Integrability and positivity in quantum field theory on noncommutative geometry[☆]

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Abstract

We review a sequence of papers in which we construct the $\lambda\phi_4^{\star 4}$-model and the $\lambda\phi_{2,4,6}^{\star 3}$-models on noncommutative Moyal space by a common method. Thereby we show that not only the Kontsevich model $\lambda\Phi^3$ but also the $\lambda\Phi_4^4$-model is integrable in a certain scaling limit which corresponds to infinitely large Moyal deformation parameter. Surprisingly, this limit gives rise to Schwinger functions on commutative Euclidean space. Our explicit formulae permit us to discuss reflection positivity of these Schwinger functions.

Keywords: quantum field theory, noncommutative geometry, solvable model, matrix model, Schwinger-Dyson equation, reflection positivity
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1. Introduction

The programme which we are going to review goes back to the influential paper [CDS98] by Connes, Douglas and Schwarz in which they proposed to compactify M-theory on the noncommutative torus. This paper, together with the prior analysis of Yang-Mills theory on noncommutative tori by Connes and Rieffel [CR87], motivated the one-loop computation of quantum Yang-Mills theory on the noncommutative 4-torus [KW99]. The Connes-Douglas-Schwarz paper also led Schomerus [Sch99] and shortly later Seiberg and Witten [SW99] to their discoveries that quantum field theories on noncommuatiative geometries arise in certain limits of string theory in presence of magnetic background fields. In this setting, Minwalla, van Raamsdonk and Seiberg demonstrated [MVS99] that quantum field theories on noncommutative spaces generate a severe problem in higher loop order (UV/IR-mixing). The mechanism was thoroughly analysed in two papers by Chepelev and Roiban [CR99, CR00].

[☆] *Dedicated to Alain Connes on the occasion of his 70th birthday.*

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20 years after [CDS98] it is time to look back. In the beginning of the century it seemed a fact that quantum field theories on noncommutative geometries are pathological. Today we know the opposite is true: The tools of noncommutative geometry [Con94] allow us to do something in the noncommutative regime what ordinary quantum field theory would dream to do: a rigorous construction in four dimensions. This contribution tries to review the main steps. We would like to thank Alain for having uncovered the wonderful world of noncommutative geometry, for the encouraging atmosphere in the community and for his support throughout these two decades. Happy birthday Alain, and may many more follow.

2. First steps

In 2002 we started a renormalisation group approach [Pol84] to the UV/IR-mixing problem in the matrix basis [GV88] of the Moyal space. In this basis the Laplacian is represented by a kernel operator $\Delta_{\underline{kl};\underline{mn}}$ which contains a local interaction and nearest-neighbour interaction. We found that the local part alone would produce a well-defined power-counting behaviour, which however is destroyed by the the nearest-neighbour interaction. Scaling the nearest-neighbour interaction down by a factor $0 < \omega < 1$ cures the UV/IR-mixing problem. The kernel operator with reduced nearest-neighbour interaction corresponds to the harmonic oscillator Schrödinger operator $H^\Omega = -\Delta + 4\Omega^2\|\Theta^{-1}x\|^2$ instead of the Laplacian $-\Delta$, where Θ is the deformation matrix which defines the Moyal product $\star$, and $0 < \Omega \leq 1$. Working out the details, we proved:

Theorem 1 ([GW05b]+[GW05a]). *Let $\star$ be the Moyal product between functions on $\mathbb{R}^4$ and H^Ω the harmonic oscillator Schrödinger operator. Then the scalar Euclidean quantum field theory defined by the action functional*

$$S(\phi) = \int_{\mathbb{R}^4} dx\Big(\frac{Z}{2}\phi(H^\Omega + \mu^2)\phi + \frac{Z^2\lambda}{4}\phi\star\phi\star\phi\star\phi\Big)(x) \qquad (1)$$

is renormalisable to all orders in perturbation theory.

This anounts to prove (in the class of formal power series in λ) that there exists a suitable dependence of Z, μ, λ, Ω on cut-off and normalisation conditions such that all correlation functions are well-defined. Translation invariance is explicitly broken, but will be recovered in a certain limit discussed in this review. On the other hand, the action is covariant under a duality found by Langmann and Szabo [LS02].

The one-loop renormalisation group flow of the coupling constant λ and of the harmonic oscillator frequency Ω was computed in [GW04]. This flow leaves the ratio $\frac{\Omega^2}{\lambda}$ constant. Since Ω flows into the UV-fixed point $\Omega^* = 1$, the flow of the coupling constant is bounded, i.e. there is (at one-loop) no Landau ghost in the scalar $\lambda\phi_4^{\star 4}$-model [GW05c]! This observation gave rise to the hope that the noncommutative $\lambda\phi_4^{\star 4}$-model can eventually be constructed. The first step along the usual construction strategy [Riv91], the multiscale analysis of the $\lambda\phi_4^{\star 4}$-model, was achieved in [RVW06]. Later a novel construction scheme, the loop vertex expansion [Riv07], which combines the Hubbard-Stratonovich transform with the BKAR forest formula [AR95], was developed for this purpose and used to rigorously construct the $\lambda\phi_2^{\star 4}$-model [Wan12].

The most important achievement started with a remarkable three-loop computation of the β-function of the coupling constant by Disertori and Rivasseau [DR07] in which they confirmed that at $\Omega = 1$, β vanishes to three-loop order. Eventually, Disertori, Gurau, Magnen and Rivasseau proved in [DGMR07] that the β-function vanishes to all orders in perturbation theory. The key step consists in an ingenious combination of Ward identities with Schwinger-Dyson equations. We felt that the result of [DGMR07] goes much deeper: Using these tools it must be possible to solve the model!

3. From field theory on Moyal space to matrix models

The Moyal product in D dimensions is defined by the oscillatory integral

$$(f \star g)(x) := \int_{\mathbb{R}^D \times \mathbb{R}^D} \frac{dy\, dk}{(2\pi)^D}\, f(x + \tfrac{1}{2}\Theta \cdot k)\, g(x + y) e^{\mathrm{i}\langle k, y\rangle}\ , \tag{2}$$

where $\Theta \in M_D(\mathbb{R})$ is skew-adjoint. It falls into the class of strict deformation quantisations by $\mathbb{R}^D$-action introduced by Rieffel [Rie93]. The following functions give rise to a convenient matrix basis [GV88]:

$$\begin{aligned} f_{\underline{m}\underline{n}}(x) &:= f_{m_1 n_1}(x^1 + \mathrm{i}x^2) \cdots f_{m_{D/2} n_{D/2}}(x^{D-1} + \mathrm{i}x^D)\,, \\ f_{mn}(z) &= 2(-1)^m \sqrt{\frac{m!}{n!}} \Big(\sqrt{\frac{2}{\theta}} z\Big)^{n-m} L_m^{n-m}\Big(\frac{2|z|^2}{\theta}\Big) e^{-\frac{|z|^2}{\theta}}, \quad m, n \in \mathbb{N}. \end{aligned} \tag{3}$$

Here $\underline{m} = (m_1, \dots, m_{D/2})$, a deformation matrix $\Theta = \begin{pmatrix} 0 & \theta \\ -\theta & 0 \end{pmatrix} \otimes I_{\frac{D}{2}\times\frac{D}{2}}$ is assumed, the $L_m^\alpha(t)$ are associated Laguerre polynomials of degree m in t and (x^1, x^2) is identified with $z = x^1 + \mathrm{i}x^2$. These functions satisfy $(f_{\underline{k}\underline{l}} \star f_{\underline{m}\underline{n}})(x) = \delta_{\underline{m}\underline{l}} f_{\underline{k}\underline{n}}(x)$ and $\int_{\mathbb{R}^D} dx\ f_{\underline{m}\underline{n}}(x) = \sqrt{|\det(2\pi\Theta)|}\delta_{\underline{m}\underline{n}}$. Coincidently, these functions are also eigenfunctions $(H^1 f_{\underline{m}\underline{n}})(x) = H_{\underline{m}\underline{n}} f_{\underline{m}\underline{n}}(x)$ of the harmonic oscillator Schrödinger operator H^1 at frequency $\Omega = 1$ with eigenvalues $H_{\underline{m}\underline{n}} = \frac{4}{\theta}(|\underline{m}| + |\underline{n}| + \frac{D}{2})$, where $|\underline{m}| := m_1 + \cdots + m_{D/2}$. Viewing the $f_{\underline{m}\underline{n}}$ as matrix bases and expanding scalar fields as $\phi(x) = \sum_{\underline{m},\underline{n}\in\mathbb{N}^{D/2}} \Phi_{\underline{m}\underline{n}} f_{\underline{m}\underline{n}}(x)$, the following representation of a general class of action functionals for scalar fields is achieved:

$$\begin{aligned} S(\phi) &= \frac{1}{(8\pi)^{\frac{D}{2}}} \int_{\mathbb{R}^D} dx\, \Big(\frac{1}{2}\phi \star H^1(\phi) + \sum_{p=1}^{s} \frac{\lambda_p}{p} \phi^{\star p}\Big)(x) \\ &= V\ \mathrm{Tr}\Big(E\Phi^2 + \sum_{p=1}^{s} \frac{\lambda_p}{p} \Phi^p\Big)\,, \qquad V := \Big(\frac{\theta}{4}\Big)^{\frac{D}{2}}\,, \end{aligned} \tag{4}$$

where $E = (E_{\underline{m}}\delta_{\underline{m}\underline{n}})$, with $E_{\underline{m}} := \frac{4}{\theta}(|\underline{m}| + \frac{D}{4})$, with respect to the matrix basis.

We define quantum field theory via its Euclidean approach, which to an action functional $S(\Phi)$, bounded from below, assigns a formal measure $d\nu(\Phi) := \frac{1}{\mathcal{Z}} e^{-S(\Phi)} D\Phi$. Such a measure can only make sense after regularisation which in our case consists in a restriction to finite matrix size $\underline{n} \in \mathbb{N}^{D/2}_{\mathcal{N}} := \{\underline{k} \in \mathbb{N}^{D/2} \;:\; |\underline{k}| \leq \mathcal{N}\}$ of any multiple matrix index. On the space of such matrices, $D\Phi = \prod_{\underline{n},\underline{m} \in \mathbb{N}^{D/2}_{\mathcal{N}}} d\phi_{\underline{mn}}$ is then a well-defined Lebesque measure (possibly further restricted by symmetry requirements such as $\Phi = \Phi^*$). Candidate Schwinger functions are the moments of such a measure $d\nu(\Phi)$, which are conveniently generated by the Fourier transform $\mathcal{Z}[J] := \int d\nu(\Phi) \exp(\mathrm{i}V \,\mathrm{Tr}(J\Phi))$, where J is a matrix which is either finite or has rapidly decaying entries. Up to an irrelevant constant, the Fourier transform can be evaluated symbolically to

$$\begin{aligned}
\mathcal{Z}(J) &= \exp\Big(- V \sum_{p=1}^{s} \frac{\lambda_p}{p(\mathrm{i}V)^p} \sum_{\underline{n}_1,\dots,\underline{n}_p} \frac{\partial^p}{\partial J_{\underline{n}_1\underline{n}_2} \cdots \partial J_{\underline{n}_{p-1}\underline{n}_p} \partial J_{\underline{n}_p\underline{n}_1}} \Big) \mathcal{Z}_{free}(J) \;, \\
\mathcal{Z}_{free}(J) &= \exp\Big(- \frac{V}{2} \sum_{\underline{m},\underline{n}} \frac{J_{\underline{mn}} J_{\underline{nm}}}{E_{\underline{m}} + E_{\underline{n}}} \Big) \;. \qquad (5)
\end{aligned}$$

The task of a quantum field theory consists in constructing the limit of the first line of (5) for removed regulator, in our case the limit $\mathcal{N} \to \infty$ and possibly removed oscillator potential. Such a construction involves a careful choice – called renormalisation – of the parameters (e.g. μ, Z, λ_p) in the action as function of the cut-off. This is a very difficult programme which rigorously succeeded in very few cases. Often the only achievement is the construction of the limit as a formal power series in λ_p.

