Construction of the Φ^4_4-quantum field theory on noncommutative Moyal space*

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Abstract

We review our recent construction of the ϕ^4-model on four-dimensional Moyal space. A milestone is the exact solution of the quartic matrix model $\mathcal{Z}[E,J] = \int d\Phi \exp(\mathrm{trace}(J\Phi - E\Phi^2 - \frac{\lambda}{4}\Phi^4))$ in terms of the solution of a non-linear equation for the 2-point function and the eigenvalues of E. The β-function vanishes identically. For the Moyal model, the theory of Carleman type singular integral equations reduces the construction to a fixed point problem. Its numerical solution reveals a second-order phase transition at $\lambda_c \approx -0.396$ and a phase transition of infinite order at $\lambda = 0$. The resulting Schwinger functions in position space are symmetric and invariant under the full Euclidean group. They are only sensitive to diagonal matrix correlation functions, and clustering is violated. The Schwinger 2-point function is reflection positive iff the diagonal matrix 2-point function is a Stieltjes function. Numerically this seems to be the case for coupling constants $\lambda \in [\lambda_c, 0]$.

1 Introduction

Perturbatively renormalised quantum field theory is an enormous phenomenological success, a success which lacks a mathematical understanding. The perturbation series is at best an asymptotic expansion which cannot converge at physical coupling constants. Some physical effects such as confinement are out of reach for perturbation theory. In two and partly three dimensions, methods of constructive physics [GJ87, Riv91], often combined with the Euclidean approach [Sch59, OS73, OS75], were used to rigorously establish quantum field theory models.

In four dimensions there was little success so far. It is generally believed that

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due to asymptotic freedom, non-Abelian gauge theory (i.e. Yang-Mills theory) has the chance of a rigorous construction. But this is a hard problem [JW00]. What makes it so difficult is the fact that any simpler model such as quantum electrodynamics or the $\lambda\phi^4$-model cannot be constructed in four dimensions (Landau ghost problem [LAK54a, LAK54b, LAK54c] or triviality [Aiz81, Frö82]).

One of the main difficulties is the *non-linearity* of the models under consideration. Fixed point methods provide a standard approach to non-linear problems, but they are rarely used in quantum field theory. In this contribution we review a sequence of papers [GW12b, GW13b, GW14] in which we successfully used symmetry and fixed point methods to exactly solve a toy model for a quantum field theory in four dimensions.

1. Following [GW12b], we show in sec. 2 that a Ward identity for the $U(\infty)$ group action leads to an exact solution of the quartic matrix model $\mathcal{Z} = \int \mathcal{D}[\Phi]\ \exp(\mathrm{trace}(J\Phi - E\Phi^2 - \frac{\lambda}{4}\Phi^4))$ in terms of the solution of a non-linear equation. As by-product we find that any renormalisable quartic matrix model has vanishing β-function. All these steps are completely elementary.
2. Self-dual ϕ^4_4-theory on Moyal space [GW05b, GW05c] is of that type. For extreme noncommutativity $\theta \to \infty$, and after careful discussion of thermodynamic and continuum limit, the non-linear equation is reduced to a fixed-point problem [GW12b] which has a unique non-perturbative and non-trivial solution for $\lambda < 0$ [GW14]. Sec. 3 reviews this work. The key step is the observation that a certain difference function satisfies a linear singular integral equation of Carleman type [Car22, Tri57]. We also present some numerical results, contained in work in progress [GW14], which show evidence for phase transitions.
3. Following [GW13b], we identify in sec. 4 a limit to Schwinger functions for a scalar field on $\mathbb{R}^4$. Surprisingly for a highly noncommutative model, these Schwinger functions show full Euclidean symmetry. Otherwise they have unusual properties such as absent momentum transfer in interaction processes. This seems to suggest triviality, but the numerical investigation [GW14] of the 2-point function shows scattering remnants from a noncommutative geometrical substructure. Most surprisingly, the Schwinger 2-point function seems to be reflection positive in one of its phases.

2 Exact solution of the quartic matrix model

For us a 'matrix' is a compact (Hilbert-Schmidt) operator on Hilbert space $H = L^2(I, \mu)$. Such operators $\Phi \in \mathcal{L}^2(H)$ can be represented by integral kernel operators $(\Phi v)_a = \int_I d\mu_b\, \Phi_{ab} v_b$. Then all natural matrix operations such as product, adjoint and trace have counterparts $(\Phi\Phi')_{ab} = \int_I d\mu_c\ \Phi_{ac}\Phi'_{cb}$, $(\Phi^*)_{ab} = \overline{\Phi_{ba}}$ and $\mathrm{tr}(\Phi\Phi') = \int_I d\mu_a\ (\Phi\Phi')_{aa}$ in $\mathcal{L}^2(H)$.

To define a *Euclidean quantum field theory* for a matrix $\Phi \in \mathcal{L}^2(H)$ we give ourselves an action functional

$$S[\Phi] = V\,\mathrm{tr}(E\Phi^2 + P[\Phi]) \;. \tag{1}$$

Here, $P[\Phi]$ is a polynomial in Φ with scalar coefficients, and this alone would be a familiar action in the theory of matrix models [DGZ95]. To be closer to field theory on a (compact) manifold $\mathcal{M}$ we add the analogue of the kinetic term $\int_{\mathcal{M}} dx\ (-\Delta\phi)\phi$, that is, we require the external matrix E to be an unbounded selfadjoint positive operator on H with compact resolvent. The volume V will play a crucial rôle. The construction involves several regularisation and limiting procedures. One such regularisation consists in a finite size $\mathcal{N}$ for the matrices, and V will be a certain function of $\mathcal{N}$ which together with $\mathcal{N}$ is sent to ∞.

Adding a source term to the action, we define the *partition function* as

$$\mathcal{Z}[J] = \int \mathcal{D}[\Phi]\ \exp(-S[\Phi] + V\,\mathrm{tr}(\Phi J)) \;, \tag{2}$$

where $\mathcal{D}[\Phi]$ is the extension of the Lebesgue measure from finite-rank operators to $\mathcal{L}^2(H)$ and J a test function matrix. For absent $P[\Phi] \mapsto 0$ in (1), $\mathcal{D}[\Phi]\exp(-V\,\mathrm{tr}(E\Phi^2))/\mathcal{Z}[0]$ would be the Gaußian measure of covariance determined by E. What we want, and what we achieve, is to construct (the moments of) the measure $\mathcal{D}[\Phi]\exp(-V\,\mathrm{tr}(E\Phi^2 + \frac{\lambda}{4}\Phi^4))/\mathcal{Z}[0]$ in the limit $V \to \infty$. Such a limit cannot be expected for $\mathcal{Z}$. Instead, we pass to the generating functional $\log\mathcal{Z}[J]$ of *connected correlation functions*,

$$\langle \varphi_{a_1b_1}\dots\varphi_{a_Nb_N}\rangle_c = \frac{\partial^N \log\mathcal{Z}[J]}{\partial J_{b_1a_1}\dots\partial J_{b_Na_N}}\Big|_{J=0} \;. \tag{3}$$

2.1 Ward identity and topological expansion

Unitary operators U belonging to an appropriate unitisation of the compact operators on H give rise to a transformation $\Phi \mapsto \tilde{\Phi} = U\Phi U^*$. Since the space of selfadjoint compact operators is invariant under the adjoint action, we have

$$\int \mathcal{D}[\Phi]\ \exp(-S[\Phi] + V\,\mathrm{tr}(\Phi J)) = \int \mathcal{D}[\tilde{\Phi}]\ \exp(-S[\tilde{\Phi}] + V\,\mathrm{tr}(\tilde{\Phi} J)) \;.$$

Unitary invariance $\mathcal{D}[\tilde{\Phi}] = \mathcal{D}[\Phi]$ of the Lebesgue measure implies

$$0 = \int \mathcal{D}[\Phi \Big\{ \exp(-S[\Phi] + V\,\mathrm{tr}(\Phi J)) - \exp(-S[\tilde{\Phi}] + V\,\mathrm{tr}(\tilde{\Phi} J))\Big\} \;.$$

Note that the integrand $\{\dots\}$ itself does not vanish because $\mathrm{tr}(E\Phi^2)$ and $\mathrm{tr}(\Phi J)$ are not unitarily invariant; we only have $\mathrm{tr}(P[\Phi])$=$\mathrm{tr}(P[\tilde{\Phi}])$ due to UU^*=U^*U=id together with the trace property. Linearisation of U about the identity operator leads to the *Ward identity*

$$0 = \int \mathcal{D}[\Phi] \left\{ E\Phi\Phi - \Phi\Phi E - J\Phi + \Phi J \right\} \exp(-S[\Phi] + V\,\mathrm{tr}(\Phi J)) \,. \tag{4}$$

We can always place ourselves in an orthonormal basis of H where E is diagonal (but J is not). Since E is of compact resolvent, E has eigenvalues $E_a > 0$ of finite multiplicity μ_a. We thus label the matrices by an enumeration of the (necessarily discrete) eigenvalues of E and an enumeration of the basis vectors of the finite-dimensional eigenspaces. Writing Φ in $\{\dots\}$ of (4) as functional derivative $\Phi_{ab} = \frac{\partial}{V\partial J_{ba}}$, we have proved (first obtained in [DGMR07]):

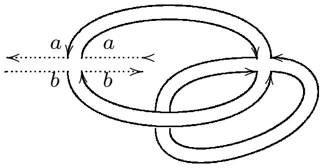
Proposition 1 *The partition function $\mathcal{Z}[J]$ of the matrix model defined by the external matrix E satisfies the $|I| \times |I|$ Ward identities*

$$0 = \sum_{n\in I} \Big(\frac{(E_a - E_p)}{V} \frac{\partial^2 \mathcal{Z}}{\partial J_{an} \partial J_{np}} + J_{pn} \frac{\partial \mathcal{Z}}{\partial J_{an}} - J_{na} \frac{\partial \mathcal{Z}}{\partial J_{np}} \Big) \,. \tag{5}$$

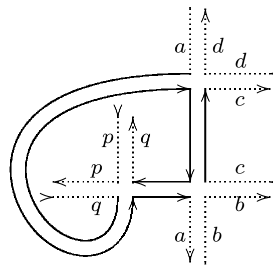
Without loss of generality we can assume that the map $I \ni m \mapsto E_m \in \mathbb{R}_+$ is injective. Namely, correlation functions will only depend on the set of eigenvalues (E_m) of E. Partitioning the index set I into equivalence classes $[m]$ which have the same E_m, the index sum over a function that only depends on E_m becomes $\sum_{m\in I} f(m) = \sum_{[m]\in[I]} \mu_{[m]} f([m])$. Therefore, at the expense of adding a measure $\mu_{[m]} = \dim\ker(E - E_m \mathrm{id})$, we can assume that $m \mapsto E_m$ is injective.

