturally in a certain limit of string theory [5], we should not expect that it is a good model for nature. In particular, the noncommutative $\mathbb{R}^D$ does not allow for gravity. For us the main purpose of this space is to develop an understanding of quantum field theory which has a broader range of applicability.

The noncommutative $\mathbb{R}^D$, $D = 2, 4, 6, \dots$, is defined as the algebra $\mathbb{R}^D_\theta$ which as a vector space is given by the space $\mathcal{S}(\mathbb{R}^D)$ of (complex-valued) Schwartz class functions of rapid decay, equipped with the multiplication rule

$$(a \star b)(x) = \int \frac{d^D k}{(2\pi)^D} \int d^D y \; a(x+\tfrac{1}{2}\theta\cdot k)\, b(x+y)\, \mathrm{e}^{\mathrm{i}k\cdot y} \;, \qquad (1)$$

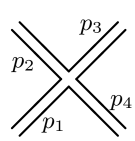
$$(\theta\cdot k)^\mu = \theta^{\mu\nu} k_\nu \;, \quad k\cdot y = k_\mu y^\mu \;, \quad \theta^{\mu\nu} = -\theta^{\nu\mu} \;.$$

The entries $\theta^{\mu\nu}$ in (1) have the dimension of an area. The physical interpretation is $\|\theta\| \approx \ell_P^2$. Much information about the noncommutative $\mathbb{R}^D$ can be found in [6].

A field theory is defined by an action functional. We obtain action functionals on $\mathbb{R}^D_\theta$ by replacing in standard action functionals the ordinary product of functions by the $\star$-product. For example, the noncommutative ϕ^4-action is given by

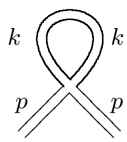
$$S[\phi] := \int d^D x \Big(\frac{1}{2} \partial_\mu \phi \star \partial^\mu \phi + \frac{1}{2} m^2 \phi \star \phi + \frac{\lambda}{4!} \phi \star \phi \star \phi \star \phi \Big) \;. \qquad (2)$$

The action (2) is then inserted into the partition function which we solve perturbatively by Feynman graphs. Due to $\int d^D x\, (a \star b)(x) = \int d^D x\, a(x) b(x)$, the propagator in momentum space is unchanged. For later purpose it is, however, convenient to write it as a double line, $\overset{p}{=\!=\!=} \;= (p^2+m^2)^{-1}$. The novelty are phase factors in the vertices, which we also write in double line notation,



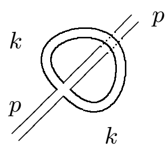
$$= \frac{\lambda}{4!}\, \mathrm{e}^{-\frac{\mathrm{i}}{2} \sum_{i<j} p_i^\mu p_j^\nu \theta_{\mu\nu}} \;. \qquad (3)$$

The double line notation reflects the fact that the vertex (3) is invariant only under cyclic permutations of the legs (using momentum conservation). The resulting Feynman graphs are *ribbon graphs* which depend crucially on how the valences of the vertices are connected. For *planar graphs* the total phase factor of the integrand is independent of internal momenta, whereas *non-planar graphs* have a total phase factor which involves internal momenta. For example, the one-loop contribution to the two-point function splits as follows into a planar part



$$= \frac{\lambda}{6} \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2+m^2} \tag{4}$$

and a non-planar part



$$= \frac{\lambda}{12} \int \frac{d^4k}{(2\pi)^4} \frac{\mathrm{e}^{\mathrm{i}p^\mu k^\nu \theta_{\mu\nu}}}{k^2+m^2} = \frac{\lambda}{48\pi^2} \sqrt{\frac{m^2}{\tilde{p}^2}}\, K_1\Big(\sqrt{m^2\tilde{p}^2}\Big)\,, \tag{5}$$

where $\tilde{p}_\mu := \theta_{\mu\nu} p^\nu$.

Planar graphs are treated as usual. Here, the contribution (4) can entirely be removed by a suitable normalisation condition for the physical mass. The contribution from the non-planar graph (5) is—at first sight—finite, which is a relict of the original motivation that noncommutativity would serve as regulator. The finiteness is important, because the momentum dependence (5) does not appear in the original action (2), which means that a divergence of the form (5) cannot be absorbed by multiplicative renormalisation.

However, the expansion of the modified Bessel function K_1 shows that the contribution (5) behaves $\sim \tilde{p}^{-2}$ for small momenta. If we insert the graph (5) declared as finite as a subgraph into a bigger graph, one easily builds examples (with an arbitrary number of external legs) which lead to non-integrable integrals at small inner momenta. This is the so-called UV/IR-mixing problem [4].