For dynamical matrix models of the type (4) the construction programme can be pushed much further than in standard quantum field theory. A famous example is the Kontsevich model [Kon92] given by $s = 3$ in (4) and $\Phi = \Phi^* \in M_{\mathcal{N}}(\mathbb{C})$. The Kontsevich model is of paramount importance because it elegantly proves Witten's conjecture [Wit91] about the equivalence of two approaches to quantum gravity in two dimensions: the Hermitean one-matrix model versus the intersection theory on the moduli space of Riemann surfaces. See [Wit92]. By a shift of the matrix Φ a standard form $\mathrm{Tr}(YM + \frac{\mathrm{i}}{6} M^3)$ of the action can be achieved. Diagonalisation $M = U^* X U$ with $X = \mathrm{diag}(x_1, x_2, \dots)$ and Jacobian $dM = \frac{(2\pi)^{\mathcal{N}(\mathcal{N}-1)/2}}{\prod_{p=1}^{\mathcal{N}} p!} \big(\prod_{j<i} (x_i - x_j)^2 \big) \big(\prod_{i=1}^{\mathcal{N}} dx_i \big) dU$ gives rise to an integral over the unitary group $U(\mathcal{N})$ which is evaluated by the Harish-Chandra–Itzykson–Zuber formula. The remaining integral over the eigenvalues x_i can be treated by several methods. One particularly elegant approach uses the fact that these integrals are unchanged under diffeomorphisms of x_i generated by $x_i^{n+1} \frac{d}{dx_i}$. The corresponding Virasoro constraints all descend from a master constraint which was solved by Makeenko-Semenoff [MS91]. We come back to this point in section 5.

A similar approach for the quartic model given by $s = 4$ in (4) does not seem to work. Inspired by Disertori et al [DGMR07] we developed in [GW14a] a new solution strategy for the quartic model. Later it turned out that this strategy can also be applied to the cubic Kontsevich model where it yields a complete and explicit solution [GSW17, GSW18].

Our strategy relies on the observation that the Fourier transform $\mathcal{Z}(J) = \int d\nu(\Phi)\exp(\mathrm{i}V\,\mathrm{Tr}(J\Phi))$ of the formal measure is invariant under a renaming $\Phi \mapsto U^*\Phi U$. This gives rise to constraints

$$0 = \sum_{\underline{n}\in\mathbb{N}^{\frac{D}{2}}} \Big(\frac{(E_{\underline{p}}-E_{\underline{a}})}{V}\frac{\partial^2\mathcal{Z}}{\partial J_{\underline{a}\underline{n}}\partial J_{\underline{n}\underline{p}}} + J_{\underline{p}\underline{n}}\frac{\partial\mathcal{Z}}{\partial J_{\underline{a}\underline{n}}} - J_{\underline{n}\underline{a}}\frac{\partial\mathcal{Z}}{\partial J_{\underline{n}\underline{p}}}\Big)\,. \tag{6}$$

These constraints were already obtained in [DGMR07]; they can be regarded as analogues of the Virasoro constraints in the Kontsevich model. In a later step it will be essential that $\underline{p}\mapsto E_{\underline{p}}$ is injective. Strictly speaking this is not the case for $D>2$ but can be achieved by passing to the 1-norms $|\underline{p}|$ at expense of a measure that reflects the multiplicities.

To use the constraints (6) we need to collect some topological information. The partition function $\mathcal{Z}(J)$ of a matrix model is formally a sum over disconnected *ribbon graphs*. Passing to the logarithm $\log\mathcal{Z}(J)$ amounts to restrict to connected ribbon graphs. Viewed as simplicial complex, a ribbon graph encodes the topology (B,g) of a genus-g Riemann surface with B disconnected boundary components $\sqcup_{\beta=1}^B\mathbb{S}^1$. Every boundary circle $\mathbb{S}^1$ carries a cycle of source matrices J, i.e. a N_β-fold cyclic product $\mathbb{J}_{\underline{p}_i\dots\underline{p}_{N_\beta}} := \prod_{i=1}^{N_\beta} J_{\underline{p}_i\underline{p}_{i+1}}$ with $N_\beta+1\equiv 1$. Consequently, $\log\mathcal{Z}(J)$ has an expansion according to the boundary structure:

$$\log\frac{\mathcal{Z}(J)}{\mathcal{Z}(0)} = \sum_{B=1}^\infty \sum_{1\le N_1\le\dots\le N_B}^\infty \sum_{\underline{p}_1^1,\underline{p}_2^1,\dots,\underline{p}_{N_B}^B\in\mathbb{N}^{\frac{D}{2}}} \frac{V^{2-B}}{S_{N_1\dots N_B}} G_{|\underline{p}_1^1\dots\underline{p}_{N_1}^1|\dots|\underline{p}_1^B\dots\underline{p}_{N_B}^B|} \times\prod_{\beta=1}^B \frac{\mathbb{J}_{\underline{p}_1^\beta\dots\underline{p}_{N_\beta}^\beta}}{(-\mathrm{i})^{N_\beta}N_\beta}\,. \tag{7}$$

The symmetry factor $S_{N_1\dots N_B}$ is obtained as follows: If ν_i of the B numbers N_β in a given tuple $(N_1,\dots,N_B)$ are equal to i, then $S_{N_1\dots N_B} = \prod_{i=1}^{N_B}\nu_i!$.

As long as we work with finite matrices we can interpret (7) as a definition of $G_{|\underline{p}_1^1\dots\underline{p}_{N_1}^1|\dots|\underline{p}_1^B\dots\underline{p}_{N_B}^B|}$. Differentiations with respect to several $J_{\underline{a}\underline{b}}$, simultaneously applied to (5) and to (7), give rise to identities called Schwinger-Dyson equations. They are the quantum analogue of equations of motion. In addition we have constraints resulting from (6). At that point we change the perspective and declare the quantum field theory as *defined by the functions* $G_{|\underline{p}_1^1\dots\underline{p}_{N_1}^1|\dots|\underline{p}_1^B\dots\underline{p}_{N_B}^B|}$ *together with the previously derived identities between them.* The advantage of this point of view is that the limit of removed regularisation is much easier for these equations than for the measure $d\nu(\Phi)$. We will show in the sequel that, at least in examples, the Schwinger-Dyson equations plus constraints completely fix the weight functions $G_{|\underline{p}_1^1\dots\underline{p}_{N_1}^1|\dots|\underline{p}_1^B\dots\underline{p}_{N_B}^B|}$ and thereby construct the quantum field theory.

A key step for this construction consists in turning the constraint (6) into a formula for the second derivative $\sum_{\underline{n}\in\mathbb{N}^{D/2}} \frac{\partial^2 \mathcal{Z}(J)}{\partial J_{\underline{an}}\partial J_{\underline{np}}}$ of the partition function, thus giving new relations for $G_{\dots}$. We have to identify the kernel of multiplication by $(E_{\underline{p}} - E_{\underline{a}})$. For injective $m \mapsto E_m$ this kernel is given by $\mathrm{W}_{\underline{a}}(J)\mathcal{Z}(J)\delta_{\underline{ap}}$ for some function $\mathrm{W}_{\underline{a}}(J)$. This function is identified by inserting (7) into $\sum_{\underline{n}\in\mathbb{N}^{D/2}} \frac{\partial^2 \mathcal{Z}(J)}{\partial J_{\underline{an}}\partial J_{\underline{np}}}$ and carefully registering the possibilities which give rise to a factor $\delta_{\underline{ap}}$. We find the following *Ward-Takahashi identity* [GW14a]:

Theorem 2.

$$-\sum_{\underline{n}\in\mathbb{N}^{\frac{D}{2}}} \frac{\partial^2 \mathcal{Z}(J)}{\partial J_{\underline{an}}\partial J_{\underline{np}}} = \delta_{\underline{ap}}\mathrm{W}_{\underline{a}}(J)\mathcal{Z}(J) + \frac{V}{E_{\underline{p}}-E_{\underline{a}}} \sum_{\underline{n}\in\mathbb{N}^{\frac{D}{2}}} \Big(J_{\underline{pn}}\frac{\partial \mathcal{Z}(J)}{\partial J_{\underline{an}}} - J_{\underline{na}}\frac{\partial \mathcal{Z}(J)}{\partial J_{\underline{np}}} \Big)\,, \tag{8}$$

$$\begin{aligned}\mathrm{W}_{\underline{a}}(J) := V^2 \sum_{(K)} \frac{\mathbb{J}_{\underline{P}^1}\cdots\mathbb{J}_{\underline{P}^K}}{S_{(K)}} \Big(&\sum_{\underline{n}\in\mathbb{N}^{D/2}} \frac{G_{|\underline{an}|\underline{P}^1|\dots|\underline{P}^K|}}{V^{K+1}} + \frac{G_{|\underline{a}|\underline{a}|\underline{P}^1|\dots|\underline{P}^K|}}{V^{K+2}} \\ &+ \sum_{r\geq 1} \sum_{\underline{q}_1,\dots,\underline{q}_r\in\mathbb{N}^{D/2}} \frac{G_{|\underline{q}_1\underline{a}\underline{q}_1\dots\underline{q}_r|\underline{P}^1|\dots|\underline{P}^K|}\mathbb{J}_{\underline{q}_1\dots\underline{q}_r}}{V^{K+1}} \Big) \\ &+ V^4 \sum_{(K),(K')} \frac{\mathbb{J}_{\underline{P}^1}\cdots\mathbb{J}_{\underline{P}^K}\mathbb{J}_{\underline{Q}^1}\cdots\mathbb{J}_{\underline{Q}^{K'}}}{S_{(K)}S_{(K')}} \frac{G_{|\underline{a}|\underline{P}^1|\dots|\underline{P}^K|}}{V^{K+1}} \frac{G_{|\underline{a}|\underline{Q}^1|\dots|\underline{Q}^{K'}|}}{V^{K'+1}}\,.\end{aligned}$$

The sums over (K) stand for $\sum_{(K)} = \sum_{K=1}^{\infty}\sum_{\underline{P}^1,\dots\underline{P}^K}$, where every $\underline{P}^\beta$ is a chain of multiple matrix indices $\underline{P}^\beta = \underline{p}_1^\beta \dots \underline{p}_{N_\beta}^\beta$ of length N_β. Similarly for $\underline{Q}_\beta$ and (K'). The symmetry factor is $S_{(K)} := S_{N_1\dots N_K}\prod_{\beta=1}^{K}(-\mathrm{i})^{N_\beta}N_\beta$.