In a perturbative expansion, Feynman graphs in matrix models are *ribbon graphs*. Viewed as simplicial complexes, they encode the topology (B, g) of a genus-g Riemann surface with B boundary components (or punctures, marked points, holes, broken/external faces). Some simple examples for $P[\Phi] = \Phi^4$ are:

$B = 1$
$g = 1$



$B = 2$
$g = 0$



Since E is diagonal, the matrix index is conserved along each strand of the ribbon graph. We have to distinguish between internal faces (with constant matrix index) and broken faces which constitute the boundary components. Such a boundary face is characterised by $N_\beta \geq 1$ external double lines to which we attach the source matrices J. Conservation of the matrix index along each strand implies that the right index of J_{ab} coincides with the left index of another J_{bc}, or of the same J_{bb}. Accordingly, the β^{th} boundary component carries a cycle $J^{N_\beta}_{p_1 \dots p_{N_\beta}} := \prod_{j=1}^{N_\beta} J_{p_j p_{j+1}}$ of N_β external source matrices, with $N_\beta + 1 \equiv 1$.

Being interested in a non-perturbative solution, we will not expand the partition function into ribbon graphs. But we keep the topological information and expand $\log \mathcal{Z}[J]$ according to the cycle structure:

$$\log \frac{\mathcal{Z}[J]}{\mathcal{Z}[0]} = \sum_{B=1}^{\infty} \sum_{1\leq N_1\leq \cdots \leq N_B}^{\infty} \sum_{p_1^\beta,\dots,p_{N_\beta}^\beta \in I} \frac{V^{2-B}}{S_{N_1\dots N_B}} G_{|p_1^1\dots p_{N_1}^1|\dots|p_1^B\dots p_{N_B}^B|} \prod_{\beta=1}^{B} \Big(\frac{J^{N_\beta}_{p_1^\beta\dots p_{N_\beta}^\beta}}{N_\beta} \Big) . \tag{6}$$

The symmetry factor $S_{N_1\dots N_B}$ is obtained as follows: If ν_i of the B numbers N_β in a given tuple $(N_1,\dots,N_B)$ are equal to i, then $S_{N_1\dots N_B} = \prod_{i=1}^{N_B} \nu_i!$.

Next we turn the Ward identity (5) for injective $m \mapsto E_m$ into a formula for the second derivative $\sum_{n\in I} \frac{\partial^2 \mathcal{Z}[J]}{\partial J_{an}\partial J_{np}}$ of the partition function. The J-cycle structure in $\log \mathcal{Z}$ creates

- singular contributions $\sim \delta_{ap}$,
- regular contributions present for all a,p:

Theorem 2

$$\begin{aligned}
\sum_{n\in I} \frac{\partial^2 \mathcal{Z}[J]}{\partial J_{an}\partial J_{np}} = \delta_{ap}\Big\{ & V^2 \sum_{(K)} \frac{J_{P_1}\cdots J_{P_K}}{S_{(K)}} \Big(\sum_{n\in I} \frac{G_{|an|P_1|\dots|P_K|}}{V^{|K|+1}} + \frac{G_{|a|a|P_1|\dots|P_K|}}{V^{|K|+2}} \\
& + \sum_{r\geq 1} \sum_{q_1,\dots,q_r\in I} \frac{G_{|q_1 a q_1\dots q_r|P_1|\dots|P_K|} J^r_{q_1\dots q_r}}{V^{|K|+1}} \Big) \\
& + V^4 \sum_{(K),(K')} \frac{J_{P_1}\cdots J_{P_K} J_{Q_1}\cdots J_{Q_{K'}}}{S_{(K)}S_{(K')}} \frac{G_{|a|P_1|\dots|P_K|}}{V^{|K|+1}} \frac{G_{|a|Q_1|\dots|Q_{K'}|}}{V^{|K'|+1}} \Big\} \mathcal{Z}[J] \\
& + \frac{V}{E_p - E_a} \sum_{n\in I} \Big(J_{pn} \frac{\partial \mathcal{Z}[J]}{\partial J_{an}} - J_{na} \frac{\partial \mathcal{Z}[J]}{\partial J_{np}} \Big) .
\end{aligned} \tag{7}$$

Proof. We identify the following four sources of a singular contribution $\sim \delta_{ap}$:

1. $\displaystyle \sum_n \frac{\partial^2}{\partial J_{an}\partial J_{np}} \sum_{q_1,q_2,\dots} G_{\dots|q_1q_2|\dots} \Big(\frac{\overset{\downarrow}{J}_{q_1q_2} \overset{\downarrow}{J}_{q_2q_1}}{2} \Big) \prod J$

2. $\displaystyle \sum_n \frac{\partial^2}{\partial J_{an}\partial J_{np}} \sum_{q_1,q_2,\dots} G_{\dots|q_1|\dots|q_2|\dots} \Big(\frac{\overset{\downarrow}{J}_{q_1q_1}}{1} \Big) \Big(\frac{\overset{\downarrow}{J}_{q_2q_2}}{1} \Big) \prod J$

3. $\displaystyle \sum_n \frac{\partial}{\partial J_{an}} \frac{\partial}{\partial J_{np}} \sum_{q_0,\dots,q_{r+1},\dots} G_{\dots|q_0q_1\dots q_rq_{r+1}|\dots} \Big(\frac{J_{q_0q_1}J_{q_1q_2}\cdots J_{q_rq_{r+1}} \overset{\downarrow}{J}_{q_{r+1}q_0}}{r+2} \Big) \prod J$
$\displaystyle = \sum_n \frac{\partial}{\partial J_{an}} \sum_{q_1,\dots,q_r,\dots} G_{\dots|pq_1\dots q_rn|\dots} \Big(\overset{\downarrow}{J}_{pq_1} J_{q_1q_2} \cdots J_{q_rn} \Big) \prod J$

4. $$\sum_n \frac{\partial^2}{\partial J_{an}\partial J_{np}}\Big[\sum_{q_1,\dots} G_{\dots|q_1|\dots}\Big(\frac{\overset{\downarrow}{J_{q_1q_1}}}{1}\Big)\prod J\Big]\Big[\sum_{q_2,\dots} G_{\dots|q_2|\dots}\Big(\frac{\overset{\downarrow}{J_{q_2q_2}}}{1}\Big)\prod J\Big]$$

All other types of derivatives, collected into $\big(\sum_{n\in I}\frac{\partial^2\mathcal{Z}[J]}{\partial J_{an}\partial J_{np}}\big)_{\rm reg}$, persist for $a\neq p$. For $p\neq a$ we clearly have

$$\Big(\sum_{n\in I}\frac{\partial^2\mathcal{Z}[J]}{\partial J_{an}\partial J_{np}}\Big)_{\rm reg}=\sum_{n\in I}\frac{\partial^2\mathcal{Z}[J]}{\partial J_{an}\partial J_{np}}\Big|_{a\neq p}=\frac{V}{E_p-E_a}\Big(J_{pn}\frac{\partial\mathcal{Z}}{\partial J_{an}}-J_{na}\frac{\partial\mathcal{Z}}{\partial J_{np}}\Big)\;, \quad (8)$$

where the last equality is the Ward identity (5), divided by $\frac{E_p-E_a}{V}\neq 0$. By a continuity argument, the rightmost term in (8) must agree with $\big(\sum_{n\in I}\frac{\partial^2\mathcal{Z}[J]}{\partial J_{an}\partial J_{np}}\big)_{\rm reg}$ also in the limit $p\to a$, and this finishes the proof. □

2.2 Schwinger-Dyson equations

We can write the action as $S=\frac{V}{2}\sum_{a,b}(E_a+E_b)\Phi_{ab}\Phi_{ba}+VS_{int}[\Phi]$, where E_a are the eigenvalues of E. Functional integration yields, up to an irrelevant constant,

$$\mathcal{Z}[J]=e^{-VS_{int}[\frac{\partial}{V\partial J}]}e^{\frac{V}{2}\langle J,J\rangle_E}\;,\qquad \langle J,J\rangle_E:=\sum_{m,n\in I}\frac{J_{mn}J_{nm}}{E_m+E_n}\;. \quad (9)$$

Instead of a perturbative expansion of $e^{-VS_{int}[\frac{\partial}{V\partial J}]}$, we apply those J-derivatives to (9) which give rise to a correlation function $G_{\dots}$ on the lhs. On the rhs of (9), these external derivatives combine with internal derivatives from $S_{int}[\frac{\partial}{V\partial J}]$ to certain identities for $G_{\dots}$. These Schwinger-Dyson equations are often of little use because they express an N-point function in terms of $(N{+}2)$-point functions.

