Renormalizable noncommutative quantum field theory

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Abstract

We discuss the noncommutative ϕ_4^4-quantum field theory as an example of a renormalizable field theory. Using a Ward identity Disertori, Gurau, Magnen and Rivasseau were able to proof the vanishing of the beta functions for the coupling constants to all orders in pertubation theory.
We extend this work and obtain from the Schwinger-Dyson equation a non-linear integral equation for the renormalised two-point function alone. The non-trivial renormalised four-point function fulfils a linear integral equation with the inhomogeneity determined by the two-point function. These integral equations might be the starting point of a nonperturbative construction of a nc quantum field theory.

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1 Introduction

This report is based on our recent work [1].

In order to improve the problems of four-dimensional quantum field theory it was suggested to include "gravity effects" through deforming space-time. The canonical deformation is particularly simple, but the resulting models suffer from the UV/IR-mixing [2]. In our previous work [3] we found a way to handle this problem. We realized that the model defined by the action

$$S = \int d^4x \Big(\frac{1}{2}\phi(-\Delta + \Omega^2 \tilde{x}^2 + \mu^2)\phi + \frac{\lambda}{4}\phi \star \phi \star \phi \star \phi \Big)(x) \tag{1}$$

is renormalisable to all orders of perturbation theory. Here, $\star$ refers to the Moyal product parametrised by the antisymmetric 4×4-matrix Θ, and $\tilde{x} = 2\Theta^{-1}x$. The model is covariant under the Langmann-Szabo duality transformation [4] and becomes self-dual at $\Omega = 1$. Certain variants have also been treated, see [5] for a review.

Evaluation of the β-functions for the coupling constants Ω, λ in first order of perturbation theory leads to a coupled dynamical system which indicates a fixed-point at $\Omega = 1$, while λ remains bounded [6, 7]. The vanishing of the β-function at $\Omega = 1$ was next proven in [8] at three-loop order and finally in [9] to all orders of perturbation theory. It implies that there is no infinite renormalisation of λ, and a non-perturbative construction seems possible [10]. The Landau ghost problem is solved.

The vanishing of the β-function to all orders has been obtained using a Ward identity [9]. We extend this work and derive an integral equation for the two-point function alone by using the Ward identity and Schwinger-Dyson equations. Usually, Schwinger-Dyson equations couple the two-point function to the four-point function. In our model, we show that the Ward identity allows to express the four-point function in terms of the two-point function, resulting in an equation for the two-point function alone. This is achieved in the first step for the bare two-point function. We are able to perform the mass and wavefunction renormalisation directly in the integral equation, giving a *self-consistent non-linear equation for the renormalised two-point function alone*.

Higher n-point functions fulfil a *linear* (inhomogeneous) Schwinger-Dyson equation, with the inhomogeneity given by m-point functions with $m < n$. This means that solving our equation for the two-point function leads to a full non-perturbative construction of this interacting quantum field theory in four dimensions.

So far we treated our equation perturbatively up to third order in λ. The solution shows an interesting number-theoretic structure.

We hope that a detailed analysis of our model will help for a non-perturbative treatment of more realistic quantum field theories.

2 Matrix Model

It is convenient to write the action (1) in the matrix base of the Moyal space, see [3, 13]. It simplifies enormously at the self-duality point $\Omega = 1$. We write down the resulting action functionals for the *bare* quantities, which involves the bare mass μ_{bare} and the wave function renormalisation $\phi \mapsto Z^{\frac{1}{2}}\phi$. For simplicity we fix the length scale to $\theta = 4$. This gives

$$S = \sum_{m,n\in\mathbb{N}^2_\Lambda} \frac{1}{2}\phi_{mn}H_{mn}\phi_{nm} + V(\phi)\;, \tag{2}$$

$$H_{mn} = Z\big(\mu^2_{bare} + |m| + |n|\big)\;, \qquad V(\phi) = \frac{Z^2\lambda}{4}\sum_{m,n,k,l\in\mathbb{N}^2_\Lambda}\phi_{mn}\phi_{nk}\phi_{kl}\phi_{lm}\;, \tag{3}$$

It is already used that this model has no renormalisation of the coupling constant [9]. All summation indices $m, n, \ldots$ belong to $\mathbb{N}^2$, with $|m| := m_1 + m_2$. The symbol $\mathbb{N}^2_\Lambda$ refers to a cut-off in the matrix size. The scalar field is real, $\phi_{mn} = \overline{\phi_{nm}}$.