Formula (8) is the core of our approach. It is a consequence of the unitary group action and the cycle structure of the partition function. The importance lies in the fact that the formula allows to kill two J-derivatives in the partition function. As we describe below, this is the key step in breaking up the tower of Schwinger-Dyson equations.

4. Solution of the $\lambda\Phi^4$-model

4.1. Schwinger-Dyson equations

We consider the quartic interaction $s = 4$ in (4). After a shift of Φ we can assume $\lambda_3 = 0$, and $\lambda_2 = 0$ and $\lambda_1 = 0$ can be assumed after redefinition of E and J. From (7) we deduce $G_{|\underline{ab}|} = \frac{-1}{V\mathcal{Z}(0)} \frac{\partial^2 \mathcal{Z}(J)}{\partial J_{\underline{ba}}\partial J_{\underline{ab}}}\Big|_{J=0}$ for the 2-point function at $\underline{a} \neq \underline{b}$. Applying these derivatives to (5) gives

$$G_{|\underline{ab}|} = \frac{1}{E_{\underline{a}}+E_{\underline{b}}} - \frac{\lambda_4}{V^3(E_{\underline{a}}+E_{\underline{b}})\mathcal{Z}(0)} \sum_{\underline{p},\underline{n}\in\mathbb{N}^{\frac{D}{2}}} \frac{\partial^2}{\partial J_{\underline{pb}}\partial J_{\underline{ba}}} \frac{\partial^2}{\partial J_{\underline{an}}\partial J_{\underline{np}}} \mathcal{Z}(J)\Big|_{J=0}\,. \tag{9}$$

The two rightmost derivatives are expressed by the Ward-Takahashi identity (8); then the remaining two derivatives are easily evaluated and give [GW14a]:

$$
\left.\begin{aligned}
G_{|\underline{a}\underline{b}|} = \frac{1}{E_{\underline{a}}+E_{\underline{b}}} - \frac{\lambda_4}{E_{\underline{a}}+E_{\underline{b}}}\frac{1}{V}\sum_{\underline{p}\in\mathbb{N}^{\frac{D}{2}}}\Big(G_{|\underline{a}\underline{b}|}G_{|\underline{a}\underline{p}|} - \frac{G_{|\underline{p}\underline{b}|}-G_{|\underline{a}\underline{b}|}}{E_{\underline{p}}-E_{\underline{a}}}\Big)
\end{aligned}\right\} \quad (10a)
$$

$$
\left.\begin{aligned}
-\frac{\lambda_4}{V^2(E_{\underline{a}}+E_{\underline{b}})}\Big(G_{|\underline{a}|\underline{a}|}G_{|\underline{a}\underline{b}|} + \frac{1}{V}\sum_{\underline{n}\in\mathbb{N}^{\frac{D}{2}}} G_{|\underline{a}\underline{n}|\underline{a}\underline{b}|} \\
+G_{|\underline{a}\underline{a}\underline{a}\underline{b}|} + G_{|\underline{b}\underline{a}\underline{b}\underline{a}|} - \frac{G_{|\underline{b}|\underline{b}|}-G_{|\underline{a}|\underline{b}|}}{E_{\underline{b}}-E_{\underline{a}}}\Big)
\end{aligned}\right\} \quad (10b)
$$

$$
\left.\begin{aligned}
-\frac{\lambda_4}{V^4(E_{\underline{a}}+E_{\underline{b}})}G_{|\underline{a}|\underline{a}|\underline{a}\underline{b}|}\;.
\end{aligned}\right\} \quad (10c)
$$

It can be checked [GW14a] that in a genus expansion $G_{\dots} = \sum_{g=0}^{\infty} V^{-2g}G^{(g)}_{\dots}$ precisely the line (10a) preserves the genus, the lines (10b) increase $g \mapsto g+1$, and the line (10c) increases $g \mapsto g+2$.

We will not rely on a genus expansion. Instead we consider a scaling limit $V \to \infty$ such that the densitised index summation $\frac{1}{V}\sum_{\underline{p}\in\mathbb{N}^{D/2}}$ remains finite. Then the exact Schwinger-Dyson equation for $G_{|\underline{a}\underline{b}|}$ coincides with its restriction (10a) to the planar sector $g=0$ – *a closed non-linear equation for* $G^{(0)}_{|\underline{a}\underline{b}|}$ *alone.* Nonetheless a non-trivial topology survives: The higher boundary components $B \geq 2$ are *not* suppressed; and in fact these contributions from $B \geq 2$ make the model interesting!

By similar calculation we derive the Schwinger-Dyson equation for higher N-point functions. This expresses the N-point function $G_{|\underline{a}\underline{b}_1\dots\underline{b}_{N-1}|}$ in terms of its summation $\displaystyle\frac{\lambda_4}{E_{\underline{a}}+E_{\underline{b}_1}}\frac{1}{V}\sum_{\underline{p}\in\mathbb{N}^{D/2}}\Big(G_{|\underline{a}\underline{p}|}G_{|\underline{a}\underline{b}_1\dots\underline{b}_{N-1}|} - \frac{G_{|\underline{p}\underline{b}_1\dots\underline{b}_{N-1}|}-G_{|\underline{a}\underline{b}_1\dots\underline{b}_{N-1}|}}{E_{\underline{p}}-E_{\underline{a}}}\Big)$ and several other functions [GW14a]. It turns out that a real theory with $\Phi = \Phi^*$ admits a short-cut which directly gives the higher N-point functions without any index summation. Since the equations for $G_{\dots}$ are real and $\overline{J_{\underline{a}\underline{b}}} = J_{\underline{b}\underline{a}}$, the reality $\mathcal{Z} = \overline{\mathcal{Z}}$ implies invariance under orientation reversal $G_{|\underline{p}_1^1\underline{p}_2^1\dots\underline{p}_{N_1}^1|\dots|\underline{p}_1^B\underline{p}_2^B\dots\underline{p}_{N_B}^B|} = G_{|\underline{p}_1^1\underline{p}_{N_1}^1\dots\underline{p}_2^1|\dots|\underline{p}_1^B\underline{p}_{N_B}^B\dots\underline{p}_2^B|}$. These identities lead to many cancellations which result in a universal algebraic recursion formula [GW14a]:

Theorem 3. *Given a quartic matrix model* $S[\Phi] = V\,\mathrm{tr}(E\Phi^2 + \frac{\lambda_4}{4}\Phi^4)$ *on* D-*dimensional Moyal space with harmonic oscillator hamiltonian. Then in a scaling limit* $V \to \infty$ *with* $\frac{1}{V}\sum_{\underline{p}\in\mathbb{N}^{D/2}}$ *finite, the* $(B=1)$*-sector of* $\log\mathcal{Z}$ *is given by*

$$
G_{|\underline{a}\underline{b}|} = \frac{1}{E_{\underline{a}}+E_{\underline{b}}} - \frac{\lambda_4}{E_{\underline{a}}+E_{\underline{b}}}\frac{1}{V}\sum_{\underline{p}\in\mathbb{N}^{\frac{D}{2}}}\Big(G_{|\underline{a}\underline{b}|}G_{|\underline{a}\underline{p}|} - \frac{G_{|\underline{p}\underline{b}|}-G_{|\underline{a}\underline{b}|}}{E_{\underline{p}}-E_{\underline{a}}}\Big)\,, \qquad (11a)
$$

$$G_{|\underline{b}_0\underline{b}_1\dots\underline{b}_{N-1}|} \tag{11b}$$
$$= (-\lambda_4)\sum_{l=1}^{\frac{N-2}{2}} \frac{G_{|\underline{b}_0\underline{b}_1\dots\underline{b}_{2l-1}|}G_{|\underline{b}_{2l}\underline{b}_{2l+1}\dots\underline{b}_{N-1}|} - G_{|\underline{b}_{2l}\underline{b}_1\dots\underline{b}_{2l-1}|}G_{|\underline{b}_0\underline{b}_{2l+1}\dots\underline{b}_{N-1}|}}{(E_{\underline{b}_0}-E_{\underline{b}_{2l}})(E_{\underline{b}_1}-E_{\underline{b}_{N-1}})}\,.$$

Corollary 4. *If the 2-point function can be normalised by an affine transformation $E \mapsto ZE + C$ together with a corresponding rescaling $\lambda_4 \mapsto Z^2\lambda_4$ of the coupling constant, then all higher N-point functions are already well-defined. In particular, the β-function is identically zero.*

The self-consistency equation (11a) was first obtained in [GW09] for the Moyal model by the graphical method proposed by [DGMR07]. There we also solved the renormalisation problem resulting from the divergent summation $\sum_{\underline{p}\in\mathbb{N}^{D/2}}$. The non-linearity of (11a) was a considerable challenge which we successfully addressed in [GW14a, GW14b].

The other topological sectors $B \geq 2$ made of $(N_1+\dots+N_B)$-point functions $G_{|\underline{b}_1^1\dots\underline{b}^1_{N_1}|\dots|\underline{b}_1^1\dots\underline{b}^B_{N_B}|}$ are similar in the following sense [GW14a]: The basic functions with all $N_i \leq 2$ satisfy an equation with index summation as (11a), but in contrast to the 2-point function these equations are linear. The other functions with one $N_i \geq 3$ are purely algebraic.

We remark that the algebraic equations for $N_i \geq 3$ have a graphical realisation in terms of non-crossing chord diagrams with additional decoration which describe the denominators $\frac{1}{E_{\underline{b}_i}-E_{\underline{b}_j}}$. The different chord structures are counted by the Catalan numbers. These functions alone would make the higher N-point functions very close to trivial. It is the inclusion of the $(2+2+\dots+2)$-point functions which gives a rich structure.

4.2. Infinite volume limit and renormalisation

We specify (11a) to the $\lambda\phi_4^{\star 4}$-model on $(D=4)$-dimensional Moyal space and combine mass term, kinetic term and renormalisation parameters into $E_{\underline{m}} = Z(\frac{|\underline{m}|}{\sqrt{V}} + \frac{\mu^2_{bare}}{2})$. The coupling constant is $\lambda_4 = Z^2\lambda$ because the vanishing β-function (Corollary 4) makes a bare coupling λ_{bare} not necessary. The matrix indices have ranges $\underline{a},\dots \in \mathbb{N}^2_{\mathcal{N}}$, i.e. pairs of natural numbers with certain cut-off. The index sum diverges for $\mathbb{N}^2_{\mathcal{N}} \mapsto \mathbb{N}^2$.