The heuristic argumentation can be made exact: Chepelev and Roiban have proven a power-counting theorem [7, 8] which relates the power-counting degree of divergence to the topology of the ribbon graph. The rough summary of the power-counting theorem is that noncommutative field theories with quadratic divergences become meaningless beyond a certain loop order. The situation is better for field theories with logarithmic UV/IR-divergences, e.g. supersymmetric models. These can be formulated to any loop order. However, the logarithmic IR-divergences at exceptional external momenta are still present so that the correlation functions are unbounded: For every $\delta > 0$ one finds non-exceptional momenta such that $\big|\langle\phi(p_1)\dots\phi(p_n)\rangle\big| > \frac{1}{\delta}$. In the remainder of this article we present an approach which solves these problems.

3 Renormalisation group approach to noncommutative scalar models

We have seen that quantum field theories on noncommutative $\mathbb{R}^D$ are not renormalisable by standard Feynman graph evaluations. One may speculate that the origin of this problem is the too naïve way one performs the continuum limit. A way to treat the limit more carefully is the use of flow equations. The idea goes back to Wilson [9]. It was then used by Polchinski [10] to give a very efficient renormalisability proof of commutative ϕ^4-theory. Applying Polchinski's method to the noncommutative ϕ^4-model, we can hope to be able to prove renormalisability to all orders, too. There is, however, a serious problem of the momentum space proof. We have to guarantee that planar graphs only appear in the distinguished interaction coefficients for which we fix the boundary condition at the renormalisation scale Λ_R. Non-planar graphs have phase factors which involve inner momenta. Polchinski's method consists in taking norms of the interaction coefficients, and these norms ignore possible phase factors. Thus, we would find that boundary conditions for non-planar graphs at Λ_R are required. Since there is an infinite number of different non-planar structures, the model is not renormalisable in this way. A more careful examination of the phase factors is also not possible because the cut-off integrals prevent the Gaußian integration required for the parametric integral representation [7, 8].

Fortunately, there is a matrix representation of the noncommutative $\mathbb{R}^D$ where the $\star$-product becomes a simple product of infinite matrices. The price for this simplification is that the propagator becomes complicated, but the difficulties can be overcome.

3.1 Matrix representation

For simplicity we restrict ourselves to the noncommutative $\mathbb{R}^2$. There exists a matrix base $\{f_{mn}(x)\}_{m,n\in\mathbb{N}}$ of the noncommutative $\mathbb{R}^2$ which satisfies

$$(f_{mn} \star f_{kl})(x) = \delta_{nk} f_{ml}(x) , \qquad \int d^2x \; f_{mn}(x) = 2\pi\theta_1 , \tag{6}$$

where $\theta_1 := \theta_{12} = -\theta_{21}$. In terms of radial coordinates $x_1 = \rho\cos\varphi,\ x_2 = \rho\sin\varphi$ one has

$$f_{mn}(\rho,\varphi) = 2(-1)^m \mathrm{e}^{\mathrm{i}(n-m)\varphi} \sqrt{\tfrac{m!}{n!}} \Big(\sqrt{\tfrac{2\rho^2}{\theta_1}}\Big)^{n-m} L_m^{n-m}\Big(\tfrac{2\rho^2}{\theta_1}\Big)\, \mathrm{e}^{-\frac{\rho^2}{\theta_1}} , \tag{7}$$

where $L_n^\alpha(z)$ are the Laguerre polynomials. See also [6]. The matrix representation was also used to obtain exactly solvable noncommutative quantum field theories [11, 12].

Now we can write down the noncommutative ϕ^4-action in the matrix base by expanding the field as $\phi(x) = \sum_{m,n\in\mathbb{N}} \phi_{mn} f_{mn}(x)$. It turns out, however, that in order to prove renormalisability we have to consider a more general action than (2) at the initial scale Λ_0. This action is obtained by adding a harmonic oscillator potential to the standard noncommutative ϕ^4-action:

$$\begin{aligned} S[\phi] &:= \int d^2x \Big(\frac{1}{2}\partial_\mu\phi \star \partial^\mu\phi + 2\Omega^2 (\tilde{x}^\mu\phi)\star(\tilde{x}_\mu\phi) + \frac{\mu_0^2}{2}\phi\star\phi \\ &\qquad\qquad + \frac{\lambda}{4!}\phi\star\phi\star\phi\star\phi\Big)(x) \\ &= 2\pi\theta_1 \sum_{m,n,k,l} \Big(\frac{1}{2}G_{mn;kl}\phi_{mn}\phi_{kl} + \frac{\lambda}{4!}\phi_{mn}\phi_{nk}\phi_{kl}\phi_{lm}\Big)\,, \end{aligned} \tag{8}$$

where $\tilde{x}^\mu := \theta^{\mu\nu}x_\nu$ and

$$G_{mn;kl} := \int \frac{d^2x}{2\pi\theta_1}\,\Big(\partial_\mu f_{mn} \star \partial^\mu f_{kl} + 4\Omega^2 (\tilde{x}^\mu f_{mn})\star(\tilde{x}_\mu f_{kl}) + \mu_0^2 f_{mn}\star f_{kl}\Big)\,. \tag{9}$$