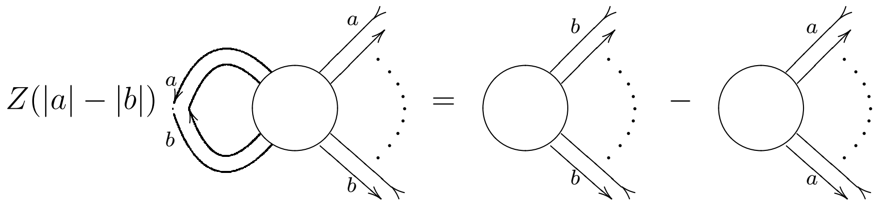
3 Ward Identity

The key step in the proof [9] that the β-function vanishes is the discovery of a Ward identity induced by inner automorphisms $\phi \mapsto U\phi U^\dagger$. Inserting into the connected graphs one special insertion vertex

$$V^{ins}_{ab} := \sum_n (H_{an} - H_{nb})\phi_{bn}\phi_{na} \tag{4}$$

is the same as the difference of graphs with external indices b and a, respectively, $Z(|a| - |b|)G^{ins}_{[ab]\ldots} = G_{b\ldots} - G_{a\ldots}$:

We write Feynman graphs in the self-dual ϕ^4_4-model as ribbon graphs on a genus-g Riemann surface with B external faces. Adding for each external face an external vertex to get a closed surface, the matrix index is constant at every face. Inserting the special vertex V^{ins}_{ab} leads, however, to an index jump from a to b in an external face which meets an external vertex. The corresponding external sources at the jumped face are thus J_{na} and J_{bm} for some other indices m, n. According to the Ward identity, this is the same as the difference between the graphs with face index b and a, respectively:



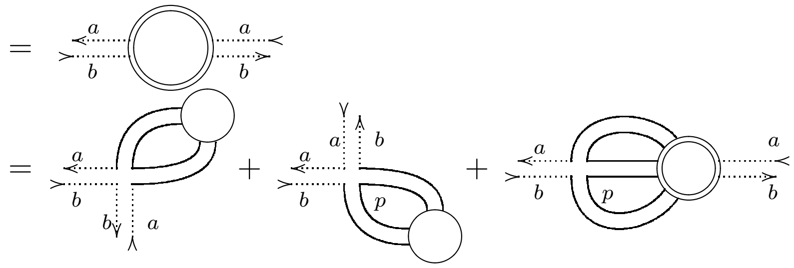
(5)

$$Z(|a| - |b|)G^{ins}_{[ab]\ldots} = G_{b\ldots} - G_{a\ldots}\;. \tag{6}$$

The dots in (6) stand for the remaining face indices. We have used $H_{an} - H_{nb} = Z(|a| - |b|)$.

4 Schwinger-Dyson equation

The Schwinger-Dyson equation for the one-particle irreducible two-point function Γ^{ab} reads

$\Gamma_{ab} =$  (7)

The sum of the last two graphs can be reexpressed in terms of the two-point function with insertion vertex,

$$\begin{aligned}\Gamma_{ab} &= Z^2\lambda \sum_p \Big(G_{ap} + G^{-1}_{ab} G^{ins}_{[ap]b}\Big) = Z^2\lambda \sum_p \Big(G_{ap} - G^{-1}_{ab} \frac{G_{bp} - G_{ba}}{Z(|p| - |a|)}\Big) \\ &= Z^2\lambda \sum_p \Big(\frac{1}{H_{ap} - \Gamma_{ap}} + \frac{1}{H_{bp} - \Gamma_{bp}} - \frac{1}{H_{bp} - \Gamma_{bp}} \frac{(\Gamma_{bp} - \Gamma_{ab})}{Z(|p| - |a|)}\Big)\,.\end{aligned} \tag{8}$$

This is a closed equation for the two-point function alone. It involves the divergent quantities Γ_{bp} and Z, μ_{bare}.

5 Renormalization

Introducing the renormalised planar two-point function Γ^{ren}_{ab} by Taylor expansion $\Gamma_{ab} = Z\mu^2_{bare} - \mu^2 + (Z-1)(|a|+|b|) + \Gamma^{ren}_{ab}$, with $\Gamma^{ren}_{00} = 0$ and $(\partial\Gamma^{ren})_{00} = 0$, we obtain a coupled system of equations for Γ^{ren}_{ab}, Z and μ_{bare}. It leads to a closed equation for the renormalised function Γ^{ren}_{ab} alone, which is further analysed in the integral representation.