It is important that all functions only depend on the spectrum of $E_{\underline{m}}$, i.e. on the 1-norms $|\underline{m}| = m_1 + m_2$ and not on m_1, m_2 separately. Renormalisation respects this degeneracy. Therefore, all index sums reduce to $\sum_{\underline{p}\in\mathbb{N}^2_{\mathcal{N}}} f(|\underline{p}|) = \sum_{|\underline{p}|=0}^{\mathcal{N}}(|\underline{p}|+1)f(|\underline{p}|)$. In these sums we study the scaling limit $V,\mathcal{N} \to \infty$ with fixed ratio $\frac{\mathcal{N}}{\sqrt{V\mu^4}} = \Lambda^2(1+\mathcal{Y})$. Note that $V = \left(\frac{\theta}{4}\right)^2 \to \infty$ is a limit of extreme noncommutativity! The new parameter $(1+\mathcal{Y})$ corresponds to a finite wavefunction renormalisation, identified later to decouple our equations, and μ will be the renormalised mass. The parameter Λ^2 represents an ultraviolet cut-off which is sent to $\Lambda \to \infty$ in the very end (continuum limit). In the scaling limit, functions of $\frac{|\underline{p}|}{\sqrt{V}} =: \mu^2(1+\mathcal{Y})p$ converge to functions of 'continuous matrix indices' $p \in [0,\Lambda^2]$, and the densitised index summation converges to a Riemann integral. After all these steps, the unrenormalised function $G^u(a,b) := \lim_{V,\mathcal{N}\to\infty} \mu^2 G_{|\underline{a}\underline{b}|}$ satisfies the following equation resulting from (11a):

$$G^u(a,b) = \frac{1}{Z\big(\frac{\mu^2_{bare}}{\mu^2} + (a+b)(1+\mathcal{Y})\big)}\Big\{1- \\ - Z^2\lambda(1+\mathcal{Y})^2 \int_0^{\Lambda^2} pdp\, \Big(G^u(a,b)G^u(a,p) - \frac{G^u(p,b)-G^u(a,b)}{(1+\mathcal{Y})Z(p-a)}\Big)\Big\}\,. \tag{12}$$

Renormalisation amounts to normalisation conditions $\Gamma(0,0) := 0$ and $(\partial\Gamma)(0,0) := 0$ for the renormalised one-particle irreducible function defined by $\Gamma(a,b) = \big(G^u(a,b)\big)^{-1} - (a+b)(1+\mathcal{Y}) - 1$. This definition can be implemented directly in (12) and amounts to a renormalisation of all Feynman graphs at once! The three equations (12) plus $\Gamma(0,0) := 0$ and $(\partial\Gamma)(0,0) := 0$ can be solved for the three quantities $\Gamma(a,b), \mu_{bare}, Z$ once a relation between $\mathcal{Y}$ and $\Gamma(a,b)$ is given. This is easy for μ_{bare} but difficult for Z because of the non-linearity in (12). We propose the following trick which postpones the non-linearity: If we multiply (12) by $\frac{Z(\frac{\mu^2_{bare}}{\mu^2}+(a+b)(1+\mathcal{Y}))}{G^u(a,b)}$, then the previously non-linear term is independent of b. So we subtract from that equation the equation at $b=0$. Our problem is then equivalent to the difference equation plus (12) at $b=0$. Chosing $\mathcal{Y} := -\lambda \lim_{b\to 0}\int_0^{\Lambda^2} dp\, \frac{G_{pb}-G_{p0}}{b}$ we obtain $\frac{Z^{-1}}{(1+\mathcal{Y})} = 1 - \lambda\int_0^{\Lambda^2} dp\, G_{p0}$ and a *linear* integral equation for the difference function $D(a,b) := \frac{a}{b}(G(a,b)-G(a,0))$ to the boundary. The non-linearity restricts to the boundary function $G(a,0)$ where the second index is put to zero. Assuming $a \mapsto G(a,b)$ Hölder-continuous, we can pass to Cauchy principal values. In terms of the *finite Hilbert transform*

$$\mathcal{H}^\Lambda_a[f(\bullet)] := \frac{1}{\pi}\lim_{\epsilon\to 0}\Big(\int_0^{a-\epsilon} + \int_{a+\epsilon}^{\Lambda^2}\Big)\frac{f(q)\,dq}{q-a}\,, \tag{13}$$

the integral equation becomes

$$\Big(\frac{b}{a} + \frac{1+\lambda\pi a\mathcal{H}^\Lambda_a\big[G(\bullet,0)\big]}{aG(a,0)}\Big)D(a,b) - \lambda\pi\mathcal{H}^\Lambda_a\big[D(\bullet,b)\big] = -G(a,0)\,. \tag{14}$$

Equation (14) is a well-known singular integral equation of Carleman type [Car22, Tri57] which can be algebraically solved by techniques for boundary values of holomorphic functions:

Theorem 5 ([GW14b]). *The matrix 2-point function $G(a,b)$ of the $\lambda\phi^{\star 4}_4$-model is in infinite volume limit given in terms of the boundary 2-point function $G(a,b)$ by the equation*

$$G(a,b) = \frac{\sin(\tau_b(a))}{|\lambda|\pi a} \mathrm{e}^{\mathrm{sign}(\lambda)(\mathcal{H}^\Lambda_0[\tau_0(\bullet)] - \mathcal{H}^\Lambda_a[\tau_b(\bullet)])}\,, \tag{15}$$

$$\tau_b(a) := \underset{[0,\pi]}{\arctan}\left(\frac{|\lambda|\pi a}{b + \frac{1+\lambda\pi a\mathcal{H}_a^\Lambda[G(\bullet,0)]}{G(a,0)}}\right),$$

provided that $\lambda < 0$. For $\lambda > 0$ the solution can only be obtained up to a multiplicative correction $\big(1+\frac{Ca+bF(b)}{\Lambda^2-a}\big)$ on the rhs of (15), where C is a undetermined constant and $b\,F(b)$ an undetermined function of b vanishing at $b = 0$.

Some remarks:

- We proved this theorem in 2012 for $\lambda > 0$ under the assumption that the multiplicative correction is absent, but knew that it could arise in principle. That no such term arises for $\lambda < 0$ was later shown in [GW14b].
- Eq. (15) implies $G(a, b) \geq 0$ for $\lambda < 0$. This is a truly non-perturbative result; individual Feynman graphs show no positivity at all! Of course positivity should also hold for $\lambda > 0$, but couldn't be proved.
- As in [GW09], the equation for $G(a, b)$ can be solved perturbatively. This reproduces exactly [GW14a] the Feynman graph calculation! Matching at $\lambda = 0$ requires C, F to be flat functions of λ (all derivatives vanish at zero).
- Because of $\mathcal{H}_a^\Lambda[G(\bullet, 0)] \overset{a\to\Lambda^2}{\longrightarrow} -\infty$, the naïve arctan series is dangerous for $\lambda > 0$. Unless there are cancellations, we expect zero radius of convergence!
- The partition function $\mathcal{Z}$ is undefined for $\lambda < 0$. But the Schwinger-Dyson equations for $G(a, b)$ and for higher functions, and with them $\log \mathcal{Z}$, extend to $\lambda < 0$. These extensions are unique but probably not analytic in a neighbourhood of $\lambda = 0$.

It remains to identify the boundary function $G(a, 0)$ which is determined by (12) at $b = 0$. The equation involves subtle cancellations so that we employ another strategy based on a symmetry argument: Given the boundary function $G(a, 0)$, the Carleman theory computes the full 2-point function $G(a, b)$ via (15). In particular, we get $G(0, b)$ as function of $G(a, 0)$. But the 2-point function is symmetric, $G(a, b) = G(b, a)$, and the special case $a = 0$ leads to the following self-consistency equation:

Proposition 6. *The limit $\theta \to \infty$ of $\lambda\phi_4^4$-theory on Moyal space is for $\lambda \leq 0$ determined by the solution of the fixed point equation $G = TG$,*

$$G(b, 0) \equiv G(0, b) = \frac{1}{1+b}\exp\left(-\lambda\int_0^b dt\int_0^{\Lambda^2}\frac{dp}{(\lambda\pi p)^2 + \big(t+\frac{1+\lambda\pi p\mathcal{H}_p^\Lambda[G(\bullet,0)]}{G(p,0)}\big)^2}\right). \tag{16}$$

At this point we can eventually send $\Lambda \to \infty$. Any solution of (16) is automatically smooth and monotonously decreasing. We proved in [GW14b] that any solution of (16) which maintains the symmetry $G(a,b)=G(b,a)$ necessarily solves the true equation (12). This solution then provides all higher correlation functions via the universal algebraic recursion formulae (11b), or via the linear equations for the basic $(N_1+\ldots+N_B)$-point functions [GW14a].

4.3. Existence of a solution

Remains to prove existence, if possible also uniqueness, of a solution of (16). This is relatively easy for $\lambda>0$ [GW14a] because of obvious bounds $0\leq G(0,b)\leq \frac{1}{1+b}$ and similarly for the first and second derivative. The case $\lambda<0$ is harder and was achieved in [GW16]. It involves the fixed point problem for the function $f(x)=\log G(0,x)$ which takes the form

$$Tf(b) := -\log(1+b) + \int_0^\infty \frac{dt}{\pi t}\Big(\arctan\frac{b+Rf(t)}{|\lambda|\pi t} - \arctan\frac{Rf(t)}{|\lambda|\pi t}\Big)\,, \qquad (17)$$

$$Rf(a) := \frac{1-|\lambda|\pi a\mathcal{H}^\infty_a[e^{f(\bullet)}]}{e^{f(a)}}\,.$$

We prove in [GW16]:

Theorem 7. *Consider the Banach space*

$$LB := \Big\{ f\in \mathcal{C}^1(\mathbb{R}_+)\;:\; f(0)=0\;,\;|f'(x)|\leq \frac{C}{1+x}\text{ for some } C\geq 0\Big\} \qquad (18)$$

of logarithmically bounded differentiable functions, equipped with the norm $\|f\|_{LB}:=|f(0)|+\sup_{x\geq 0}\big|(1+x)f'(x)\big|$. *Then*

$$\mathcal{K}_\lambda = \Big\{ f\in LB\;:\; f(0)=0\;,\qquad -\frac{1-|\lambda|}{1+x}\leq f'(x)\leq -\frac{1-\frac{|\lambda|}{1-2|\lambda|}}{1+x}\Big\} \qquad (19)$$

is a norm-closed subset of LB on which the map T given in (17) is defined. For any $f\in\mathcal{K}_\lambda$ and $-\frac{1}{6}\leq\lambda\leq 0$ one has:

i) $Tf\in\mathcal{K}_\lambda$.
ii) $T:\mathcal{K}_\lambda\to\mathcal{K}_\lambda$ *is norm-continuous.*
iii) The restriction of $T\mathcal{K}_\lambda$ to any interval $[0,\Lambda^2]$ is relatively compact in norm-topology.

The Schauder fixed point theorem then guarantees that T has a fixed point $f_*=Tf_*\in\mathcal{K}\big|_{[0,\Lambda^2]}$ which we denote $\log G(0,b):=f_*(b)$.