In the field-theoretical matrix models under consideration, the Ward identity (7) lets this tower of Schwinger-Dyson equations collapse. To see this we consider the 2-point function $G_{|ab|}$ for $a\neq b$. According to (6), $G_{|ab|}$ is obtained by deriving (9) with respect to J_{ba} and J_{ab}:

$$\begin{aligned}
G_{|ab|}&=\frac{1}{V\mathcal{Z}[0]}\frac{\partial^2\mathcal{Z}[J]}{\partial J_{ba}\partial J_{ab}}\Big|_{J=0} \qquad \text{(disconnected part of } \mathcal{Z} \text{ does not contribute for } a\neq b)\\
&=\frac{1}{V\mathcal{Z}[0]}\Big\{\frac{\partial}{\partial J_{ba}}e^{-VS_{int}\big[\frac{\partial}{V\partial J}\big]}\frac{\partial}{\partial J_{ab}}e^{\frac{V}{2}\langle J,J\rangle_E}\Big\}_{J=0}\\
&=\frac{1}{(E_a+E_b)\mathcal{Z}[0]}\Big\{\frac{\partial}{\partial J_{ba}}e^{-VS_{int}\big[\frac{\partial}{V\partial J}\big]}J_{ba}e^{\frac{V}{2}\langle J,J\rangle_E}\Big\}_{J=0}\\
&=\frac{1}{E_a+E_b}+\frac{1}{(E_a+E_b)\mathcal{Z}[0]}\Big\{\Big(\Phi_{ab}\frac{\partial(-VS_{int})}{\partial\Phi_{ab}}\Big)\Big[\frac{\partial}{V\partial J}\Big]\Big\}\mathcal{Z}[J]\Big|_{J=0}\;.
\end{aligned}\quad (10)$$

Now observe that $\frac{\partial(-VS_{int})}{\partial\Phi_{ab}}$ contains, for any $P[\Phi]$, the derivative $\sum_n\frac{\partial^2}{\partial J_{an}\partial J_{np}}$ which we know from (7). In case of the quartic matrix model $P[\Phi]=\frac{\lambda}{4}\Phi^4$ we have $\frac{\partial(-VS_{int})}{\partial\Phi_{ab}} = -\lambda V \sum_{n,p\in I} \Phi_{bp}\Phi_{pn}\Phi_{na}$, hence

$$\Big(\Phi_{ab}\frac{\partial(-VS_{int})}{\partial\Phi_{ab}}\Big)\Big[\frac{\partial}{V\partial J}\Big] = -\frac{\lambda}{V^3}\sum_{p,n\in I}\frac{\partial^2}{\partial J_{pb}\partial J_{ba}}\frac{\partial^2}{\partial J_{an}\partial J_{np}}\;,$$

and the Schwinger-Dyson equation (10) for $G_{|ab|}$ becomes with (7)

$$\begin{aligned}
G_{|ab|} &= \frac{1}{E_a+E_b} - \frac{\lambda}{V^3(E_a+E_b)\mathcal{Z}[0]}\sum_{p\in I}\frac{\partial^2}{\partial J_{pb}\partial J_{ba}}\sum_{n\in I}\frac{\partial^2\mathcal{Z}}{\partial J_{an}\partial J_{np}}\Big|_{J=0}\\
&= \frac{1}{E_a+E_b} - \frac{\lambda}{V(E_a+E_b)\mathcal{Z}[0]}\frac{\partial^2}{\partial J_{ab}\partial J_{ba}}\Big\{\\
&\quad\Big(\sum_{n\in I}\frac{G_{|an|}}{V} + \sum_{n,q,r\in I}\frac{G_{|an|qr|}}{V^2}\frac{J_{qr}J_{rq}}{2} + \sum_{n,q,r\in I}\frac{G_{|an|q|r|}}{V^3}\frac{J_{qq}}{1}\frac{J_{rr}}{1}\\
&\quad + \frac{G_{|a|a|}}{V^2} + \sum_{q,r\in I}\frac{G_{|a|a|qr|}}{V^3}\frac{J_{qr}J_{rq}}{2} + \sum_{q,r\in I}\frac{G_{|a|a|q|r|}}{V^4}\frac{J_{qq}}{1}\frac{J_{rr}}{1}\\
&\quad + \sum_{q,r\in I}\frac{G_{|qaqr|}}{V}J_{qr}J_{rq} + V^2\frac{G_{|a|q|}}{V^2}\frac{J_{qq}}{1}\frac{G_{|a|r|}}{V^2}\frac{J_{rr}}{1}\Big)\mathcal{Z}[J]\Big\}\Big|_{J=0}\\
&\quad - \frac{\lambda}{V^2(E_a+E_b)\mathcal{Z}[0]}\sum_{p\in I}\frac{\Big(\frac{\partial^2\mathcal{Z}[J]}{\partial J_{ab}\partial J_{ba}} + \delta_{pb}\frac{\partial^2\mathcal{Z}[J]}{\partial J_{aa}\partial J_{bb}} - \frac{\partial^2\mathcal{Z}[J]}{\partial J_{pb}\partial J_{bp}}\Big)}{E_p-E_a}\Bigg|_{J=0}\;. \qquad (11)
\end{aligned}$$

Taking $\frac{\partial^2\mathcal{Z}[J]}{\partial J_{pb}\partial J_{bp}} = (VG_{|pb|} + \delta_{pb}G_{|p|b|})\mathcal{Z}[0] + \mathcal{O}(J)$ and $\frac{\partial J_{rr}}{\partial J_{ab}} = 0$ for $a \neq b$ into account, we have proved:

Proposition 3 *The 2-point function of a quartic matrix model with action $S = V\,\mathrm{tr}(E\Phi^2 + \frac{\lambda}{4}\Phi^4)$ satisfies for injective $m \mapsto E_m$ the Schwinger-Dyson equation*

$$G_{|ab|} = \frac{1}{E_a+E_b} - \frac{\lambda}{E_a+E_b}\frac{1}{V}\sum_{p\in I}\Big(G_{|ab|}G_{|ap|} - \frac{G_{|pb|}-G_{|ab|}}{E_p-E_a}\Big) \qquad \Big\} \qquad (12a)$$

$$\left.\begin{aligned}&-\frac{\lambda}{V^2(E_a+E_b)}\Big(G_{|a|a|}G_{|ab|} + \frac{1}{V}\sum_{n\in I}G_{|an|ab|}\\ &\qquad\qquad + G_{|aaab|} + G_{|baba|} - \frac{G_{|b|b|}-G_{|a|b|}}{E_b-E_a}\Big)\end{aligned}\right\} \qquad (12b)$$

$$-\frac{\lambda}{V^4(E_a+E_b)}G_{|a|a|ab|}\;. \qquad \Big\} \qquad (12c)$$

It can be checked [GW12b] that in a genus expansion $G_{\dots} = \sum_{g=0}^{\infty} V^{-2g}G^{(g)}_{\dots}$ (which is probably not convergent but Borel summable), precisely the line (12a) preserves the genus, the lines (12b) increase $g \mapsto g+1$ and the line (12c) increases $g \mapsto g+2$. In particular, in a scaling limit $V \to \infty$ with $\frac{1}{V}\sum_{p\in I}$ finite, the exact

Schwinger-Dyson equation for $G_{|ab|}$ coincides with its restriction (12a) to the planar sector $g=0$, a closed non-linear equation for $G^{(0)}_{|ab|}$ alone:

$$G^{(0)}_{|ab|} = \frac{1}{E_a+E_b} - \frac{\lambda}{E_a+E_b}\frac{1}{V}\sum_{p\in I}\Big(G^{(0)}_{|ab|}G^{(0)}_{|ap|} - \frac{G^{(0)}_{|pb|}-G^{(0)}_{|ab|}}{E_p-E_a}\Big)\,. \tag{13}$$

We have derived in 2007/08 this self-consistency equation for the Moyal model by the graphical method proposed by [DGMR07]. In this form, (13) is meaningless because $\sum_{p\in I}$ diverges. In 2009 we solved the renormalisation problem, namely the renormalisation of infinitely many Feynman graphs at once [GW09]. This renormalisation increases the non-linearity. In [GW09] we have solved (13) perturbatively to $\mathcal{O}(\lambda^3)$. After several years of setbacks with the non-perturbative solution, a breakthrough came in 2012: The equation (13) can be turned into an equation which is linear in the difference $G^{(0)}_{|ab|}-G^{(0)}_{|a0|}$ to the boundary and non-linear only in $G^{(0)}_{|a0|}$!