We view Ω as a regulator and refer to the action (8) as describing a regularised ϕ^4-model. The action (8) could also be obtained by restricting a complex ϕ^4-model with magnetic field [11, 12] to the real part. One finds

$$\begin{aligned} G_{mn;kl} &= \Big(\mu_0^2 + \frac{2}{\theta_1}(1+\Omega^2)(n+m+1)\Big)\delta_{nk}\delta_{ml} \\ &\quad - \frac{2}{\theta_1}(1-\Omega^2)\Big(\sqrt{(n+1)(m+1)}\,\delta_{n+1,k}\delta_{m+1,l} - \sqrt{nm}\,\delta_{n-1,k}\delta_{m-1,l}\Big)\,. \end{aligned} \tag{10}$$

The kinetic matrix $G_{mn;kl}$ has the important property that $G_{mn;kl} = 0$ unless $m+k = n+l$. The same relation is induced for the propagator $\Delta_{nm;lk}$ defined by $\sum_{k,l=0}^\infty G_{mn;kl}\Delta_{lk;sr} = \sum_{k,l=0}^\infty \Delta_{nm;lk}G_{kl;rs} = \delta_{mr}\delta_{ns}$. In order to evaluate the propagator we first diagonalise the kinetic matrix $G_{mn;kl}$:

$$G_{m,m+\alpha;l+\alpha,l} = \sum_{y\in\mathbb{N}} U^{(\alpha)}_{my}\big(\mu_0^2 + \tfrac{4\Omega}{\theta}(2y+\alpha+1)\big)U^{(\alpha)}_{yl}\,, \tag{11}$$

$$\begin{aligned} U^{(\alpha)}_{ny} &= \sqrt{\binom{\alpha+n}{n}\binom{\alpha+y}{y}\Big(\frac{1-\Omega}{1+\Omega}\Big)^{2n+2y+\alpha+1}\Big(\frac{4\Omega}{1-\Omega^2}\Big)^{\alpha+1}} \\ &\quad \times M_n\Big(y; 1+\alpha, \frac{(1-\Omega)^2}{(1+\Omega)^2}\Big)\,, \end{aligned} \tag{12}$$

where $M_n(y;\beta,c) = {}_2F_1\Big(\genfrac{}{}{0pt}{}{-n,-y}{\beta}\Big|1-c\Big)$ are the (orthogonal) Meixner polynomials [13]. A lengthy calculation gives

$$\begin{aligned} \Delta_{mn;kl} &= \frac{\theta_1}{2(1+\Omega^2)} \delta_{m+k,n+l} \sum_{v=\frac{|m-l|}{2}}^{\frac{\min(m+l,k+n)}{2}} B\big(\tfrac{1}{2}+\tfrac{\mu_0^2\theta_1}{8\Omega}+\tfrac{1}{2}(m+k)-v, 1+2v\big) \\ &\quad \times \sqrt{\binom{n}{v+\frac{n-k}{2}}\binom{k}{v+\frac{k-n}{2}}\binom{m}{v+\frac{m-l}{2}}\binom{l}{v+\frac{l-m}{2}}} \Big(\frac{1-\Omega}{1+\Omega}\Big)^{2v} \\ &\quad \times {}_2F_1\Bigg(\begin{matrix} 1+2v\,,\ \frac{1}{2}+\frac{\mu_0^2\theta_1}{8\Omega}-\frac{1}{2}(m+k)+v \\ \frac{3}{2}+\frac{\mu_0^2}{2\sqrt{1-\omega}\,\mu^2}+\frac{1}{2}(m+k)+v \end{matrix} \Bigg| \frac{(1-\Omega)^2}{(1+\Omega)^2}\Bigg) . \end{aligned} \tag{13}$$

Here, $B(a,b)$ is the Beta-function and $F\big({a,b \atop c};z\big)$ the hypergeometric function.