For the proof of i) one first has to control $\frac{\mathcal{H}^\infty_a[e^{f(\bullet)}]}{e^{f(a)}}$ for $f\in\mathcal{K}_\lambda$, which succeeds although no reasonable bound for $\mathcal{H}^\infty_a[e^{f(\bullet)}]$ alone is possible. Then an upper bound $(Rf)(a)\leq 1+|\lambda|\pi a\cot(|\lambda|\pi)$ is releatively easy to prove, whereas a lower bound of $(Rf)(a)$ by a piece-wise linear function in a is tedious. The proof of uniform continuity $\|Tf-Tg\|_{LB}\leq c(\lambda)\|f-g\|_{LB}$ in ii) is also involved. The constant $c(\lambda)$ ranges from $1+\frac{1}{e}$ at $\lambda=0$ to about 4 at $\lambda=-\frac{1}{6}$ and slightly misses contractivity which would imply uniqueness by the Banach fixed point theorem. Maybe a better control of the asymptotic behaviour of $(\log G(0,x))'$ for large x rescues contractivity. So far we have to employ the Schauder fixed point theorem where step iii) follows from a variant of the Arzelà-Ascoli theorem.

A numerical iteration of T converges [GW14b] and supports the conjecture of a unique fixed point $f_* = Tf_*$. As discussed later in sec. 6, for reflection positivity of the 2-point function we need to know that $x \mapsto G(0,x)$ is a Stieltjes function. This is true for the boundaries of the region $\mathcal{K}_\lambda$ and very plausible for the numerically found fixed point, but a rigorous proof is still missing.

Note added in proof. In [PW18] a new solution strategy of eq. (11a) was found (for the matrix E of the $\lambda\Phi^4_2$-model on Moyal space). It gives rise to the same equation (15) but a new equation to determine the angle function $\tau_b(a)$. It was possible to guess the perturbative solution of the $\tau_b(a)$-equation and to resum it to the Lambert-W function. This yields an explicit exact solution of $G(a,b)$ for any coupling constant $\lambda > -\frac{1}{2\log 2}$ (for 2D) in terms of the Lambert function and another function for which an integral representation was derived. The latter function expands into Nielsen polylogarithms, Lambert-W expands into logarithms. It should be possible to extend [PW18] to four dimensions and to complete for the the $\lambda\Phi^4_4$-model the programme described below for the $\lambda\Phi^3$-model.

5. Solution of the $\lambda\Phi^3$-model

In a recent joint work with Akifumi Sako from Tokyo we completely solved the matricial $\lambda\Phi^3$-model, i.e. the renormalised Kontsevich model. Considerable progress with this model has already been achieved long ago in a series of papers [GS06a, GS06b, GS08] of H.G. with H. Steinacker. Formulae for the renormalised 2-point function and the 1+1-point functions are given in [GS06a], but a complete solution for all functions is new.

Renormalisation requires the following ansatz for the action functional:

$$\begin{aligned} S(\Phi) &:= \frac{1}{(8\pi)^{\frac{D}{2}}} \int_{\mathbb{R}^D} dx \Big(\frac{Z}{2}\phi \star H^1(\phi) + \kappa\phi + \frac{\nu}{2} H^1(\phi) + \frac{\zeta}{4} H^1(H^1(\phi)) \\ &\qquad + \frac{\mu^2_{bare}}{2}\phi\star\phi + \frac{\lambda_{bare}Z^{\frac{3}{2}}}{3}\phi\star\phi\star\phi\Big)(x) \\ &= V\Big(\sum_{\underline{n},\underline{m}\in\mathbb{N}^{D/2}_{\mathcal{N}}} Z\Phi_{\underline{m}\underline{n}}\Phi_{\underline{n}\underline{m}} \frac{E_{\underline{m}}+E_{\underline{n}}}{2} + \sum_{\underline{n}\in\mathbb{N}^{D/2}_{\mathcal{N}}} (\kappa + \nu E_{\underline{n}} + \zeta E^2_{\underline{n}})\Phi_{\underline{n}\underline{n}} \\ &\qquad + \frac{\lambda_{bare}Z^{\frac{3}{2}}}{3} \sum_{\underline{n},\underline{m},\underline{l}\in\mathbb{N}^{D/2}_{\mathcal{N}}} \Phi_{\underline{n}\underline{m}}\Phi_{\underline{m}\underline{l}}\Phi_{\underline{l}\underline{n}}\Big), \end{aligned} \tag{20}$$

where $\kappa, \nu, \zeta, Z, \mu^2_{bare}, \lambda_{bare}$ are functions of $(V, \mathcal{N})$ and renormalised parameters (λ_r, μ^2). In the step to the last two lines we have absorbed $E_{\underline{n}} \mapsto E_{\underline{n}} + \frac{\mu^2_{bare}}{2Z}$ and redefined κ, ν, ζ. In low dimension not all parameters are necessary: $\nu = \zeta = 0$, $Z = 1$, $\mu^2_{bare} = \mu^2$, and $\lambda_{bare} = \lambda$ for $D = 2$; $\zeta = 0$, $Z = 1$ and $\lambda_{bare} = \lambda$ for $D = 4$. As before, $V := (\frac{\theta}{4})^{D/2}$ for Moyal space and $H_{\underline{m}\underline{n}} := E_{\underline{m}} + E_{\underline{n}}$.

5.1. Schwinger-Dyson equations and their solution for $B = 1$

Combining $V\,\mathrm{Tr}((\nu E + \zeta E^2)\Phi)$ in (20) with the term $\mathrm{i}V\,\mathrm{Tr}(J\Phi)$ from the Fourier transform, the Ward-Takahashi identity (8) becomes

$$-\sum_{\underline{n}\in\mathbb{N}_{\mathcal{N}}^{D/2}} \frac{\partial^2 \mathcal{Z}(J)}{\partial J_{\underline{b}\underline{n}}\partial J_{\underline{n}\underline{a}}} = \delta_{\underline{a}\underline{b}} \mathrm{W}_{\underline{a}}(J)\mathcal{Z}(J) - \frac{V}{Z}(\nu + \zeta H_{\underline{a}\underline{b}}) \frac{\partial \mathcal{Z}(J)}{\mathrm{i}\partial J_{|\underline{b}\underline{a}|}} + \sum_{\underline{n}\in\mathbb{N}_{\mathcal{N}}^{D/2}} \frac{V}{Z(E_{\underline{a}} - E_{\underline{b}})} \Big(J_{\underline{a}\underline{n}} \frac{\partial}{\partial J_{\underline{b}\underline{n}}} - J_{\underline{n}\underline{b}} \frac{\partial}{\partial J_{\underline{n}\underline{a}}} \Big) \mathcal{Z}(J) \;, \qquad (21)$$

where $\mathrm{W}_{\underline{a}}$ is the same as in (8). Applying the derivative in the 1-point $G_{|\underline{a}|} = \frac{1}{V\mathcal{Z}(0)} \frac{\partial \mathcal{Z}(J)}{\mathrm{i}\partial J_{\underline{a}\underline{a}}}\big|_{J=0}$ to (5) gives (without using (21))

$$G_{|\underline{a}|} = \frac{1}{ZH_{\underline{a}\underline{a}}} \Big\{ -\kappa - \nu E_{\underline{a}} - \zeta E_{\underline{a}}^2 - \lambda_{bare} Z^{\frac{3}{2}} \Big(G_{|\underline{a}|}^2 + \frac{1}{V} \sum_{\underline{m}\in\mathbb{N}_{\mathcal{N}}^{D/2}} G_{|\underline{a}\underline{m}|} + \frac{G_{|\underline{a}|\underline{a}|}}{V^2} \Big) \Big\}. \qquad (22)$$

The same steps as in the derivation of (10) yield

$$G_{|\underline{a}\underline{b}|} = \frac{1}{ZH_{\underline{a}\underline{b}}} \Big(1 + \lambda_{bare} Z^{\frac{1}{2}} \frac{(G_{|\underline{a}|} - G_{|\underline{b}|})}{E_{\underline{a}} - E_{\underline{b}}} + \lambda_{bare} Z^{\frac{1}{2}} (\nu + \zeta H_{\underline{a}\underline{b}}) G_{|\underline{a}\underline{b}|} \Big) \;. \qquad (23)$$

Higher N-point functions are algebraically expressed by $(N{-}1)$-point functions where their finiteness requires the following identities between the renormalisation constants: $\lambda_r = \sqrt{Z}\lambda_{bare}$, $F_{\underline{a}} := E_{\underline{a}} - \frac{1}{2}\lambda_r \nu$ and $\lambda_r \zeta = Z - 1$ for finite $\lambda_r, F_{\underline{a}}$. Then

$$G_{|\underline{a}\underline{b}|} = \frac{1}{F_{\underline{a}} + F_{\underline{b}}} + \lambda_r \frac{G_{|\underline{a}|} - G_{|\underline{b}|}}{F_{\underline{a}}^2 - F_{\underline{b}}^2} \;, \quad G_{|\underline{a}_1\underline{a}_2\dots\underline{a}_N|} = \lambda_r \frac{G_{|\underline{a}_1\underline{a}_3\dots\underline{a}_N|} - G_{|\underline{a}_2\underline{a}_3\dots\underline{a}_N|}}{(F_{\underline{a}_1}^2 - F_{\underline{a}_2}^2)} \;. \qquad (24)$$

Inserting the first identity into (22) gives

$$W_{|\underline{a}|}^2 + 2\lambda_r \nu W_{|\underline{a}|} = \frac{4}{Z} F_{\underline{a}}^2 - \Big(4\frac{\lambda_r \kappa}{Z} + \Big(1 + \frac{1}{Z}\Big)(\lambda_r \nu)^2 \Big) - \frac{2\lambda_r^2}{V} \sum_{\underline{n}\in\mathbb{N}_{\mathcal{N}}^{D/2}} \frac{W_{|\underline{a}|} - W_{|\underline{n}|}}{F_{\underline{a}}^2 - F_{\underline{n}}^2} - \frac{4\lambda_r^2}{V^2} G_{|\underline{a}|\underline{a}|} \;, \qquad (25)$$

where $\frac{W_{|\underline{a}|}}{2\lambda_r} := G_{|\underline{a}|} + \frac{F_{\underline{a}}}{\lambda_r}$ (and thus $G_{|\underline{a}\underline{b}|} = \frac{1}{2} \frac{W_{|\underline{a}|} - W_{|\underline{b}|}}{F_{\underline{a}}^2 - F_{\underline{b}}^2}$). The last term in (25) vanishes for $V \to \infty$ so that a closed equation for W is obtained. The remaining parameters ν, κ, Z are fixed by normalisation conditions, depending on dimension D.