A similar calculation gives the Schwinger-Dyson equation for higher N-point functions:

$$\begin{aligned}
&G_{|ab_1\dots b_{N-1}|}\\
&= -\frac{\lambda}{E_a+E_{b_1}}\Big(\frac{1}{V}\sum_{p\in I}\Big(G_{|ap|}G_{|ab_1\dots b_{N-1}|} - \frac{G_{|pb_1\dots b_{N-1}|}-G_{|ab_1\dots b_{N-1}|}}{E_p-E_a}\Big)\\
&\qquad - \sum_{l=1}^{\frac{N-2}{2}} G_{|b_1\dots b_{2l}|}\frac{G_{|b_{2l+1}\dots b_{N-1}a|}-G_{|b_{2l+1}\dots b_{N-1}b_{2l}|}}{E_{b_{2l}}-E_a}\Big) \qquad (14a)\\
&-\frac{\lambda}{V^2(E_a+E_{b_1})}\Big(G_{|a|a|}G_{|ab_1\dots b_{N-1}|} + \sum_{k=1}^{N-1} G_{|b_1\dots b_k a b_k\dots b_{N-1}a|}\\
&\qquad + G_{|aaab_1\dots b_{N-1}|} + \frac{1}{V}\sum_{n\in I} G_{|an|ab_1\dots b_{N-1}|}\\
&\qquad - \sum_{k=1}^{N-1}\frac{G_{|b_1\dots b_k|b_{k+1}\dots b_{N-1}b_k|}-G_{|b_1\dots b_k|b_{k+1}\dots b_{N-1}a|}}{E_{b_k}-E_a}\Big) \qquad (14b)\\
&-\frac{\lambda}{V^4(E_a+E_{b_1})}G_{|a|a|ab_1\dots b_{N-1}|}\,. \qquad (14c)
\end{aligned}$$

Again, the first lines (14a) preserve the genus, whereas $g\mapsto g+1$ in (14b) and $g\mapsto g+2$ in (14c). The planar sector $G^{(0)}_{|ab_1\dots b_{N-1}|}$, exact for $V\to\infty$ with $\frac{1}{V}\sum_{p\in I}$ finite, is a linear inhomogeneous equation with inductively known parameters.

It turns out that a real theory with $\Phi=\Phi^*$ admits a short-cut which directly gives the higher N-point functions without any index summation. Since the equations for $G_{\dots}$ are real and $\overline{J_{ab}}=J_{ba}$, the reality $\mathcal{Z}=\overline{\mathcal{Z}}$ implies (in addition to invariance under cyclic permutations) invariance under orientation reversal

$$G_{|p_0^1p_1^1\dots p^1_{N_1-1}|\dots|p_0^Bp_1^B\dots p^B_{N_B-1}|} = G_{|p_0^1p^1_{N_1-1}\dots p_1^1|\dots|p_0^Bp^B_{N_B-1}\dots p_1^B|}\,. \tag{15}$$

Whereas empty for $G_{|ab|}$, in $(E_a+E_{b_1})G_{ab_1b_2\dots b_{N-1}} - (E_a+E_{b_{N-1}})G_{ab_{N-1}\dots b_2b_1}$ the identities (15) lead to many cancellations which result in a universal algebraic recursion formula:

Proposition 4

$$
\begin{aligned}
G_{|b_0b_1\dots b_{N-1}|} &= (-\lambda)\sum_{l=1}^{\frac{N-2}{2}} \frac{G_{|b_0b_1\dots b_{2l-1}|}G_{|b_{2l}b_{2l+1}\dots b_{N-1}|} - G_{|b_{2l}b_1\dots b_{2l-1}|}G_{|b_0b_{2l+1}\dots b_{N-1}|}}{(E_{b_0}-E_{b_{2l}})(E_{b_1}-E_{b_{N-1}})} \\
&\quad + \frac{(-\lambda)}{V^2}\sum_{k=1}^{N-1} \frac{G_{|b_0b_1\dots b_{k-1}|b_kb_{k+1}\dots b_{N-1}|} - G_{|b_kb_1\dots b_{k-1}|b_0b_{k+1}\dots b_{N-1}|}}{(E_{b_0}-E_{b_k})(E_{b_1}-E_{b_{N-1}})}\;. \qquad (16)
\end{aligned}
$$

The last line of (16) increases the genus and is absent in $G^{(0)}_{|b_0b_1\dots b_{N-1}|}$. Instead of giving the general proof, let us look at the case $N=4$. Then (14), multiplied by $E_a - E_{b_1}$, reads

$$
\begin{aligned}
&(E_a-E_b)G_{|abcd|} \\
&= (-\lambda)\Big(\frac{1}{V}\sum_{p\in I}\Big(G_{|ap|}G_{|abcd|} - \frac{G_{|pbcd|}-G_{|abcd|}}{E_p-E_a}\Big) - G_{|bc|}\frac{G_{|da|}-G_{|dc|}}{E_c-E_a}\Big) \\
&- \frac{\lambda}{V^2}\Big(G_{|a|a|}G_{|abcd|} + G_{|babcda|} + G_{|bcacda|} + G_{|bcdada|} + G_{|aaabcd|} + \frac{1}{V}\sum_{p\in I} G_{|ap|abcd|} \\
&\qquad\qquad - \frac{G_{|b|cdb|}-G_{|b|cda|}}{E_b-E_a} - \frac{G_{|bc|dc|}-G_{|bc|da|}}{E_c-E_a} - \frac{G_{|d|bcd|}-G_{|a|bcd|}}{E_d-E_a}\Big) \\
&- \frac{\lambda}{V^4}G_{|a|a|abcd|}\;. \qquad (17)
\end{aligned}
$$

Write down the same equation but with $b\leftrightarrow d$, and take the difference between these equations. Then most terms cancel because by (15) we have the equalities $G_{|abcd|} = G_{|adcb|}$, $G_{|pbcd|} = G_{|pdcb|}$, $G_{|babcda|} = G_{|dcbaba|}$, $G_{|bcacda|} = G_{|dcacba|}$, $G_{|bcdada|} = G_{|dadcba|}$, $G_{|aaabcd|} = G_{|aaadcb|}$, $G_{|ap|abcd|} = G_{|ap|adcb|}$, $G_{|b|cdb|} = G_{|dcb|b|}$, $G_{|bc|dc|} = G_{|dc|bc|}$, $G_{|bcd|d|} = G_{|d|cbd|}$ and $G_{|a|a|abcd|} = G_{|a|a|adcb|}$. Altogether, the difference $(17)-(17)_{b\leftrightarrow d}$ reads after cancellation

$$
\begin{aligned}
(E_d-E_b)G_{|abcd|} &= (-\lambda)\frac{G_{|ab|}G_{|cd|}-G_{|ad|}G_{|cb|}}{E_c-E_a} \\
&\quad - \frac{\lambda}{V^2}\Big(\frac{G_{|b|cda|}-G_{|a|cdb|}}{E_b-E_a} + \frac{G_{|bc|da|}-G_{|ba|dc|}}{E_c-E_a} + \frac{G_{|a|bcd|}-G_{|d|bca|}}{E_d-E_a}\Big)\;,
\end{aligned}
$$

and this is (16) for $N=4$.

For completeness, we list in the appendix the Schwinger-Dyson equation for $B=2$ boundary components.

We make the following key observation: An affine transformation $E\mapsto ZE+C$ together with a corresponding rescaling $\lambda\mapsto Z^2\lambda$ leaves the algebraic equations (16) as well as (65) and (66) invariant:

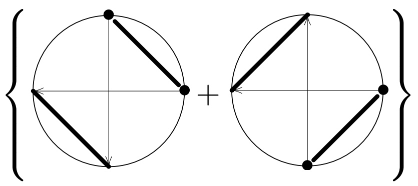
Theorem 5 *Given a real quartic matrix model with $S = V\,\mathrm{tr}(E\Phi^2 + \frac{\lambda}{4}\Phi^4)$ and $m \mapsto E_m$ injective, which determines the set $G_{|p^1_1\dots p^1_{N_1}|\dots|p^B_1\dots p^B_{N_B}|}$ of $(N_1+\ldots+N_B)$-point functions. Assume that the basic functions with all $N_i \leq 2$ are turned finite by $E_a \mapsto Z(E_a + \frac{\mu^2}{2} - \frac{\mu^2_{bare}}{2})$ and $\lambda \mapsto Z^2\lambda$. Then all functions with one $N_i \geq 3$*

1. *are finite without further need of a renormalisation of λ, i.e. all renormalisable quartic matrix models have vanishing β-function,*
2. *are given by universal algebraic recursion formulae in terms of renormalised basic functions with $N_i \leq 2$.* □

The theorem tells us that vanishing of the β-function for the self-dual Φ^4_4-model on Moyal space (proved in [DGMR07] to all orders in perturbation theory) is generic to all quartic matrix models, and the result even holds non-perturbatively!