3.2 The Polchinski equation for matrix models

We summarise here our derivation [14] of the Polchinski equation for the noncommutative ϕ^4-theory in the matrix base. According to Polchinski's derivation of the exact renormalisation group equation [10] we consider the following cut-off partition function:

$$\begin{aligned} Z[J,\Lambda] &= \int \Big(\prod_{a,b} d\phi_{ab}\Big) \exp\big(-S[\phi,J,\Lambda]\big) , \\ S[\phi,J,\Lambda] &= (2\pi\theta_1)\Big(\sum_{m,n,k,l} \frac{1}{2}\phi_{mn} G^K_{mn;kl}(\Lambda)\,\phi_{kl} + \sum_{m,n,k,l} \phi_{mn} F_{mn;kl}[\Lambda] J_{kl} \\ &\qquad + \sum_{m,n,k,l} \frac{1}{2} J_{mn} E_{mn;kl}[\Lambda] J_{kl} + L[\phi,\Lambda] + C[\Lambda]\Big) , \\ G^K_{mn;kl}(\Lambda) &= \prod_{i\in\{m,n,k,l\}} K\big[\tfrac{i}{\Lambda^2\theta_1}\big]^{-1} G_{mn;kl} . \end{aligned} \tag{14}$$

with $L[0,\Lambda]=0$. The cut-off function $K(x)$ is a smooth decreasing function with $K(x)=1$ for $0\leq x\leq 1$ and $K(x)=0$ for $x\geq 2$. Accordingly, we define

$$\Delta^K_{nm;lk}(\Lambda) = \prod_{i\in\{m,n,k,l\}} K\big[\tfrac{i}{\Lambda^2\theta_1}\big] \Delta_{nm;lk} . \tag{15}$$

The function $C[\Lambda]$ is the vacuum energy and the matrices E and F, which are not necessary in the commutative case, must be introduced because the propagator Δ is non-local. It is in general not possible to separate the support of the sources J from the support of the Λ-variation of K. We would obtain the original problem (without cut-off) for the choice

$$\begin{aligned} L[\phi,\infty] &= \sum_{m,n,k,l} \frac{\lambda}{4!} \phi_{mn}\phi_{nk}\phi_{kl}\phi_{lm} , \\ C[\infty] &= 0 , \qquad E_{mn;kl}[\infty] = 0 , \qquad F_{mn;kl}[\infty] = \delta_{ml}\delta_{nk} . \end{aligned} \tag{16}$$

However, we shall expect divergences in the partition function which require a renormalisation, i.e. additional (divergent) counterterms in $L[\phi,\infty]$.

Following Polchinski [10] we first ask ourselves how to choose L, C, E, F in order to make $Z[J,\Lambda]$ *independent* of Λ. After straightforward calculation one finds the answer

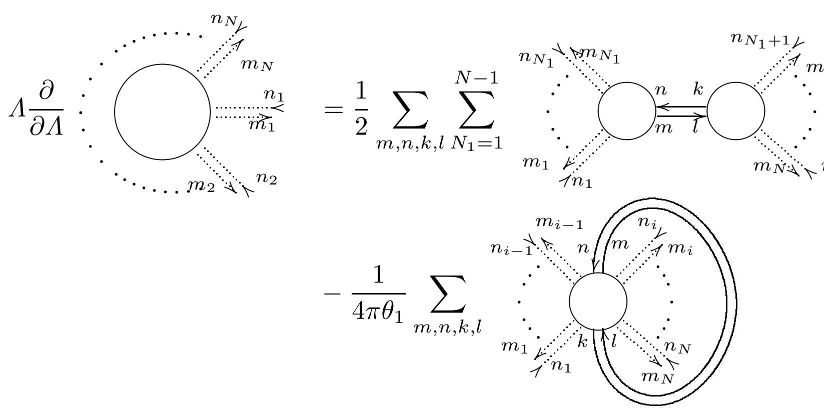
$$\Lambda \frac{\partial L[\phi,\Lambda]}{\partial \Lambda} = \sum_{m,n,k,l} \frac{1}{2} \Lambda \frac{\partial \Delta^K_{nm;lk}(\Lambda)}{\partial \Lambda} \left(\frac{\partial L[\phi,\Lambda]}{\partial \phi_{mn}} \frac{\partial L[\phi,\Lambda]}{\partial \phi_{kl}} - \frac{1}{2\pi\theta_1} \Big[\frac{\partial^2 L[\phi,\Lambda]}{\partial \phi_{mn}\, \partial \phi_{kl}} \Big]_\phi \right), \quad (17)$$

where $\big[f[\phi]\big]_\phi := f[\phi] - f[0]$. The corresponding differential equations for C, E, F are easy to integrate [14]. Now, instead of computing Green's functions from $Z[J,\infty]$ we can equally well start from $Z[J,\Lambda_R]$, where it leads to Feynman graphs with vertices given by the Taylor expansion coefficients $A^{(V)}_{m_1n_1;\dots;m_Nn_N}$ in

$$L[\phi,\Lambda] = \lambda \sum_{V=1}^{\infty} \big(2\pi\theta_1\lambda\big)^{V-1} \sum_{N=2}^{\infty} \frac{1}{N!} \sum_{m_i,n_i} A^{(V)}_{m_1n_1;\dots;m_Nn_N}[\Lambda] \phi_{m_1n_1} \cdots \phi_{m_Nn_N} . \quad (18)$$