The steps are the same as before in the $\lambda\Phi_4^4$-model. All functions depend only on the 1-norms of multiple matrix indices so that index sums reduce to

$\sum_{\underline{m}\in\mathbb{N}_{\mathcal{N}}^{D/2}} f(|\underline{m}|) = \sum_{|\underline{m}|=0}^{\mathcal{N}} \binom{|\underline{m}|+\frac{D}{2}-1}{\frac{D}{2}-1} f(|\underline{m}|)$. Next the limit $\mathcal{N}, V \to \infty$ is taken subject to fixed ratio $\frac{\mathcal{N}}{V^{\frac{2}{D}}} = \mu^2\Lambda^2$. This limit maps 1-norms of matrix indices into $|\underline{a}| \mapsto V^{\frac{2}{D}}\mu^2 a$ with $a \in [0,\Lambda^2]$, in particular $F_{\underline{a}} \mapsto \mu^2(a+\frac{1}{2})$. Sums over matrix indices converge to a Riemann integrals over $[0,\Lambda^2]$. The occurrence of $F_{\underline{a}}^2$ in (25) and (24) then suggests a substitution $A(a) = (2a+1)^2$. In these variables and in mass-dimensionless quantities, (25) becomes

$$(W(A))^2 + 2\lambda_r \nu W(A) + \int_1^{(1+2\Lambda^2)^2} dT\, \rho(T)\, \frac{W(A)-W(T)}{A-T} - \frac{A}{Z} = \text{const}, \quad (26)$$

$$\rho(T) := \frac{\lambda_r^2(\sqrt{T}-1)^{\frac{D}{2}-1}}{2^{\frac{D}{2}-4}\Gamma(\frac{D}{2})\sqrt{T}}\;, \qquad G(a) \equiv \frac{W((2a+1)^2)-(2a+1)}{2\lambda_r}\;.$$

The recursion (24) can be explicitly solved. For $N > 1$ one has [GSW17]:

$$G(a_1,\dots,a_N) = \frac{\lambda_r^{N-2}}{2} \sum_{k=1}^{N} W((2a_k+1)^2) \prod_{l=1,l\neq k}^{N} \frac{1}{(a_k+\frac{1}{2})^2-(a_l+\frac{1}{2})^2}\;. \quad (27)$$

Thus it remains to solve (26). We can take advantage of the fact that (26) is, for $Z = 1$ and $\nu = 0$, *exactly the master constraint in the Kontsevich model from which all Virasoro constaints descend*. This master constraint was solved by Makeenko-Semenoff [MS91] by viewing ρ and W as boundary values of holomorphic functions on $\mathbb{C}\setminus[1,(1+\Lambda^2)^2]$. The solution technique is thus not unrelated to the solution of the Carleman equation (14) by (15) in the $\lambda\Phi^4_4$-model. The Makeenko-Semenoff solution can easily be adapted to include Z, ν and gives [GSW17]

$$W(A) := \frac{\sqrt{A+c}}{\sqrt{Z}} - \lambda_r\nu + \frac{1}{2}\int_1^{(1+\Lambda^2)^2} dT \frac{\rho(T)}{(\sqrt{A+c}+\sqrt{T+c})\sqrt{T+c}}\;, \quad (28)$$

for some function $c(\lambda_r,\nu,Z)$. Inserting $\rho(T)$ from (26) shows that the integral diverges, if Z, ν where absent, in the limit $\Lambda \to \infty$. This is the usual divergence in quantum field theory which is avoided by a careful choice of ν, Z according to normalisation conditions. The standard normalisation conditions on the 1- and 2-point functions translate into

$$W(1) \overset{D\geq 2}{=} 1\;, \qquad W'(1) \overset{D\geq 4}{=} \frac{1}{2}\;, \qquad W''(1) \overset{D=6}{=} -\frac{1}{4}\;. \quad (29)$$

These conditions determine $\nu(\lambda_r,\Lambda)$ $Z(\lambda_r,\Lambda)$ (unless 0 and 1 for small D) as well as $c(\lambda_r) = \lim_{\Lambda\to\infty} c(\lambda_r,\Lambda)$. For $D = 6$ the solution reads [GSW18] (for $\Lambda \to \infty$ where possible)

$$\begin{aligned} W(A) &= \sqrt{A+c}\sqrt{1+c} - c \\ &\quad + \frac{1}{2}\int_1^{\infty} \frac{dT\,\rho(T)\,(\sqrt{A+c}-\sqrt{1+c})^2}{(\sqrt{A+c}+\sqrt{T+c})(\sqrt{1+c}+\sqrt{T+c})^2\sqrt{T+c}}\;, \end{aligned} \quad (30a)$$

$$-c = \int_1^\infty \frac{dT\ \rho(T)}{(\sqrt{1+c}+\sqrt{T+c})^3\sqrt{T+c}}\ . \tag{30b}$$

$$\frac{1}{\sqrt{Z(\Lambda)}} = \sqrt{1+c} + \frac{1}{2}\int_1^{(1+\Lambda^2)^2} dT \frac{\rho(T)}{(\sqrt{1+c}+\sqrt{T+c})^2\sqrt{T+c}}\ . \tag{30c}$$

We refer to [GSW17] for $D = 2$ and to [GSW18] for $D = 4$. The last identity (which implies $Z(\Lambda) \in [0, 1]$, as it should) is needed in the β-function

$$\beta_\lambda := \Lambda^2 \frac{d\lambda_{bare}(\Lambda)}{d\Lambda^2} = \frac{2\lambda_r^3\Lambda^6}{\left(\sqrt{1+c}+\sqrt{(2\Lambda^2+1)^2+c}\right)^2\sqrt{(2\Lambda^2+1)^2+c}} \overset{\Lambda\to\infty}{\longrightarrow} \frac{\lambda_r^3}{4}\ .$$

Since β_λ has the same sign as λ_r, $|\lambda_{bare}(\Lambda^2)|$ increases with Λ^2 and tends to ∞ for $\lambda \to \infty$. This is the opposite of asymptotic freedeom; nevertheless the model can be rigorously constructed! This came as surprise to us. The vanishing of the β-function in the $\lambda\Phi^4_4$-model was originally thought to be essential for constructing the model – but it isn't.

The consistency relation (30a) is responsible for complexity of this quantum field theory. Inserting $\rho(T)$ from (26) gives the transcendental equation (for $D = 6$)

$$\lambda_r^2 = \frac{(-4c)}{1 - 2\sqrt{1+c} + 2(1+c)\log(1+\frac{1}{\sqrt{1+c}})}\ . \tag{31}$$

The functions $G(\dots)$ are then expressed via (30a) and (27) in terms of the inverse solution $c(\lambda_r^2)$ which exists by the inverse function theorem. It is easy to invert (30a) as a formal power series in λ^2. We have demonstrated in [GSW18] that the resulting perturbative expansion of $G(a)$ perfectly agrees with the renormalised Feynman graph calculation.

5.2. Solution for $B > 1$ boundary components

Schwinger-Dyson equations for $(N_1+\ldots+N_B)$-point functions easily give, as long as one $N_i > 1$, a recursion

$$\begin{aligned} &G_{|\underline{a}_1^1\dots\underline{a}_{N_1}^1|\dots|\underline{a}_1^B\dots\underline{a}_{N_B}^B|} \\ &= \lambda_r \frac{G_{|\underline{a}_1^1\underline{a}_3^1\dots\underline{a}_{N_1}^1|\underline{a}_1^2\dots\underline{a}_{N_2}^2|\dots|\underline{a}_1^B\dots\underline{a}_{N_B}^B|} - G_{|\underline{a}_2^1\underline{a}_3^1\dots\underline{a}_{N_1}^1|\underline{a}_1^2\dots\underline{a}_{N_2}^2|\dots|\underline{a}_1^B\dots\underline{a}_{N_B}^B|}}{F^2_{\underline{a}_1^1} - F^2_{\underline{a}_2^1}} \end{aligned} \tag{32}$$

which is solved in terms of $(1+\ldots+1)$-point functions and after passing to the scaling limit by

$$\begin{aligned} &G(a_1^1,\dots,a_{N_1}^1|\dots|a_1^B,\dots,a_{N_B}^B) \\ &= \sum_{k_1=1}^{N_1}\cdots\sum_{k_B=1}^{N_B} G(a_{k_1}^1|\dots|a_{k_B}^B)\prod_{\beta=1}^{B}\prod_{l_\beta=1,l_\beta\neq k_\beta}^{N_\beta}\frac{\lambda_r}{(a_{k_\beta}^\beta+\frac{1}{2})^2-(a_{l_\beta}^\beta+\frac{1}{2})^2}\ . \end{aligned} \tag{33}$$

For the (1+...+1)-point function one derives the Schwinger-Dyson equation

$$(W_{|\underline{a}^1|}+\nu\lambda_r)G_{|\underline{a}^1|\underline{a}^2|\dots|\underline{a}^B|}+\frac{\lambda_r^2}{V}\sum_{\underline{n}\in\mathbb{N}_{\mathcal{N}}^{D/2}}\frac{G_{|\underline{a}^1|\underline{a}^2|\dots|\underline{a}^B|}-G_{|\underline{n}|\underline{a}^2|\dots|\underline{a}^B|}}{(F^2_{\underline{a}^1}-F^2_{\underline{n}})} \tag{34}$$
$$=-\lambda_r\sum_{\beta=2}^{B}G_{|\underline{a}^1\underline{a}^\beta\underline{a}^\beta|\underline{a}^2|\dots\overset{\beta}{\check{}}\dots|\underline{a}^B|}-\frac{\lambda_r}{V^2}G_{|\underline{a}^1|\underline{a}^1|\underline{a}^2|\dots|\underline{a}^B|}$$
$$-\lambda_r\sum_{p=1}^{B-2}\sum_{2\le i_1<\dots<i_p\le B}G_{|\underline{a}^1|\underline{a}^{i_1}|\dots|\underline{a}^{i_p}|}G_{|\underline{a}^1|\underline{a}^{j_1}|\dots|\underline{a}^{j_{B-p-1}}|}\,.$$

Taking the scaling limit and transforming variables a^i to A^i gives [GSW17]

$$(W(A^1)+\tilde{\lambda}\tilde{\nu})G(A^1|A\triangleleft^{\{2,\dots B\}})$$
$$+\frac{1}{2}\int_1^{(1+\Lambda^2)^2}dT\rho(T)\frac{G(A^1|A\triangleleft^{\{2,\dots B\}})-G(T|A\triangleleft^{\{2,\dots B\}})}{(A-T)}$$
$$=-\lambda_r\sum_{\beta=2}^{B}G(A^1,A^\beta,A^\beta|A\triangleleft^{\{2\dots\overset{\beta}{\check{}}\dots B\}})$$
$$-\lambda_r\sum_{\substack{J\subset\{2,\dots,B\}\\1\le|J|\le B-2}}G(A^1|A\triangleleft^J)G(A^1|A\triangleleft^{\{2,\dots,B\}\setminus J})\,, \tag{35}$$

where the measure $\rho(T)$ was defined in (26) and $G(A|A\triangleleft^{\{i_1,\dots,i_p\}}):=G(A|A^{i_1}|\dots|A^{i_p})$.