We replace the indices in $a, b, \ldots \mathbb{N}$ by continuous variables in $\mathbb{R}_+$. Equation (8) depends only on the length $|a| = a_1 + a_2$ of indices. The Taylor expansion respects this feature, so that we replace $\sum_{p\in\mathbb{N}^2_\Lambda}$ by $\int_0^\Lambda |p|\, dp$. After a convenient change of variables $|a| =: \mu^2 \frac{\alpha}{1-\alpha}$, $|p| =: \mu^2 \frac{\rho}{1-\rho}$ and

$$\Gamma^{ren}_{ab} =: \mu^2 \frac{1-\alpha\beta}{(1-\alpha)(1-\beta)} \Big(1 - \frac{1}{G_{\alpha\beta}}\Big)\,, \tag{9}$$

and using an identity resulting from the symmetry $G_{0\alpha} = G_{\alpha 0}$, we arrive at [1]:

Theorem 1 *The renormalised planar connected two-point function $G_{\alpha\beta}$ of self-dual noncommutative ϕ^4_4-theory satisfies the integral equation*

$$G_{\alpha\beta} = 1 + \lambda\Big(\frac{1-\alpha}{1-\alpha\beta}\big(\mathcal{M}_\beta - \mathcal{L}_\beta - \beta\mathcal{Y}\big) + \frac{1-\beta}{1-\alpha\beta}\big(\mathcal{M}_\alpha - \mathcal{L}_\alpha - \alpha\mathcal{Y}\big) \\ + \frac{1-\beta}{1-\alpha\beta}\Big(\frac{G_{\alpha\beta}}{G_{0\alpha}} - 1\Big)\big(\mathcal{M}_\alpha - \mathcal{L}_\alpha + \alpha\mathcal{N}_{\alpha 0}\big) \\ - \frac{\alpha(1-\beta)}{1-\alpha\beta}\big(\mathcal{L}_\beta + \mathcal{N}_{\alpha\beta} - \mathcal{N}_{\alpha 0}\big) + \frac{(1-\alpha)(1-\beta)}{1-\alpha\beta}(G_{\alpha\beta}-1)\mathcal{Y}\Big) , \qquad (10)$$

where $\alpha, \beta \in [0,1)$*,*

$$\mathcal{L}_\alpha := \int_0^1 d\rho \, \frac{G_{\alpha\rho} - G_{0\rho}}{1-\rho} , \qquad \mathcal{M}_\alpha := \int_0^1 d\rho \, \frac{\alpha \, G_{\alpha\rho}}{1-\alpha\rho} , \qquad \mathcal{N}_{\alpha\beta} := \int_0^1 d\rho \, \frac{G_{\rho\beta} - G_{\alpha\beta}}{\rho - \alpha} ,$$

and $\mathcal{Y} = \lim_{\alpha\to 0} \frac{\mathcal{M}_\alpha - \mathcal{L}_\alpha}{\alpha}$.

6 Pertubation expansion

These integral equations are the starting point for a perturbative solution. In this way, the renormalised correlation functions are directly obtained, without Feynman graph computation and further renormalisation steps. We obtain

$$G_{\alpha\beta} = 1 + \lambda\Big\{ A(I_\beta - \beta) + B(I_\alpha - \alpha)\Big\} \\ + \lambda^2\Big\{ AB\big((I_{\overset{\alpha}{\bullet}} - \alpha) + (I_{\overset{\beta}{\bullet}} - \beta) + (I_\alpha - \alpha)(I_\beta - \beta) + \alpha\beta(\zeta(2)+1)\big) \\ + A\big(\beta I_{\overset{\beta}{\bullet}} - \beta I_\beta\big) - \alpha AB\big((I_\beta)^2 - 2\beta I_\beta + I_\beta\big) \\ + B\big(\alpha I_{\overset{\alpha}{\bullet}} - \alpha I_\alpha\big) - \beta BA\big((I_\alpha)^2 - 2\alpha I_\alpha + I_\alpha\big)\Big\} + \mathcal{O}(\lambda^3) , \qquad (11)$$

where $A := \frac{1-\alpha}{1-\alpha\beta}$, $B := \frac{1-\beta}{1-\alpha\beta}$ and the following iterated integrals appear:

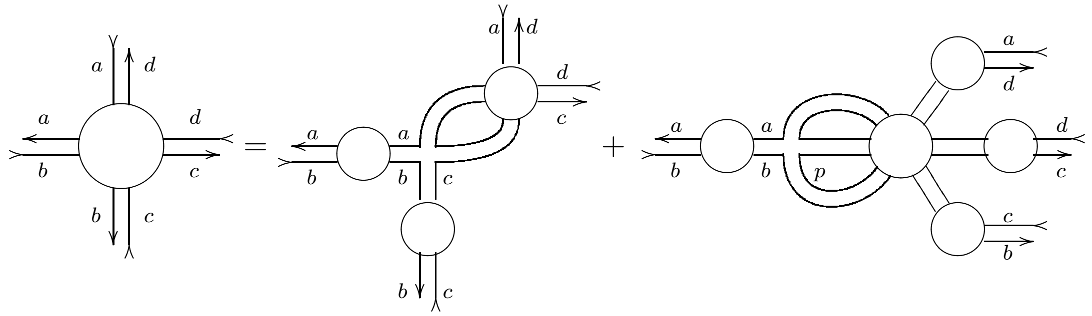
$$I_\alpha := \int_0^1 dx \, \frac{\alpha}{1-\alpha x} = -\ln(1-\alpha) , \\ I_{\overset{\alpha}{\bullet}} := \int_0^1 dx \, \frac{\alpha \, I_x}{1-\alpha x} = \mathrm{Li}_2(\alpha) + \frac{1}{2}\big(\ln(1-\alpha)\big)^2 . \qquad (12)$$

We conjecture that $G_{\alpha\beta}$ is at any order a polynomial with rational coefficients in α, β, A, B and iterated integrals labelled by rooted trees.

7 Four-point Schwinger-Dyson equation

The knowledge of the two-point function allows a successive construction of the whole theory. As an example we treat the planar connected four-point function G_{abcd}. The starting point is again the Schwinger-Dyson equation:

Following the a-face in direction of the arrow, there is a distinguished vertex at which the first ab-line starts. For this vertex there are two possibilities for the matrix index of the diagonally opposite corner to the a-face: either c or a summation vertex p:

 (13)

We write the first contribution as a product of the vertex $Z^2\lambda$, the left connected two-point function, the downward two-point function and an insertion, and reexpress it by means of the Ward-identity. After amputation of the external two-point functions we obtain the Schwinger-Dyson equation for the *renormalised* 1PI four-point function $G_{abcd} = G_{ab}G_{bc}G_{cd}G_{da}\Gamma^{ren}_{abcd}$ as follows:

$$\Gamma^{ren}_{abcd} = Z\lambda\frac{1}{|a|-|c|}\Big(\frac{1}{G_{ad}} - \frac{1}{G_{cd}}\Big) + Z\lambda\sum_p \frac{1}{|a|-|p|}G_{pb}\Big(\frac{G_{dp}}{G_{ad}}\Gamma^{ren}_{pbcd} - \Gamma^{ren}_{abcd}\Big)\,. \tag{14}$$

In terms of the 1PI function we get

$$\begin{aligned}
Z^{-1}\Gamma^{ren}_{abcd} &= \lambda\Big(1 - \frac{\Gamma^{ren}_{ad} - \Gamma^{ren}_{cd}}{|a|-|c|}\Big) \\
&\quad + \lambda\sum_p \frac{|a|+|d|+\mu^2-\Gamma^{ren}_{ad}}{|p|+|b|+\mu^2-\Gamma^{ren}_{pb}}\,\frac{\dfrac{\Gamma^{ren}_{pbcd}-\Gamma^{ren}_{abcd}}{|p|-|a|}}{|p|+|d|+\mu^2-\Gamma^{ren}_{pd}} \\
&\quad + \lambda\Gamma^{ren}_{abcd}\sum_p \frac{1-\dfrac{\Gamma^{ren}_{ad}-\Gamma^{ren}_{pd}}{|a|-|p|}}{(|p|+|b|+\mu^2-\Gamma^{ren}_{pb})(|p|+|d|+\mu^2-\Gamma^{ren}_{pd})}\,.
\end{aligned} \tag{15}$$

Passing to the integral representation and the variables α and β, we find for $\Gamma_{\alpha\beta\gamma\delta} := \Gamma^{ren}_{abcd}$ an integral equation, which manipulated appropriately allows again to take the limit $\xi \to 1$ after insertion of the expression for the wave function renormalisation constant.