These vertices are connected with each other by internal lines $\Delta^K_{nm;lk}(\Lambda)$ and to sources J_{kl} by external lines $\Delta^K_{nm;lk}(\Lambda_0)$. Since the summation variables are cut-off in the propagator (15), loop summations are finite, provided that the interaction coefficients $A^{(V)}_{m_1n_1;\dots;m_Nn_N}[\Lambda]$ are bounded. Thus, renormalisability amounts to prove that for certain initial conditions (parametrised by finitely many parameters!) the evolution of L according to (17) does not produce any divergences.

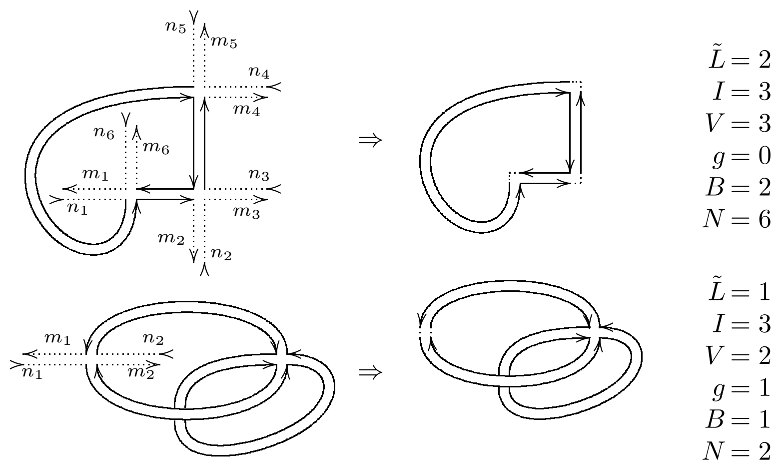
Inserting the expansion (18) into (17) and restricting to the part with N external legs we get the graphical expression



(19)

Combinatorical factors are not shown and symmetrisation in all indices $m_i n_i$ has to be performed. On the rhs of (19) the two valences mn and kl of subgraphs are connected to the ends of a *ribbon* which symbolises the differentiated propagator $\underset{m \quad l}{\overset{n \quad k}{\rightleftarrows}} = \Lambda \frac{\partial}{\partial \Lambda} \Delta^K_{nm;lk}$. We see that for the simple fact that the fields ϕ_{mn} carry two indices, the effective action is expanded into ribbon graphs.

In the expansion of L there will occur very complicated ribbon graphs with crossings of lines which cannot be drawn any more in a plane. A general ribbon graph can, however, be drawn on a *Riemann surface* of some *genus* g. In fact, a ribbon graph *defines* the Riemann surfaces topologically through the *Euler characteristic* χ. We have to regard here the external lines of the ribbon graph as amputated (or closed), which means to directly connect the single lines m_i with n_i for each external leg $m_i n_i$. A few examples may help to understand this procedure:



(20)

The genus is computed from the number $\tilde{L}$ of single-line loops, the number I of internal (double) lines and the number V of vertices of the graph according to Euler's formula $\chi = 2 - 2g = \tilde{L} - I + V$. The number B of boundary components of a ribbon graph is the number of those loops which carry at least one external leg. There can be several possibilities to draw the graph and its Riemann surface, but $\tilde{L}, I, V, B$ and thus g remain unchanged. Indeed, the Polchinski equation (17) interpreted as in (19) tells us which external legs of the vertices are connected. It is completely irrelevant how the ribbons are drawn between these legs. In particular, there is no distinction between overcrossings and undercrossings.

We expect that non-planar ribbon graphs with $g > 0$ and/or $B > 1$ behave differently under the renormalisation flow than planar graphs having $B = 1$ and $g = 0$. This suggests to introduce a further grading in g, B in the interactions coefficients $A^{(V,B,g)}_{m_1 n_1;\dots;m_N n_N}$. Technically, our strategy is to apply the summations in (19) either to the propagator or the subgraph only and to maximise the other object over the summation indices. For that purpose one has to introduce further characterisations of a ribbon graph which disappear at the end, see [14].