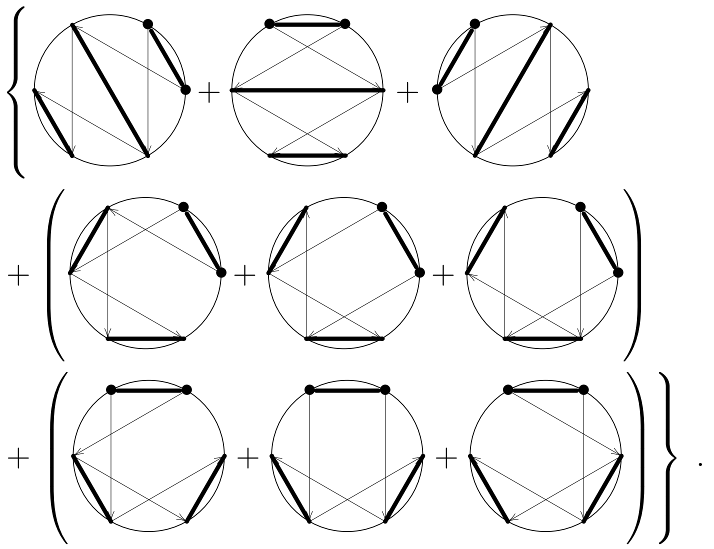
The universal recursion formula (16) computes the planar N-point function $G_{|b_0\dots b_{N-1}|}$ at $B = 1$ as a sum of fractions with products of 2-point functions in the numerator and products of differences of eigenvalues of E in the denominator. This structure admits an interesting graphical interpretation. We draw the indices $b_0, \dots b_{N-1}$ in cyclic order on the circle S^1 and represent a factor $G_{b_ib_j}$ as a chord connecting b_i with b_j and a factor $\frac{1}{E_{b_i}-E_{b_j}}$ as an arrow from b_i to b_j:

$$G^{(0)}_{|b_0b_1b_2b_3|} = (-\lambda)\frac{G_{b_0b_1}G_{b_2b_3}-G_{b_0b_3}G_{b_2b_1}}{(E_{b_0}-E_{b_2})(E_{b_1}-E_{b_3})} = (-\lambda)$$



,

$$G_{b_0\dots b_5} = (-\lambda)^2$$



. (18)

The chords form the non-crossing chord diagrams counted by the Catalan number $C_{\frac{N}{2}} = \frac{N!}{(\frac{N}{2}+1)!\frac{N}{2}!}$. The arrows form two disjoint trees, one connecting the even vertices and one connecting the odd vertices. By rational fraction expansion it is possible to achieve that each tree intersects the chord only in the vertices.

The assignment of trees to a given chord diagram is, in general, not unique. A canonical choice is not known to us.

2.3 Digression: Quantum gravity in two dimensions

Two-dimensional quantum gravity (see [DGZ95, ADJ97] for reviews) can be interpreted as the enumeration of random triangulations of surfaces. Its asymptotic behaviour is captured by the matrix model partition function

$$\mathcal{Z} = \int \mathcal{D}[\Phi]\ \exp\Big(-\mathcal{N} \sum_n t_n\ \mathrm{tr}(\Phi^n)\Big)\ , \tag{19}$$

where the integral is over $(\mathcal{N}\times\mathcal{N})$-Hermitean matrices Φ and the t_n are scalar coefficients. In the limit $\mathcal{N}\to\infty$, this series in (t_n) is evaluated in terms of the τ-function for the Korteweg-de Vries (KdV) hierarchy. There is another approach to topological gravity in which the partition function is a series in (t_n) with coefficients given by intersection numbers of complex curves. Witten conjectured [Wit91] that the partition functions of the two approaches coincide. This conjecture was proved by Kontsevich [Kon82] who achieved the computation of the intersection numbers in terms of weighted sums over ribbon graphs (fat Feynman graphs), which he proved to be generated from the Airy function matrix model (Kontsevich model)

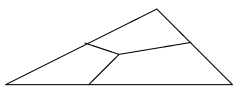
$$\mathcal{Z}[E] = \frac{\displaystyle\int \mathcal{D}[\Phi]\ \exp\big(-\tfrac{1}{2}\mathrm{tr}(E\Phi^2) + \tfrac{\mathrm{i}}{6}\mathrm{tr}(\Phi^3)\big)}{\displaystyle\int \mathcal{D}[\Phi]\ \exp\big(-\tfrac{1}{2}\mathrm{tr}(E\Phi^2)\big)}\ . \tag{20}$$

The external matrix $E = E^* > 0$ is related by $t_n = (2n{-}1)!!\ \mathrm{tr}(E^{-(2n-1)})$ to the series (t_n). The limit $\mathcal{N}\to\infty$ of $\mathcal{Z}[E]$ gives the KdV evolution equation, thus proving Witten's conjecture.

We have proved that also the *quartic matrix model*

$$\mathcal{Z}[E,J,\lambda] = \frac{\displaystyle\int \mathcal{D}[\Phi]\ \exp\big(-\mathrm{tr}(E\Phi^2) + \mathrm{tr}(J\Phi) - \tfrac{\lambda}{4}\mathrm{tr}(\Phi^4)\big)}{\displaystyle\int \mathcal{D}[\Phi]\ \exp\big(-\mathrm{tr}(E\Phi^2) - \tfrac{\lambda}{4}\mathrm{tr}(\Phi^4)\big)} \tag{21}$$

is in the large-$\mathcal{N}$ limit exactly solvable in terms of the solution of a non-linear equation (13). Any triangulation can be subdivided into a quadrangulation



(and vice versa). From Witten's uniqueness argument [Wit91], 2D quantum gravity should have equivalent descriptions as cubic (20) and quartic (21) matrix model. Understanding the precise relation between (20) and (21) would be of high interest:

1. In contrast to (21), the cubic action (20) lacks manifest positivity due to its purely imaginary coupling constant.
2. A quartic action admits a Hubbard-Stratonovich transform which is the key ingredient of a new approach to constructive quantum field theory [Riv07b] that avoids the cluster expansion.
3. Conversely, the integrability of (20) might provide valuable information about the solution of the self-consistency equation (13).

Coloured tensor models (see [GP12, Riv13] for recent reviews) extend these methods to quantum gravity in $D \geq 3$. They became a very active domain of research after understanding [Gur10] of the analogue of the large-$\mathcal{N}$ behaviour of matrix models [tHo74]. They have Schwinger-Dyson equations (see e.g. [Bon12]) and action of the $U(\infty)$ group. A first promising result is the recent derivation of closed equations for the 2-point functions of rank 3 and 4 tensorial group field theory [Sam14].

3 Φ^4_4-theory on Moyal space as a fixed point problem

3.1 Preliminaries

Taking the renormalisation group [WK74] serious, we would expect that General Relativity, because not renormalisable, is irrelevant and hence scaled away. The existence of gravity thus tells us that the scaling must stop at some length scale, and from the weakness of the gravitational coupling constant one deduces the value of that scale: the Planck length 10^{-35} m. There, the geometry of nature is expected to differ from the familiar structure of a differentiable manifold. One of many candidates for Planck scale physics is noncommutative geometry [Con94], a vast reformulation of geometry and topology in the language of operator algebras. The focus is shifted from manifolds to generalisations of the algebra of functions. This concept proved very successful in understanding the geometry of the Standard Model of particle physics as Riemannian geometry of a space which is the product of a manifold with a discrete space [Con96, CC96].

A large class of examples of noncommutative geometries comes from deformations of the algebra of functions on manifolds. Schwartz functions on Euclidean space $\mathbb{R}^4$ admit an $\mathbb{R}^4$-group action by translation. As shown by Rieffel [Rie93], this group action induces a noncommutative associative product on the space of Schwartz functions, the Moyal product:

$$(f \star g)(x) = \int_{\mathbb{R}^4 \times \mathbb{R}^4} \frac{dy\, dk}{(2\pi)^4} f(x+\tfrac{1}{2}\Theta k)\; g(x+y)\; e^{\mathrm{i}\langle k,y\rangle}\;, \quad \Theta = -\Theta^t \in M_4(\mathbb{R})\;. \quad (22)$$

Whether or not the Moyal space $(\mathbb{R}^4, \star)$ is relevant for Planck scale physics is pure speculation (although a refinement can be justified by uncertainty relations for position operators [DFR95]). In any case the Moyal space is a nice toy model on which it is easy to formulate and to study (quantum) field theories. To formulate a Euclidean quantum field theory on Moyal space it is, at first sight, enough to replace in the action of a usual field theory the pointwise product of functions by the $\star$-product. The simplest example is the $\phi_4^{\star 4}$-model with action

$$S[\phi] = \int_{\mathbb{R}^4} dx \,\Big(\frac{1}{2}\phi \star (-\Delta + \mu^2)\phi + \frac{\lambda}{4}\phi \star \phi \star \phi \star \phi\Big)(x) \,. \tag{23}$$

The resulting Feynman rules [Fil96] lead to situations where a multiple insertion of non-planar subgraphs gives rise to divergences of arbitrarily high degree (ultraviolet/infrared mixing [MVS00]). See [CR00] for a thorough investigation of this problem. Relativistic quantum field theories on noncommutative Minkowski space are much more difficult [BDFP02]. Here the UV/IR-mixing problem occurs in different types of graphs [Bah10].