The solution of equation (35) goes over 8 pages in [GSW17]. For $B=2$ where (35) simplifies considerably one finds the algebraic solution

$$G(a^1|a^2)=\frac{4\tilde{\lambda}^2}{\sqrt{(2a^1{+}1)^2{+}c}\cdot\sqrt{(2a^2{+}1)^2{+}c}\cdot(\sqrt{(2a^1{+}1)^2{+}c}+\sqrt{(2a^2{+}1)^2{+}c})^2}\,, \tag{36}$$

which was already given in [GS06a]. For $B\ge 3$ we found the key ansatz

$$G(A^1|\dots|A^B)=\frac{(-2\tilde{\lambda})^{3B-4}}{\rho_0}\sum_{M=0}^{B-3}\gamma_B^M\frac{d^M}{dt^M}\prod_{\beta=1}^{B}\frac{1}{(A^\beta+c-2t)^{\frac{3}{2}}}\,, \tag{37}$$

where $\rho_0:=\lim_{\Lambda\to\infty}\frac{1}{\sqrt{Z(\Lambda)}}-\frac{1}{2}\int_1^{(1+\Lambda^2)^2}\frac{dT\rho(T)}{(T+c)^{3/2}}$. This leads to $\gamma_3^M=\delta_{M,0}$ and a complicated system of non-linear recursion equations for γ_B^M. The solution of the first of them contains intriguing combinatorial factors which are found to be described by Bell polynomials $B_{n,k}(\{x_1,\dots,x_{n-k-1}\})$. The ansatz

$$\gamma_B^M=\frac{1}{\rho_0^{B-3}}\sum_{K=0}^{B-3-M}\frac{(B{-}3{+}K)!}{(B{-}3{-}M)!M!}B_{B-3-M,K}\Big(\Big\{-\frac{(2r{+}1)!!\rho_r}{(r+1)\rho_0}\Big\}_{r=1}^{B-2-M-K}\Big)\,, \tag{38}$$

with $\rho_l := -\frac{1}{2}\int_1^\infty \frac{dT\rho(T)}{(T+c)^{l+3/2}}$ for $l \geq 1$, is then confirmed in a lengthy induction proof. The solution (38) is inserted into (37) and rewritten in terms of the generating function of Bell polynomials. Eventually the following result is established:

Theorem 8 ([GSW17]). *The scaling limit of the* $(1+\ldots+1)$*-point function of* $\lambda\Phi^3$ *matricial quantum field theory is given for* $B \geq 3$ *by*

$$G(a^1|\ldots|a^B) = \frac{d^{B-3}}{dt^{B-3}}\Big(\frac{(-2\lambda_r)^{3B-4}}{(R(t))^{B-2}\prod_{\beta=1}^{B}((2a^\beta+1)^2+c-2t)^{\frac{3}{2}}}\Big)\Big|_{t=0}, \tag{39}$$

$$R(t) := \lim_{\Lambda\to\infty}\Big(\frac{1}{\sqrt{Z(\Lambda)}} - \int_1^{(1+\Lambda^2)^2}\frac{dT\rho(T)}{\sqrt{T+c}}\frac{1}{(\sqrt{T+c}+\sqrt{T+c-2t})\sqrt{T+c-2t}}\Big).$$

In this way a complete construction of the scaling limit of the renormalised Kontsevich model $\lambda\Phi^3_D$ in dimensions $D \in \{2,4,6\}$ is achieved.

6. Schwinger functions and reflection positivity

6.1. Reverting the matrix representation

In sections 4 and 5 we have constructed the connected matrix correlation functions $G_{|\underline{q}_1^1\cdots\underline{q}_{N_1}^1|\cdots|\underline{q}_1^B\cdots\underline{q}_{N_B}^B|}$ of the $(\theta\to\infty)$-limit of $\lambda\phi_4^{\star 4}$-theory and $\lambda\phi_D^{\star 3}$-theory on Moyal space. Now we revert the introduction of the matrix basis (3) to obtain Schwinger functions in position space:

$$\begin{aligned}S_c(\mu x_1,\ldots,\mu x_N) := \lim_{V\mu^D\to\infty}\lim_{\Lambda\to\infty}\frac{1}{(8\pi)^{\frac{D}{2}}}\sum_{N_1+\cdots+N_B=N}\sum_{\underline{q}_i^\beta\in\mathbb{N}^2_{\mathcal{N}}} G_{|\underline{q}_1^1\cdots\underline{q}_{N_1}^1|\cdots|\underline{q}_1^B\cdots\underline{q}_{N_B}^B|}\\ \times\sum_{\sigma\in\mathcal{S}_N}\prod_{\beta=1}^{B}\frac{f_{\underline{q}_1\underline{q}_2}(x_{\sigma(s_\beta+1)})\cdots f_{\underline{q}_{N_\beta}\underline{q}_1}(x_{\sigma(s_\beta+N_\beta)})}{V\mu^D N_\beta},\end{aligned} \tag{40}$$

where $s_\beta := N_1+\ldots+N_{\beta-1}$ and $\mathcal{N} = \Lambda^2\mu^2V^{\frac{2}{D}}$ for $\lambda\Phi^3$ and $\mathcal{N} = \Lambda^2\mu^2V^{\frac{2}{D}}(1+\mathcal{Y})$ for $\lambda\Phi^4$. The $G_{\ldots}$ are made dimensionless by appropriate rescaling in μ. There are two delicate points with this definition: First, we perform the limits $\lim_{V\mu^D\to\infty}, \lim_{\Lambda\to\infty}$ in different order than before and second the convention $\frac{\delta J_{\underline{mn}}}{\delta J(\xi)} := \mu^D f_{\underline{mn}}(\xi)$ is made.

The next step consists in representing $G_{\ldots|\underline{a}_1^\beta\ldots\underline{a}_{N_\beta}^\beta|\ldots}$, for every boundary component, as a Laplace transform in $\frac{1}{\sqrt{V\mu^D}}(|\underline{a}_1^\beta|+\cdots+|\underline{a}_{N_\beta}^\beta|)$ and Fourier transform in $\frac{1}{\sqrt{V\mu^D}}(|\underline{a}_{i+1}^\beta|-|\underline{a}_i^\beta|)$. For example,

$$G_{|\underline{a}\underline{b}|} = \int_0^\infty dt\int_{-\infty}^\infty d\omega\; \mathcal{G}(t,\omega)e^{-\frac{t}{\sqrt{V\mu^4}}(|\underline{a}|+|\underline{b}|)-\mathrm{i}\frac{\omega}{\sqrt{V\mu^4}}(|\underline{a}|-|\underline{b}|)}. \tag{41}$$

Using generating functions for Laguerre polynomials, the following identity can be established [GW13]:

$$
\begin{aligned}
&\sum_{m_1,\dots,m_N=0}^{\infty} \frac{1}{\theta}\prod_{i=1}^{N} f_{m_i m_{i+1}}(x_i) z_i^{m_i} \\
&= \frac{2^N}{\theta(1-\prod_{i=1}^N(-z_i))} \exp\Big(- \frac{\sum_{i=1}^N \|x_i\|^2}{\theta} \frac{1+\prod_{i=1}^N(-z_i)}{1-\prod_{i=1}^N(-z_i)}\Big) \\
&\times \exp\Big(-\frac{2}{\theta}\sum_{1\leq k<l\leq N}\Big(\big(\langle x_k,x_l\rangle - \mathrm{i} x_k\times x_l\big)\frac{\prod_{j=k+1}^{l}(-z_j)}{1-\prod_{i=1}^N(-z_i)} \\
&\qquad\qquad + \big(\langle x_k,x_l\rangle + \mathrm{i} x_k\times x_l\big)\frac{\prod_{j=l+1}^{N+k}(-z_j)}{1-\prod_{i=1}^N(-z_i)}\Big)\Big).
\end{aligned}
\tag{42}
$$

The z_i are of the form $z \sim \exp(-\frac{t+\mathrm{i}\omega}{\sqrt{V\mu^D}})$ as in (41). At this point the limit $V\mu^D \to \infty$ can be taken where z_i converges to 1. Thus for odd N the limit is zero, whereas for N even one has $\lim_{\theta\to\infty}\theta(1-\prod_{i=1}^N(-z_i)) = \frac{4Nt}{\mu^2}$. The vector product and all Fourier variables ω drop out, and the scalar products (42) arrange with the norms to $\mu^2\|x_1-x_2+\cdots-x_N\|^2$. Absence of the Fourier variables means that all matrix indices per boundary component are equal. The Laplace transform is easily reverted after introduction of an auxiliary p-integration per boundary component. The final result is:

Theorem 9 ([GW13]). *The connected N-point Schwinger functions of the $\lambda\phi_4^{\star 4}$ and $\lambda\phi_D^{\star 3}$ models on extreme Moyal space $\theta\to\infty$ in D dimensions are given by*

$$
\begin{aligned}
&S_c(\mu x_1,\dots,\mu x_N) \\
&= \frac{1}{(8\pi)^{\frac{D}{2}}} \sum_{\substack{N_1+\dots+N_B=N\\ N_\beta \text{ even}}} \sum_{\sigma\in\mathcal{S}_N} \Big(\prod_{\beta=1}^{B} \frac{4^{N_\beta}}{N_\beta}\int_{\mathbb{R}^4} \frac{dp_\beta}{4\pi^2\mu^4}\, e^{\mathrm{i}\big\langle \frac{p_\beta}{\mu}, \sum_{i=1}^{N_\beta}(-1)^{i-1}\mu x_{\sigma(s_\beta+i)}\big\rangle}\Big) \\
&\qquad \times G\Big(\underbrace{\tfrac{\|p_1\|^2}{2\mu^2},\dots,\tfrac{\|p_1\|^2}{2\mu^2}}_{N_1}\Big|\dots\Big|\underbrace{\tfrac{\|p_B\|^2}{2\mu^2},\dots,\tfrac{\|p_B\|^2}{2\mu^2}}_{N_B}\Big).
\end{aligned}
\tag{43}
$$

For the $\lambda\phi_4^{\star 4}$ model one must replace $\mu^2 \mapsto (1+\mathcal{Y})\mu^2$ in the last line of (43) to account for the finite wavefuntion renormalisation.

Some comments:

- Only a restricted sector of the underlying matrix model contributes to position space: The external matrix indices of the same boundary component are equal. This is reminiscent of quantum chromodynamics where only colour-singlets play a physical rôle, whereas internally coloured quarks and gluons interact. In this sense, (43) corresponds to a confinement of noncommutativity.

- Schwinger functions are symmetric and invariant under the full Euclidean group. This comes truly surprising since $\theta \neq 0$ breaks both translation invariance and manifest rotation invariance. The limit $\theta \to \infty$ was expected to make this symmetry violation even worse!

- The most interesting sector is the case where every boundary component has $N_\beta = 2$ indices. It is described by the $(2+\ldots+2)$-point functions $G\big(\frac{\|p_1\|^2}{2\mu^2}\, \frac{\|p_1\|^2}{2\mu^2}\big|\ldots\big|\frac{\|p_B\|^2}{2\mu^2}\, \frac{\|p_B\|^2}{2\mu^2}\big)$. This $(2+\ldots+2)$-sector describes the propagation and interaction of B (at the moment Euclidean) particles without any momentum exchange. Such a behaviour is necessary in any integrable model [Mos75, Kul76]. It is tempting to speculate that also for the $\lambda\phi_4^{\star 4}$-model there might be an integrable structure behind that is responsible for that model being solvable (see [PW18]) and for absence of momentum transfer. One would also like to make the relation between the integrable Kontsevich model and the observed structure precise.