3.3 ϕ^4-theory on noncommutative $\mathbb{R}^2$

First one estimates the A-functions by integrating (17) perturbatively between an initial scale Λ_0 to be sent to ∞ later on and the renormalisation scale Λ_R:

Lemma 1. *The homogeneous parts* $A^{(V,B,g)}_{m_1n_1;\dots;m_Nn_N}$ *of the coefficients of the effective action describing a regularised* ϕ^4*-theory on* $\mathbb{R}^2_\theta$ *in the matrix base are for* $2 \leq N \leq 2V{+}2$ *and* $\sum_{i=1}^N (m_i{-}n_i) = 0$ *bounded by*

$$\begin{aligned}&\big|A^{(V,B,g)}_{m_1n_1;\dots;m_Nn_N}[\Lambda,\Lambda_0,\Omega,\rho_0]\big|\\ &\quad\leq \big(\Lambda^2\theta_1\big)^{2-V-B-2g}\Big(\frac{1}{\Omega}\Big)^{3V-\frac{N}{2}+B+2g-2} P^{2V-\frac{N}{2}}\Big[\ln\frac{\Lambda_0}{\Lambda_R}\Big]\,. \end{aligned}\tag{21}$$

We have $A^{(V,B,g)}_{m_1n_1;\dots;m_Nn_N} \equiv 0$ *for* $N > 2V{+}2$ *or* $\sum_{i=1}^N (m_i{-}n_i) \neq 0$*. By* $P^q[x]$ *we denote a polynomial in* x *of degree* q*.*

The proof of (21) for general matrix models by induction goes over 20 pages! The formula specific for the ϕ^4-model on $\mathbb{R}^2_\theta$ follows from the asymptotic behaviour of the cut-off propagator (15), (13) and a certain index summation, see [14, 15].

We see from (21) that the only divergent function is

$$\begin{aligned}A^{(1,1,0)}_{m_1n_1;m_2n_2} &= A^{(1,1,0)}_{00;00}\delta_{m_1n_2}\delta_{m_2n_1}\\ &\quad + \Big(A^{(1,1,0)}_{m_1n_1;m_2n_2}[\Lambda,\Lambda_0,\rho^0] - A^{(1,1,0)}_{00;00}\delta_{m_1n_2}\delta_{m_2n_1}\Big)\,, \end{aligned}\tag{22}$$

which is split into the distinguished divergent function

$$\rho[\Lambda,\Lambda_0,\Omega,\rho^0] := A^{(1,1,0)}_{00;00}[\Lambda,\Lambda_0,\Omega,\rho^0] \tag{23}$$

for which we impose the boundary condition $\rho_R := \rho[\Lambda_R,\Lambda_0,\Omega,\rho^0] = 0$ and a convergent part with boundary condition at Λ_0.

The limit $\Omega \to 0$ in (21) is singular. In fact the estimation for $\Omega = 0$ with an optimal choice of the ρ-coefficients (different than (23)!) would grow with $\big(\Lambda\sqrt{\theta_1}\big)^{V-\frac{N}{2}-B-2g+2}$. Since the exponent of Λ can be arbitrarily large, there would be an infinite number of divergent interaction coefficients, which means that the ϕ^4-model is not renormalisable when keeping $\Omega = 0$.

In order to pass to the limit $\Lambda_0 \to \infty$ one has to control the total Λ_0-dependence of the functions $A^{(V,B,g)}_{m_1n_1;\dots;m_Nn_N}[\Lambda,\Lambda_0,\Omega[\Lambda_0],\rho^0[\Lambda_R,\Lambda_0,\rho_R]]$. This leads again to a differential equation in Λ, see [15]. It is then not difficult to see that the regularised ϕ^4-model with $\Omega > 0$ is renormalisable. It turns out that one can even prove more [15]: On can endow the parameter Ω for the oscillator frequency with an Λ_0-dependence so that in the limit $\Lambda_0 \to \infty$ one obtains a standard ϕ^4-model without the oscillator term:

Theorem 1. *The ϕ^4-model on $\mathbb{R}^2_\theta$ is (order by order in the coupling constant) renormalisable in the matrix base by adjusting the bare mass $\Lambda_0^2\rho[\Lambda_0]$ to give $A^{(1,1,0)}_{00;00}[\Lambda_R] = 0$ and by performing the limit $\Lambda_0 \to \infty$ along the path of regulated models characterised by $\Omega[\Lambda_0] = \big(1+\ln\frac{\Lambda_0}{\Lambda_R}\big)^{-1}$. The limit $A^{(V,B,g)}_{m_1n_1;\dots;m_Nn_N}[\Lambda_R,\infty] := \lim_{\Lambda_0\to\infty} A^{(V,B,g)}_{m_1n_1;\dots;m_Nn_N}[\Lambda_R,\Lambda_0,\Omega[\Lambda_0],\rho^0[\Lambda_0]]$ of the expansion coefficients of the effective action $L[\phi,\Lambda_R,\Lambda_0,\Omega[\Lambda_0],\rho^0[\Lambda_0]]$ exists and satisfies*