The Moyal algebra $(\mathcal{S}(\mathbb{R}^4), \star)$ has a matrix basis [GV88, VG88, GGISV03]

$$\phi(x) = \sum_{\underline{m},\underline{n}\in\mathbb{N}^2} \Phi_{\underline{mn}} f_{\underline{mn}}(x), \qquad f_{\underline{mn}}(x) = f_{m_1n_1}(x^0,x^1) f_{m_2n_2}(x^3,x^4)\,,$$

$$f_{mn}(y^0,y^1) = 2(-1)^m \sqrt{\frac{m!}{n!}}\Big(\sqrt{\frac{2}{\theta}}y\Big)^{n-m} L_m^{n-m}\Big(\frac{2|y|^2}{\theta}\Big) e^{-\frac{|y|^2}{\theta}}\,, \tag{24}$$

where L^n_m are Laguerre polynomials, $y \equiv y^0 + \mathrm{i}y^1$ and $\underline{m} = (m_1, m_2)$. Without loss of generality we assume the only non-vanishing components of Θ to be $\theta := \Theta_{12} = -\Theta_{21} = \Theta_{34} = -\Theta_{43}$. The functions $f_{\underline{mn}}$ satisfy

$$(f_{\underline{kl}} \star f_{\underline{mn}})(x) = \delta_{\underline{ml}} f_{\underline{kn}}(x)\,, \qquad \int_{\mathbb{R}^4} dx\; f_{\underline{mn}}(x) = (2\pi\theta)^2 \delta_{\underline{mn}}\,.$$

Therefore, the $\phi_4^{\star 4}$-interaction in (23) becomes a matrix product (we write ϕ for a function and Φ for a matrix):

$$S[\phi] = (2\pi\theta)^2 \sum_{\underline{k},\underline{l},\underline{m},\underline{n}\in\mathbb{N}^2} \Big(\tfrac{1}{2}\Phi_{\underline{kl}}(\Delta_{\underline{kl};\underline{mn}} + \mu^2 \delta_{\underline{kn}}\delta_{\underline{lm}})\Phi_{\underline{mn}} + \frac{\lambda}{4}\Phi_{\underline{kl}}\Phi_{\underline{lm}}\Phi_{\underline{mn}}\Phi_{\underline{nk}}\Big)\,. \tag{25}$$

The matrix kernel $\Delta_{\underline{kl};\underline{mn}}$ of the Laplacian $(-\Delta)$, viewed as map from $\mathbb{N}^4$ to $\mathbb{N}^4$, consists of a local interaction plus nearest neighbour interaction.

In [GW05b] we studied the renormalisation group flow of the $\phi_4^{\star 4}$-model in matrix representation (using a power-counting theorem [GW05a] for matrix models with kernel $\Delta_{\underline{kl};\underline{mn}}$). We noticed that the marginal parts of the local term and of the nearest neighbour term in $\Delta_{\underline{kl};\underline{mn}}$ have *different flows*. To absorb these different flows a 4^{th} relevant/marginal operator in the action functional is necessary. This operator corresponds to a harmonic oscillator potential:

$$S[\phi] = 64\pi^2 \int d^4x \Big(\frac{Z}{2}\phi \star \big(-\Delta + \Omega^2 (2\Theta^{-1}x)^2 + \mu^2_{bare}\big)\phi + \frac{\lambda Z^2}{4}\phi\star\phi\star\phi\star\phi\Big)(x)\,. \tag{26}$$

We proved in [GW05b] that the corresponding Euclidean quantum field theory is renormalisable to all orders in perturbation theory. This result was reestablished by various methods, see [Riv07a] for a review.

Presence of the harmonic oscillator term $\Omega \neq 0$ breaks translation invariance. Conversely, this term achieves covariance under Langmann-Szabo duality transformation [LS02] which consists in exchanging $x \leftrightarrow p$ and $\phi(x) \leftrightarrow \hat{\phi}(p)$ followed by Fourier transform back to the original variables. Remarkably, this transformation leaves $\displaystyle\int dx\ \phi\star\phi\star\phi\star\phi$ invariant, and it exchanges $\int dx\ \phi(-\Delta)\phi$ with $\int dx\ \phi|2\Theta^{-1}x|^2\phi$. Presence of the oscillator term gives rise to an interesting spectral noncommutative geometry [GW13a] (see also [GW12a]) which is conceptually simpler than the isospectral deformation [GGISV03] of $\mathbb{R}^4$. Most importantly, the oscillator term cures the Landau ghost problem [LAK54a, LAK54b, LAK54c] of usual ϕ^4_4-theory: We have discovered in [GW04, GW05c] that the one-loop renormalisation group flows of Ω and λ influence each other in such a way that the running coupling constant $\lambda(\Lambda)$ remains finite at any scale Λ. Even more, at the self-duality point $\Omega = 1$ the β-function of the $\lambda\Phi^4_4$-coupling vanishes to all orders in perturbation theory [DGMR07]. This result was obtained by an ingenious combination of Ward identities and Schwinger-Dyson equations (see [DR07] for an explicit three-loop calculation). In [GW12b] we have generalised the method of Disertori-Gurau-Magnen-Rivasseau [DGMR07] to the whole class of quartic matrix models (reviewed in sec. 2). Vanishing of the β-function is often connected with integrability, and together with the absent Landau ghost problem a non-perturbatively constructed ϕ^4_4-model on Moyal space came into reach. The first milestone was the derivation of the self-consistency equation (13) and the understanding of its renormalisation in [GW09]. It took us several years to fully understand this equation, and it is only recently that we finished the solution/construction of the Moyal space ϕ^4_4-model [GW12b]. In the sequel we review this construction.

3.2 Renormalisation and integral representation

At the self-duality point $\Omega = 1$, the matrix kernel $\Delta^{\Omega=1}_{\underline{kl};\underline{mn}}$ of the Schrödinger operator $H = -\Delta + \|2\Theta^{-1}x\|^2$ becomes purely local and turns the action (26) in matrix basis (24) into a (field-theoretical) quartic matrix model with action

$$S[\Phi] = V\Big(\sum_{\underline{m},\underline{n}\in\mathbb{N}^2_{\mathcal{N}}} E_{\underline{m}}\,\Phi_{\underline{mn}}\Phi_{\underline{nm}} + \frac{Z^2\lambda}{4}\sum_{\underline{m},\underline{n},\underline{k},\underline{l}\in\mathbb{N}^2_{\mathcal{N}}}\Phi_{\underline{mn}}\Phi_{\underline{nk}}\Phi_{\underline{kl}}\Phi_{\underline{lm}}\Big)\;, \tag{27}$$
$$E_{\underline{m}} = Z\Big(\frac{|\underline{m}|}{\sqrt{V}} + \frac{\mu^2_{bare}}{2}\Big)\;, \qquad |\underline{m}| := m_1 + m_2 \leq \mathcal{N}\;,\quad V = \Big(\frac{\theta}{4}\Big)^2\;.$$

Our general results on quartic matrix models imply that the planar 2-point function $G^{(0)}_{|\underline{a}\underline{b}|}$ satisfies the self-consistency equation (13),

$$G^{(0)}_{|\underline{a}\underline{b}|} = \frac{1}{E_{\underline{a}}+E_{\underline{b}}} - \frac{Z^2\lambda}{E_{\underline{a}}+E_{\underline{b}}}\frac{1}{V}\sum_{\underline{p}\in\mathbb{N}^2_{\mathcal{N}}}\Big(G^{(0)}_{|\underline{a}\underline{b}|}G^{(0)}_{|\underline{a}\underline{p}|} - \frac{G^{(0)}_{|\underline{p}\underline{b}|}-G^{(0)}_{|\underline{a}\underline{b}|}}{E_{\underline{p}}-E_{\underline{a}}}\Big) . \qquad (28)$$

We have introduced a cut-off $\mathbb{N}^2_{\mathcal{N}}$ in the matrix size; the index sum diverges for $\mathbb{N}^2_{\mathcal{N}} \mapsto \mathbb{N}^2$. As usual, the renormalisation strategy consists in adjusting Z, μ_{bare} in such a way that the limit $\mathbb{N}^2_{\mathcal{N}} \mapsto \mathbb{N}^2$ exists. This will be achieved by normalisation conditions for the 1PI function $\Gamma_{\underline{a}\underline{b}}$ defined by $G^{(0)}_{|\underline{a}\underline{b}|} =: (H_{\underline{a}\underline{b}} - \Gamma_{\underline{a}\underline{b}})^{-1}$, where $H_{\underline{a}\underline{b}} := E_{\underline{a}} + E_{\underline{b}}$. We express (28) in terms of $\Gamma_{\underline{a}\underline{b}}$,

$$\Gamma_{\underline{a}\underline{b}} = -\frac{\lambda Z^2}{V}\sum_{\underline{p}\in\mathbb{N}^2_{\mathcal{N}}}\Big(\frac{1}{H_{\underline{a}\underline{p}}-\Gamma_{\underline{a}\underline{p}}} + \frac{1}{H_{\underline{p}\underline{b}}-\Gamma_{\underline{p}\underline{b}}} - \frac{1}{(H_{\underline{p}\underline{b}}-\Gamma_{\underline{p}\underline{b}})}\frac{\Gamma_{\underline{p}\underline{b}}-\Gamma_{\underline{a}\underline{b}}}{\frac{Z}{\sqrt{V}}(|\underline{p}|-|\underline{a}|)}\Big) , \qquad (29)$$

and write $\Gamma_{\underline{a}\underline{b}}$ as first-order Taylor formula with remainder $\Gamma^{ren}_{\underline{a}\underline{b}}$,

$$\Gamma_{\underline{a}\underline{b}} = Z\mu^2_{bare} - \mu^2 + \tfrac{(Z-1)}{\sqrt{V}}(|\underline{a}|+|\underline{b}|) + \Gamma^{ren}_{\underline{a}\underline{b}} , \quad \Gamma^{ren}_{\underline{0}\underline{0}} = 0 , \quad (\partial\Gamma^{ren})_{\underline{0}\underline{0}} = 0 .$$

Equation (29) for $\Gamma_{\underline{a}\underline{b}}\big[\Gamma^{ren}_{\underline{a}\underline{b}}, \mu^2_{bare}, Z\big]$ together with $\Gamma^{ren}_{\underline{0}\underline{0}} = 0$ and $(\partial\Gamma^{ren})_{\underline{0}\underline{0}}$ constitute three equations to determine the three functions $\Gamma^{ren}_{\underline{a}\underline{b}}, \mu^2_{bare}, Z$. Eliminating μ^2_{bare}, Z thus gives rise to a *closed equation for the renormalised function* $\Gamma^{ren}_{\underline{a}\underline{b}}$ *alone*. For this elimination it is important to note that the equations for $\Gamma^{ren}_{\underline{a}\underline{b}}, \mu^2_{bare}, Z$ depend on $\underline{a}, \underline{b}$ only via the norms $|\underline{a}|, |\underline{b}|$ which parametrise the spectrum of E. Therefore, $\Gamma_{\underline{a}\underline{b}}$ is actually a function only of $|\underline{a}|, |\underline{b}|$, and consequently the index sum reduces to $\sum_{\underline{p}\in\mathbb{N}^2_{\mathcal{N}}} f(|\underline{p}|) = \sum_{|\underline{p}|=0}^{\mathcal{N}}(|\underline{p}|+1)f(|\underline{p}|)$.