Theorem 2 *The renormalised planar 1PI four-point function $\Gamma_{\alpha\beta\gamma\delta}$ of self-dual noncommutative ϕ^4_4-theory (with continuous indices $\alpha, \beta, \gamma, \delta \in [0,1)$) satisfies the integral equa-*

tion

$$\Gamma_{\alpha\beta\gamma\delta} = \lambda \cdot \frac{\Big(1 - \dfrac{(1-\alpha)(1-\gamma\delta)(G_{\alpha\delta}-G_{\gamma\delta})}{G_{\gamma\delta}(1-\delta)(\alpha-\gamma)} + \displaystyle\int_0^1 \rho\, d\rho \frac{(1-\beta)(1-\alpha\delta)G_{\beta\rho}G_{\delta\rho}}{(1-\beta\rho)(1-\delta\rho)} \frac{\Gamma_{\rho\beta\gamma\delta}-\Gamma_{\alpha\beta\gamma\delta}}{\rho-\alpha}\Big)}{G_{\alpha\delta} + \lambda\Big((\mathcal{M}_\beta - \mathcal{L}_\beta - \mathcal{Y})G_{\alpha\delta} + \displaystyle\int_0^1 d\rho \frac{G_{\alpha\delta}G_{\beta\rho}(1-\beta)}{(1-\delta\rho)(1-\beta\rho)} + \displaystyle\int_0^1 \rho\, d\rho \frac{(1-\beta)(1-\alpha\delta)G_{\beta\rho}}{(1-\beta\rho)(1-\delta\rho)} \frac{(G_{\rho\delta}-G_{\alpha\delta})}{(\rho-\alpha)}\Big)} \qquad (16)$$

In lowest order we find

$$\Gamma_{\alpha\beta\gamma\delta} = \lambda - \lambda^2 \Big(\frac{(1-\gamma)(I_\alpha-\alpha)-(1-\alpha)(I_\gamma-\gamma)}{\alpha-\gamma} + \frac{(1-\delta)(I_\beta-\beta)-(1-\beta)(I_\delta-\delta)}{\beta-\delta}\Big) + \mathcal{O}(\lambda^3)\,. \qquad (17)$$

Note that $\Gamma_{\alpha\beta\gamma\delta}$ is cyclic in the four indices, and that $\Gamma_{0000} = \lambda + \mathcal{O}(\lambda^3)$.

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References

[1] H. Grosse and R. Wulkenhaar, "Progress in solving a noncommutative quantum field theory in four dimensions," arXiv:0909.1389 [hep-th].

[2] S. Minwalla, M. Van Raamsdonk and N. Seiberg, "Noncommutative perturbative dynamics," JHEP **0002** (2000) 020 [arXiv:hep-th/9912072].

[3] H. Grosse and R. Wulkenhaar, "Renormalisation of ϕ^4-theory on noncommutative $\mathbb{R}^4$ in the matrix base," Commun. Math. Phys. **256** (2005) 305 [arXiv:hep-th/0401128].

[4] E. Langmann and R. J. Szabo, "Duality in scalar field theory on noncommutative phase spaces," Phys. Lett. B **533** (2002) 168 [arXiv:hep-th/0202039].

[5] V. Rivasseau, "Non-commutative renormalization" In: *Quantum Spaces (Séminaire Poincaré X)*, eds. B. Duplantier and V. Rivasseau, Birkhäuser Verlag Basel (2007) 19–109 [arXiv:0705.0705 [hep-th]].

[6] H. Grosse and R. Wulkenhaar, "The β-function in duality-covariant noncommutative ϕ^4-theory," Eur. Phys. J. C **35** (2004) 277 [arXiv:hep-th/0402093].

[7] H. Grosse and R. Wulkenhaar, "Renormalisation of ϕ^4-theory on non-commutative $\mathbb{R}^4$ to all orders," Lett. Math. Phys. **71** (2005) 13.

[8] M. Disertori and V. Rivasseau, "Two and three loops beta function of non commutative ϕ^4_4 theory," Eur. Phys. J. C **50** (2007) 661 [arXiv:hep-th/0610224].

[9] M. Disertori, R. Gurau, J. Magnen and V. Rivasseau, "Vanishing of beta function of non commutative ϕ^4_4 theory to all orders," Phys. Lett. B **649** (2007) 95 [arXiv:hep-th/0612251].

[10] V. Rivasseau, "Constructive Matrix Theory," JHEP **0709** (2007) 008 [arXiv:0706.1224 [hep-th]].

[11] A. Connes and D. Kreimer, "Hopf algebras, renormalization and noncommutative geometry," Commun. Math. Phys. **199** (1998) 203 [arXiv:hep-th/9808042].

[12] M. Kontsevich and D. Zagier, "Periods" In: *Mathematics unlimited—2001 and beyond,* Springer-Verlag Berlin (2001) 771–808.

[13] H. Grosse and R. Wulkenhaar, "Renormalisation of ϕ^4 theory on noncommutative $\mathbb{R}^2$ in the matrix base," JHEP **0312**, 019 (2003) [arXiv:hep-th/0307017].