$$\begin{aligned}
&\Big|\lambda\big(2\pi\theta_1\lambda\big)^{V-1} A^{(V,B,g)}_{m_1n_1;\dots;m_Nn_N}[\Lambda_R,\infty] \\
&\qquad - \big(2\pi\theta_1\lambda\big)^{V-1} A^{(V,V^e,B,g,\iota)}_{m_1n_1;\dots;m_Nn_N}\Big[\Lambda_R,\Lambda_0,\frac{1}{(1+\ln\frac{\Lambda_0}{\Lambda_R})},\rho^0[\Lambda_0]\Big]\Big| \\
&\quad \leq \frac{\Lambda_R^4}{\Lambda_0^2}\Big(\frac{\lambda}{\Lambda_R^2}\Big)^V \Big(\frac{(1+\ln\frac{\Lambda_0}{\Lambda_R})}{\Lambda_R^2\theta_1}\Big)^{B+2g-1} P^{5V-N-1}\Big[\ln\frac{\Lambda_0}{\Lambda_R}\Big]\,.
\end{aligned} \tag{24}$$

In this way we have proven that the real ϕ^4-model on $\mathbb{R}^2_\theta$ is perturbatively renormalisable when formulated in the matrix base.

It was important to observe that the cut-off action at Λ_0 is (due to the cut-off) not translation-invariant. We are therefore free to break the translational symmetry of the action at Λ_0 even more by adding a harmonic oscillator potential for the fields ϕ. There exists a Λ_0-dependence of the oscillator frequency Ω with $\lim_{\Lambda_0\to\infty}\Omega = 0$ such that the effective action at Λ_R is convergent (and thus bounded) order by order in the coupling constant in the limit $\Lambda_0 \to \infty$. This means that the partition function of the original (translation-invariant) ϕ^4-model without cut-off and with suitable divergent bare mass can equally well be solved by Feynman graphs with propagators cut-off at Λ_R and vertices given by the bounded expansion coefficients of the effective action at Λ_R. Hence, this model is renormalisable, and in contrast to the naïve Feynman graph approach in momentum space [8] there is no problem with exceptional configurations. This makes clear that the adaptation of Polchinski's renormalisation programme is the preferred method for noncommutative field theories.

3.4 ϕ^4-theory on noncommutative $\mathbb{R}^4$

The renormalisation of ϕ^4-theory on $\mathbb{R}^4_\theta$ in the matrix base is performed similarly [16]. We choose a coordinate system in which $\theta_1 = \theta_{12} = -\theta_{21}$ and $\theta_2 = \theta_{34} = -\theta_{43}$ are the only non-vanishing components of θ. Moreover, we assume $\theta_1 = \theta_2$ for simplicity. Then we expand the scalar field according to $\phi(x) = \sum_{m_1,n_1,m_2,n_2\in\mathbb{N}} \phi_{\substack{m_1\,n_1\\ m_2\,n_2}} f_{m_1n_1}(x_1,x_2) f_{m_2n_2}(x_3,x_4)$. The action (8) with integration over $\mathbb{R}^4$ leads then to a kinetic term generalising (10) and a propagator generalising (13). Using estimates on the asymptotic behaviour of that propagator one proves the four-dimensional generalisation of Lemma 1 on the power-counting degree of the N-point functions. For $\Omega > 0$ one finds that all non-planar graphs ($B > 1$ and/or $g > 0$) and all graphs with $N \geq 6$ external legs are convergent.

The remaining infinitely many planar two- and four-point functions have to be split into a divergent ρ-part and a convergent complement. Using some sort of locality for the propagator (13), which is a consequence of its derivation from Meixner polynomials, one proves that