- We are aware of the problem that the absence of momentum transfer in four dimensions is a sign of *triviality*. Typical triviality proofs rely on clustering, analyticity in Mandelstam representation or absence of bound states. All this needs verification.

6.2. Osterwalder-Schrader axioms

Under conditions identified by Osterwalder-Schrader [OS73, OS75], Schwinger functions of a Eulidean quantum field theory permit an analytical continuation to Wightman functions [SW64] of a true relativistic quantum field theory. In simplified terms, the reconstruction theorem of Osterwalder-Schrader for a field theory on $\mathbb{R}^D$ reads:

Theorem 10 ([OS73, OS75]). *Assume the Schwinger functions* $S(x_1,\ldots,x_N)$ *satisfy*

(OS0) factorial growth,
(OS1) Euclidean invariance,
(OS2) reflection positivity,
(OS3) permutation symmetry.

Then the $\mathcal{S}(\xi_1,\ldots\xi_{N-1})\big|_{\xi_i^0>0}$*, with* $\xi_i = x_i - x_{i+1}$*, are Laplace-Fourier transforms of Wightman functions in a relativistic quantum field theory. If in addition the* $S(x_1,\ldots,x_N)$ *satisfy*

(OS4) clustering

then the Wightman functions satisfy clustering, too.

The Schwinger functions (43) clearly satisfy (OS1)+(OS3). Clustering (OS4) is not realised. Factorial growth (OS0) is obvious for the $\lambda\Phi^3$-model due to (39) and (33). It shouldn't be a problem either for $\lambda\Phi^4_4$. Thus the remaining problem is (OS2) reflection positivity: *For each assignment $N \mapsto f_N \in \mathcal{S}(\mathbb{R}^{ND})$ of test functions, one has*

$$\sum_{M,N} \int dx\, dy\; S(x_1,\dots,x_N,y_1,\dots,y_M)\overline{f_N(x_1^r,\dots,x_N^r)} f_M(y_1,\dots,y_M) \geq 0\;,$$

where $(x^0,x^1,\dots x^{D-1})^r := (-x^0,x^1,\dots x^{D-1})$. Let $\hat{S}$ be the Fourier transform of the Schwinger function S, viewed as function of its independent momenta. Then reflection positivity implies for a special choice of test functions that the temporal Fourier transform of $\hat{S}$ (in all independent energies) is, for any spatial momenta, a *positive definite function*. Such functions are described by

Theorem 11 (Hausdorff-Bernstein-Widder). *For a continuous/smooth function F on $(\mathbb{R}_+)^N \ni t = (t^1,\dots,t^N)$ are equivalent:*

1. *F is positive definite, i.e.* $\sum_{i,j=1}^K \overline{c_i} c_j F(t_i+t_j) \geq 0$
2. *F is the joint Laplace transform of a positive measure*
3. *F is completely monotonic, i.e.* $(-1)^{k_1+\dots+k_N}\partial_{t^1}^{k_1}\dots\partial_{t^N}^{k_N} F(t) \geq 0$.

Knowing that the Schwinger functions, considered as function of time differences, are Laplace transforms constitutes the main part of the Osterwalder-Schrader theorem.

Thanks to our explicit formulae (39) and (33) of all Schwinger functions a direct verification of complete monotonicity in the $\lambda\Phi^3$-model is realistic. For the 2-point function this amounts to prove that $a \mapsto G(a,a) \equiv \int_0^\infty \frac{d\varrho(m^2)}{a+m^2}$ is a Stieltjes function, i.e. the Stieltjes transform of a positive measure $d\varrho(m^2)$. The easiest way to convice oneself that this condition is necessary (sufficiency is clear) is to compare it with Källén-Lehmann spectral representation of a Wightman 2-point function. Stieltjes functions have a holomorphic extension to the complex plane minus the negative reals. The imaginary part of the jump across the cut $]-\infty,0]$ is proportional to the Stieltjes measure. It is then straightforward to determine whether $a \mapsto G(a,a)$ is Stieltjes for the $\lambda\Phi^3_D$-model. Somewhat surprisingly, this is the case for $D=4$ and $D=6$, *but not for $D=2$!* We cite the result in $D=6$:

Theorem 12 ([GSW17]). *The diagonal 2-point function of the renormalised 6-dimensional Kontsevich model $\lambda\phi_6^{\star 3}$ on Moyal space with harmonic oscillator potential is, for real coupling constant and in large-$(\mathcal{N},V)$ limit, a Stieltjes function. This Stieltjes measure $\varrho(t)$ has support $[1-\sqrt{-c},1+\sqrt{-c}] \cup [2,\infty[$ consisting of an isolated region near $t=1$ and the unbounded interval $t \geq 2$. The precise relation is*

$$\tilde{G}\Big(\frac{p^2}{2\mu^2},\frac{p^2}{2\mu^2}\Big)$$

$$= \frac{\lambda_r^2}{4\pi(\sigma^2-1)} \int_0^\pi d\phi \; \frac{\begin{array}{c}\big\{2\frac{\log(1+\sigma)}{\sigma} - 1 + \sigma(\sigma-1)\tan^2\phi \\ -\tan\phi\big(1+\sigma^2\tan^2\phi\big)\big(\arctan_{[0,\pi]}(\sigma\tan\phi) - \phi\big)\big\}\end{array}}{1 - \frac{\sqrt{\sigma^2-1}}{\sigma}\cos\phi + \frac{p^2}{\mu^2}}$$

$$+ \frac{\lambda_r^2}{4}\int_2^\infty dt \; \frac{t(t-2)/(t-1)^3}{t+\frac{p^2}{\mu^2}} \;, \tag{44}$$

where $\sigma := \frac{1}{\sqrt{1+c}} \in [1, -2W_{-1}(-\frac{1}{2\sqrt{e}}) - 1]$ *is the inverse solution of* $\lambda_r^2 = \frac{4(\sigma^2-1)}{\sigma^2-2\sigma+2\log(1+\sigma)} \in [1, \frac{8W_{-1}(-\frac{1}{2\sqrt{e}})}{1+2W_{-1}(-\frac{1}{2\sqrt{e}})}]$. *Here,* $W_{-1}(z)$ *for* $z \in [-\frac{1}{e}, 0]$ *is the lower real branch of the Lambert-*W *function.*

The 2-point function $G(a,a)$ is never Stieltjes for $\lambda \in \mathrm{i}\mathbb{R}$ where the partition function has a chance to exist. Positivity only holds for $\lambda \in \mathbb{R}$ where the action is unbounded from below, rendering the partition function meaningless. We have numerical evidence [GW14b] and partial analytic results that exactly the same is true for the $\lambda\phi_4^{\star 4}$-model: The 2-point function is definitely not reflection positive in the stable case $\lambda > 0$, whereas for $\lambda < 0$ positivity seems to hold.

7. Outlook

Reflection positivity of the 2-point function is necessary, but alone not sufficient for a reconstruction of the Wightman theory. All (disconnected) Schwinger N-point functions must be reflection positive. Work on this question is not yet completed. We have a simple argument that reflection positivity *does not hold* for the whole set of Schwinger functions (43) for the $\lambda\phi^{\star 3}$-model. The reason is the fast decay in a^β established in (39) which contradicts complete monotonicity in Theorem 11. The situation is probably not much better for $\lambda\phi_4^{\star 4}$.

We are therefore exploring another approach. The transition (40) from connected matrix correlation functions $\big\langle\Phi_{\underline{a}_1\underline{b}_1}\dots\Phi_{\underline{a}_N\underline{b}_N}\big\rangle_c$ to Schwinger functions by reverting the Moyal matrix basis can be formulated as

$$S_c(x_1,\dots,x_N) := \sum_{\underline{a}_1,\underline{b}_1,\dots\underline{a}_N,\underline{b}_N} f_{\underline{a}_1\underline{b}_1}(x_1)\cdots f_{\underline{a}_N\underline{b}_N}(x_N)\big\langle\Phi_{\underline{a}_1\underline{b}_1}\dots\Phi_{\underline{a}_N\underline{b}_N}\big\rangle_c \;. \tag{45}$$

From the point of view of *noncommutative geometry* [Con94] initiated by Alain Connes, this is probably not what one should do. Topology of a noncommutative space is encoded in a noncommutative algebra $\mathcal{A}$, whereas *geometry* needs spectral triples $(\mathcal{A},\mathcal{H},\mathcal{D})$ for which a metric structure is defined on (an appropriate subspace of) *states* on $\mathcal{A}$ via Connes' distance formula [Con94]

$$\mathrm{dist}(\omega_1,\omega_2) = \sup\{|\omega_1(a)-\omega_2(a)| \;:\; \|[\mathcal{D},a]\| \leq 1\} \;. \tag{46}$$

Therefore, a more consistent definition whould be of the form

$$S_c(\omega_N) = \sum_{\underline{a}_1,\underline{b}_1,\dots\underline{a}_N,\underline{b}_N} \omega_N(f_{\underline{a}_1\underline{b}_1}\otimes\cdots\otimes f_{\underline{a}_N\underline{b}_N})\big\langle\Phi_{\underline{a}_1\underline{b}_1}\dots\Phi_{\underline{a}_N\underline{b}_N}\big\rangle_c \;, \tag{47}$$

where ω_N is a state on the N-fold tensor product $\mathcal{A}_\Theta^{\otimes N}$ of the Moyal algebra. Comparison shows that previously we used the pointwise evaluation at x_i, $\omega_{x_1,\dots,x_N}(f_{\underline{a}_1\underline{b}_1} \otimes \cdots \otimes f_{\underline{a}_N\underline{b}_N}) := f_{\underline{a}_1\underline{b}_1}(x_1)\cdots f_{\underline{a}_N\underline{b}_N}(x_N)$. However, pointwise evaluation *is not a state on* $\mathcal{A}_\Theta$ because positivity is violated: any diagonal $f_{\underline{a}\underline{a}}$ is a projection in $\mathcal{A}_\Theta$, in particular positive, but $L_m^0(t)$ (arising via (3) has m zeros and changes signs between them.

The space of states on the Moyal algebra $\mathcal{A}_\Theta$ is very rich, in fact we are not aware of a classification. The use of states permits another way to force translation invariance. As pointed out in [BDKP03], the tensor product of Moyal algebras factorises into $\mathcal{A}_\Theta^{\otimes N} = \mathcal{A}_\Theta \otimes \mathcal{A}_\Theta^{\otimes N-1}$, where the first tensor factor describes the center-of-motion coordinate and the second one depends only on coordinate differences. Every state on the center-of-motion algebra gives rise to translation-invariant Schwinger functions even for finite Moyal deformation parameter θ. The big question is whether states on $\mathcal{A}_\Theta^{\otimes N-1}$ exist which also guarantee reflection positivity.

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