We study a particular scaling limit in which matrix size $\mathcal{N}$ and volume V are simultaneously sent to ∞ such that the ratio $\frac{\mathcal{N}}{\sqrt{V\mu^4}} = \Lambda^2(1+\mathcal{Y})$ is kept fixed. Note that $V = \big(\frac{\theta}{4}\big)^2 \to \infty$ is a limit of extreme noncommutativity! The new parameter $(1+\mathcal{Y})$ corresponds to a finite wavefunction renormalisation, identified later to decouple our equations. The parameter Λ^2 represents an ultraviolet cut-off which is sent to $\Lambda \to \infty$ in the very end (continuum limit). In the scaling limit, functions of $\frac{|\underline{p}|}{\sqrt{V}} =: \mu^2(1+\mathcal{Y})p$ converge to functions of 'continuous matrix indices' $p \in [0, \Lambda^2]$. In the same way, $\Gamma^{ren}_{\underline{a}\underline{b}}$ converges to a function $\mu^2\Gamma_{ab}$ with $a, b \in [0, \Lambda^2]$, and the discrete sum converges to a Riemann integral

$$\frac{1}{V}\sum_{|\underline{p}|=0}^{\mathcal{N}}(|\underline{p}|+1)f\big(\tfrac{|\underline{p}|}{\sqrt{V}}\big) \longrightarrow \mu^4(1+\mathcal{Y})^2\int_0^{\Lambda^2} p\,dp\; f\big(\mu^2(1+\mathcal{Y})p\big) .$$

This limit makes the restriction to the planar sector (13) of (12) exact.

After elimination of μ^2_{bare}, but before elimination of Z, our equation for Γ_{ab} becomes

$$\begin{aligned}
&(Z-1)(1+\mathcal{Y})(a+b)+\Gamma_{ab} \\
&= -\lambda(1+\mathcal{Y})^2\int_0^{\Lambda^2} p\,dp\,\Big(\frac{Z^2}{(a+p)(1+\mathcal{Y})+1-\Gamma_{ap}} - \frac{Z^2}{p(1+\mathcal{Y})+1-\Gamma_{0p}}\Big) \\
&- \lambda(1+\mathcal{Y})^2\int_0^{\Lambda^2} p\,dp\,\Big(\frac{Z}{(b+p)(1+\mathcal{Y})+1-\Gamma_{pb}} - \frac{Z}{p(1+\mathcal{Y})+1-\Gamma_{p0}} \\
&\qquad\qquad - \frac{Z}{(b+p)(1+\mathcal{Y})+1-\Gamma_{pb}}\,\frac{\Gamma_{pb}-\Gamma_{ab}}{(1+\mathcal{Y})(p-a)} \\
&\qquad\qquad + \frac{Z}{p(1+\mathcal{Y})+1-\Gamma_{p0}}\,\frac{\Gamma_{p0}}{p(1+\mathcal{Y})}\Big)\;. \qquad (30)
\end{aligned}$$

Applying $\frac{d}{db}\big|_{a=b=0}$ we get Z in terms of Γ_{ab} (and its derivative). Inserted back one gets a highly non-linear integro-differential equation. Fortunately we can reduce the non-linearity by subtracting from (30) the same equation taken at $b=0$. This subtraction eliminates the second line of (30) containing Z^2. In terms of $G_{ab} := \big((a+b)(1+\mathcal{Y})+1-\Gamma_{ab}\big)^{-1}$, this difference equation reads

$$\frac{Z^{-1}}{(1+\mathcal{Y})}\Big(\frac{1}{G_{ab}}-\frac{1}{G_{a0}}\Big) = b-\lambda\int_0^{\Lambda^2} p\,dp\,\frac{\frac{G_{pb}}{G_{ab}}-\frac{G_{p0}}{G_{a0}}}{p-a}\;. \qquad (31)$$

Differentiation $\frac{d}{db}\big|_{a=b=0}$ of (31) yields Z in terms of G_{ab} and its derivative. The resulting derivative G' can be avoided by adjusting

$$\mathcal{Y} := -\lambda\lim_{b\to 0}\int_0^{\Lambda^2} dp\;\frac{G_{pb}-G_{p0}}{b}\;.$$

This choice leads to $\dfrac{Z^{-1}}{(1+\mathcal{Y})} = 1-\lambda\displaystyle\int_0^{\Lambda^2} dp\;G_{p0}$, which is a perturbatively divergent integral for $\Lambda\to\infty$. Inserting Z^{-1} and $\mathcal{Y}$ back into (31) we end up in a *linear* integral equation for the difference function $D_{ab} := \frac{a}{b}(G_{ab}-G_{a0})$ to the boundary:

$$\Big(\frac{b}{a}+\frac{1}{aG_{a0}}\Big)D_{ab}+G_{a0} = \lambda\int_0^{\Lambda^2} dp\,\Big(\frac{D_{pb}-D_{ab}\frac{G_{p0}}{G_{a0}}}{p-a}\Big)\;. \qquad (32)$$

The non-linearity restricts to the boundary function G_{a0} where the second index is put to zero. Assuming $a\mapsto G_{ab}$ Hölder-continuous, we can pass to Cauchy principal values. In terms of the *finite Hilbert transform*

$$\mathcal{H}^{\Lambda}_a[f(\bullet)] := \frac{1}{\pi}\lim_{\epsilon\to 0}\Big(\int_0^{a-\epsilon}+\int_{a+\epsilon}^{\Lambda^2}\Big)\frac{f(q)\,dq}{q-a}\;, \qquad (33)$$

the integral equation (32) becomes

$$\Big(\frac{b}{a}+\frac{1+\lambda\pi a\mathcal{H}^\Lambda_a\big[G_{\bullet 0}\big]}{aG_{a0}}\Big)D_{ab}-\lambda\pi\mathcal{H}^\Lambda_a\big[D_{\bullet b}\big]=-G_{a0}\;. \tag{34}$$

3.3 The Carleman solution

Equation (34) is a well-known singular integral equation of Carleman type [Car22, Tri57]:

Theorem 6 ([Tri57], transformed from $[-1,1]$ to $[0,\Lambda^2]$) *The singular linear integral equation*

$$h(a)y(a)-\lambda\pi\mathcal{H}^\Lambda_a[y]=f(a)\;,\qquad a\in\,]0,\Lambda^2[\;,$$

is for $h(a)$ continuous on $]0,\Lambda^2[$, Hölder-continuous near $0,\Lambda^2$, and $f\in L^p$ for some $p>1$ (determined by $\vartheta(0)$ and $\vartheta(\Lambda^2)$) solved by

$$\begin{aligned}y(a)&=\frac{\sin(\vartheta(a))e^{-\mathcal{H}^\Lambda_a[\pi-\vartheta]}}{\lambda\pi a}\Big(a\,f(a)e^{\mathcal{H}^\Lambda_a[\pi-\vartheta]}\cos(\vartheta(a))\\&\qquad\qquad+\mathcal{H}^\Lambda_a\Big[e^{\mathcal{H}^\Lambda_\bullet[\pi-\vartheta]}\bullet f(\bullet)\sin(\vartheta(\bullet))\Big]+C\Big) \qquad (35a)\\&\overset{*}{=}\frac{\sin(\vartheta(a))e^{\mathcal{H}^\Lambda_a[\vartheta]}}{\lambda\pi}\Big(f(a)e^{-\mathcal{H}^\Lambda_a[\vartheta]}\cos(\vartheta(a))\\&\qquad\qquad+\mathcal{H}^\Lambda_a\Big[e^{-\mathcal{H}^\Lambda_\bullet[\vartheta]}f(\bullet)\sin(\vartheta(\bullet))\Big]+\frac{C'}{\Lambda^2-a}\Big)\;, \qquad (35b)\end{aligned}$$

where $\vartheta(a)=\underset{[0,\pi]}{\arctan}\Big(\frac{\lambda\pi}{h(a)}\Big)$, $\sin(\vartheta(a))=\frac{|\lambda\pi|}{\sqrt{(h(a))^2+(\lambda\pi)^2}}\geq 0$ *and C,C' are arbitrary constants.*

The possibility of $C,C'\neq 0$ is due to the fact that the finite Hilbert transform has a kernel, in contrast to the infinite Hilbert transform with integration over $\mathbb{R}$. The two formulae (35a) and (35b) are formally equivalent, but the solutions belong to different function classes and normalisation conditions may (and will) make a choice.