$$
\begin{aligned}
A^{\text{planar}}_{\substack{m_1\, n_1\\ m_2\, n_2};\substack{k_1\, l_1\\ k_2\, l_2}} - \delta_{m_1 l_1}\delta_{n_1 k_1}\delta_{m_2 l_2}\delta_{n_2 k_2}\Big(& A^{\text{planar}}_{\substack{0\,0\\0\,0};\substack{0\,0\\0\,0}} \\
& + m_1\Big(A^{\text{planar}}_{\substack{1\,0\\0\,0};\substack{0\,1\\0\,0}} - A^{\text{planar}}_{\substack{0\,0\\0\,0};\substack{0\,0\\0\,0}}\Big) + n_1\Big(A^{\text{planar}}_{\substack{0\,1\\0\,0};\substack{1\,0\\0\,0}} - A^{\text{planar}}_{\substack{0\,0\\0\,0};\substack{0\,0\\0\,0}}\Big) \\
& + m_2\Big(A^{\text{planar}}_{\substack{0\,0\\1\,0};\substack{0\,0\\0\,1}} - A^{\text{planar}}_{\substack{0\,0\\0\,0};\substack{0\,0\\0\,0}}\Big) + n_2\Big(A^{\text{planar}}_{\substack{0\,0\\0\,1};\substack{0\,0\\1\,0}} - A^{\text{planar}}_{\substack{0\,0\\0\,0};\substack{0\,0\\0\,0}}\Big)\Big) \\
& - \big(\sqrt{(m_1{+}1)(n_1{+}1)}\delta_{m_1+1,l_1}\delta_{n_1+1,k_1}\delta_{m_2 l_2}\delta_{n_2 k_2} \\
& \quad + \sqrt{m_1 n_1}\delta_{m_1-1,l_1}\delta_{n_1-1,k_1}\delta_{m_2 l_2}\delta_{n_2 k_2}\big) A^{\text{planar}}_{\substack{1\,1\\0\,0};\substack{0\,0\\0\,0}} \\
& - \big(\sqrt{(m_2{+}1)(n_2{+}1)}\delta_{m_2+1,l_2}\delta_{n_2+1,k_2}\delta_{m_1 l_1}\delta_{n_1 k_1} \\
& \quad + \sqrt{m_2 n_2}\delta_{m_2-1,2_1}\delta_{n_2-1,k_2}\delta_{m_1 l_1}\delta_{n_1 k_1}\big) A^{\text{planar}}_{\substack{0\,0\\1\,1};\substack{0\,0\\0\,0}}\ ,
\end{aligned}
\tag{25}
$$

$$
A^{\text{planar}}_{\substack{m_1\, n_1\\ m'_1\, n'_1};\dots;\substack{m_4\, n_4\\ m'_4\, n'_4}} - \big(\tfrac{1}{6}\delta_{\substack{n_1\, m_2\\ n'_1\, m'_2}}\delta_{\substack{n_2\, m_3\\ n'_2\, m'_4}}\delta_{\substack{n_3\, m_4\\ n'_3\, m'_4}}\delta_{\substack{n_4\, m_1\\ n'_4\, m'_1}} + 5\text{ perm's}\big) A^{\text{planar}}_{\substack{0\,0\\0\,0};\dots;\substack{0\,0\\0\,0}}\ , \tag{26}
$$

are convergent functions, thus identifying

$$
\begin{aligned}
\rho_1 &:= A^{\text{planar}}_{\substack{0\,0\\0\,0};\substack{0\,0\\0\,0}}\ , \\
\rho_2 &:= A^{\text{planar}}_{\substack{1\,0\\0\,0};\substack{0\,1\\0\,0}} - A^{\text{planar}}_{\substack{0\,0\\0\,0};\substack{0\,0\\0\,0}} = A^{\text{planar}}_{\substack{0\,0\\1\,0};\substack{0\,0\\0\,1}} - A^{\text{planar}}_{\substack{0\,0\\0\,0};\substack{0\,0\\0\,0}}\ , \\
\rho_3 &:= A^{\text{planar}}_{\substack{1\,1\\0\,0};\substack{0\,0\\0\,0}} = A^{\text{planar}}_{\substack{0\,0\\1\,1};\substack{0\,0\\0\,0}} \\
\rho_4 &:= A^{\text{planar}}_{\substack{0\,0\\0\,0};\substack{0\,0\\0\,0};\substack{0\,0\\0\,0};\substack{0\,0\\0\,0}}
\end{aligned}
\tag{27}
$$

as the distinguished divergent ρ-functions for which we impose boundary conditions at Λ_R. Details will be given in [16].

The function ρ_3 has no commutative analogue. Due to (25) it corresponds to a normalisation condition for the frequency parameter Ω in (10). This means that in contrast to the two-dimensional case we cannot remove the oscillator potential with the limit $\Lambda_0 \to \infty$. In other words, the oscillator potential in (8) is a necessary companionship to the $\star$-product interaction. This observation is in agreement with the UV/IR-entanglement first observed in [4]. Whereas the UV/IR-problem prevents the renormalisation of ϕ^4-theory on $\mathbb{R}^4_\theta$ in momentum space [8], we have found a self-consistent solution of the problem by providing the unique (due to properties of the Meixner polynomials) renormalisable extension of the action.

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