In principle, (35) provides the solution G_{ab} of (34), where the angle function

$$\vartheta_b(a):=\underset{[0,\pi]}{\arctan}\left(\frac{\lambda\pi a}{b+\frac{1+\lambda\pi a\mathcal{H}^\Lambda_a[G_{\bullet 0}]}{G_{a0}}}\right) \tag{36}$$

plays a key rôle. This solution involves multiple Hilbert transforms which are difficult to control. A better strategy starts from the observation that the angle (36) satisfies, for $b=0$, again a Carleman type singular integral equation

$$\lambda\pi\cot\vartheta_0(a)G_{a0}-\lambda\pi\mathcal{H}^\Lambda_a[G_{\bullet 0}]=\tfrac{1}{a}$$

with solution

$$\begin{aligned}
G_{a0} &= \frac{e^{-\mathcal{H}^\Lambda_a[\pi-\vartheta_0]}\sin(\vartheta_0(a))}{\lambda\pi a}\Big(e^{\mathcal{H}^\Lambda_a[\pi-\vartheta_0]}\cos(\vartheta_0(a)) \\
&\qquad\qquad + \mathcal{H}^\Lambda_a\big[e^{\mathcal{H}^\Lambda_\bullet[\pi-\vartheta_0]}\sin(\vartheta_0(\bullet))\big] + C\Big) && \text{(37a)}\\
&\overset{*}{=} \frac{e^{\mathcal{H}^\Lambda_a[\vartheta_0]}\sin(\vartheta_0(a))}{\lambda\pi}\Big(\frac{e^{-\mathcal{H}^\Lambda_a[\vartheta_0]}\cos(\vartheta_0(a))}{a} \\
&\qquad\qquad + \mathcal{H}^\Lambda_a\Big[\frac{e^{-\mathcal{H}^\Lambda_\bullet[\vartheta_0]}\sin(\vartheta_0(\bullet))}{\bullet}\Big] + \frac{C'}{\Lambda^2-a}\Big)\;. && \text{(37b)}
\end{aligned}$$

Tricomi's identities [Tri57, §4.4(28+18)], which can be arranged as

$$e^{\pm\mathcal{H}^\Lambda_a[\vartheta_b]}\cos(\vartheta_b(a)) \mp \mathcal{H}^\Lambda_a\big[e^{\pm\mathcal{H}^\Lambda_\bullet[\vartheta_b]}\sin(\vartheta_b(\bullet))\big] = 1\;,$$

and rational fraction expansion $\mathcal{H}^\Lambda_a\big[\frac{f(\bullet)}{\bullet}\big] = \frac{1}{a}\big(\mathcal{H}^\Lambda_a\big[f(\bullet)\big] - \mathcal{H}^\Lambda_0\big[f(\bullet)\big]\big)$ simplify (37) to

$$\begin{aligned}
G_{a0} &= \frac{e^{-\mathcal{H}^\Lambda_a[\pi-\vartheta_0]}\sin(\vartheta_0(a))}{\lambda\pi a}\big(C-1\big) && \text{(38a)}\\
&\overset{*}{=} \frac{e^{\mathcal{H}^\Lambda_a[\vartheta_0]}\sin(\vartheta_0(a))}{\lambda\pi a}\Big(e^{-\mathcal{H}^\Lambda_0[\vartheta_0]}\cos(\vartheta_0(0)) + \frac{C'a}{\Lambda^2-a}\Big)\;. && \text{(38b)}
\end{aligned}$$

Both lines are formally equivalent, but we have to guarantee the normalisation $\lim_{a\to 0} G_{a0} = 1$. From (36) one concludes $\lim_{p\to 0}\vartheta_0(p) = \left\{\begin{array}{ll} 0 & \text{for } \lambda \geq 0 \\ \pi & \text{for } \lambda < 0\end{array}\right\}$. Consequently, $e^{-\mathcal{H}^\Lambda_0[\vartheta_0]} = \exp\big(-\frac{1}{\pi}\int_0^{\Lambda^2}\frac{dp}{p}\vartheta_0(p)\big) \xrightarrow{\lambda<0} 0$, which means that (38b) reduces for $\lambda<0$ to (38a), with $C'\mapsto C-1$. Similarly, $\lim_{a\to 0} e^{-\mathcal{H}^\Lambda_a[\pi-\vartheta_0]} \overset{\lambda\geq 0}{=} 0$, so that (38a) is only consistent with $\lambda<0$. The normalisation $\lim_{a\to 0} G_{a0} = 1$ leads with $\lim_{a\to 0}\frac{\sin\vartheta_0(a)}{|\lambda|\pi a} = 1$ to $1-C = e^{-\mathcal{H}^\Lambda_0[\pi-\vartheta_0]}$ in (38a), whereas (38b) stays as it is for $\lambda>0$. These results can be summarised as follows:

Lemma 7 *The angle function* $\tau_b(a) := \underset{[0,\pi]}{\arctan}\left(\frac{|\lambda|\pi a}{b + \frac{1+\lambda\pi a\mathcal{H}^\Lambda_a[G_{\bullet 0}]}{G_{a0}}}\right)$ *is for* $b=0$ *reverted to*

$$G_{a0} = \frac{\sin(\tau_0(a))}{|\lambda|\pi a}e^{\mathrm{sign}(\lambda)(\mathcal{H}^\Lambda_0[\tau_0(\bullet)]-\mathcal{H}^\Lambda_a[\tau_0(\bullet)])}\begin{cases} 1 & \textit{for } \lambda<0\;, \\ \big(1+\frac{Ca}{\Lambda^2-a}\big) & \textit{for } \lambda>0\;, \end{cases} \tag{39}$$

where C is an arbitrary constant.

Recall that G_{a0} forms the inhomogeneity in the Carleman equation (34). We insert (39) into the Carleman solution (35) for (34) and obtain with the addition theorem $|\lambda|\pi a\sin\big(\tau_d(a)-\tau_b(a)\big) = (b-d)\sin\tau_b(a)\sin\tau_d(a)$ after essentially the same steps as in the proof of (39):

Theorem 8 ([GW14]) *The full matrix 2-point function G_{ab} of self-dual ϕ^4_4-theory on Moyal space is in the limit $\theta \to \infty$ given in terms of the boundary 2-point function G_{a0} by the equation*

$$G_{ab} = \frac{\sin(\tau_b(a))}{|\lambda|\pi a} e^{\mathrm{sign}(\lambda)(\mathcal{H}^\Lambda_0[\tau_0(\bullet)] - \mathcal{H}^\Lambda_a[\tau_b(\bullet)])} \begin{cases} 1 & \text{for } \lambda < 0 \,, \\ \big(1 + \frac{Ca + bF(b)}{\Lambda^2 - a}\big) & \text{for } \lambda > 0 \,, \end{cases} \tag{40}$$

where C is an undetermined constant and $b\,F(b)$ an undetermined function of b vanishing at $b = 0$.

Some remarks:

- We have proved this theorem in 2012 for $\lambda > 0$ under the assumption $C' = 0$ in (35b), but knew that non-trivial solutions of the homogeneous Carleman equation parametrised by $C' \neq 0$ are possible. That no such term arises for $\lambda < 0$ (if angles are redefined $\vartheta \mapsto \tau$) is a recent result [GW14].
- We expect C, F to be Λ-dependent so that $\big(1 + \frac{Ca + bF(b)}{\Lambda^2 - a}\big) \stackrel{\Lambda \to \infty}{\longrightarrow} 1 + \tilde{C}a + b\tilde{F}(b)$.
- An important observation is $G_{ab} \geq 0$, at least for $\lambda < 0$. This is a truly non-perturbative result; individual Feynman graphs show no positivity at all!
- As in [GW09], the equation for G_{ab} can be solved perturbatively. Matching at $\lambda = 0$ requires C, F to be flat functions of λ (all derivatives vanish at zero). Because of $\mathcal{H}^\Lambda_a[G_{\bullet 0}] \stackrel{a \to \Lambda^2}{\longrightarrow} -\infty$, the naïve arctan series is dangerous for $\lambda > 0$. Unless there are cancellations, we expect zero radius of convergence!
- From (40) we deduce the finite wavefunction renormalisation

$$\mathcal{Y} := -1 - \frac{dG_{ab}}{db}\Big|_{a=b=0} = \int_0^{\Lambda^2} \frac{dp}{(\lambda\pi p)^2 + \big(\frac{1 + \lambda\pi p \mathcal{H}^\Lambda_p[G_{\bullet 0}]}{G_{p0}}\big)^2} - \begin{cases} 0 & \text{for } \lambda < 0 \,, \\ F(0) & \text{for } \lambda > 0 \,. \end{cases} \tag{41}$$

- The partition function $\mathcal{Z}$ is undefined for $\lambda < 0$. But the Schwinger-Dyson equations for G_{ab} and for higher functions, and with them $\log \mathcal{Z}$, extend to $\lambda < 0$. These extensions are unique but probably not analytic in a neighbourhood of $\lambda = 0$.

It remains to identify the boundary function G_{a0}. The Carleman equation (34) for G_{ab} was obtained from the difference $(30) - (30)_{b=0}$. Consequently, $(30)_{b=0}$ gives the second relation between G_{ab} and G_{a0} from which both are determined. Combining them we obtain a single consistency equation for G_{a0}, which in terms of $\mathcal{T}_a := |\lambda|\pi a \cot \tau_0(a)$ reads [GW12b]

$$\begin{aligned} \mathcal{T}_a &= 1 + a + \lambda\pi a \mathcal{H}^\Lambda_a[1] \\ &\quad + \int_0^{\Lambda^2} dp \left(\frac{p \exp\Big(\mathcal{H}^\Lambda_a\Big[\underset{[0,\pi]}{\arctan} \frac{|\lambda|\pi\bullet}{p + \mathcal{T}_\bullet}\Big]\Big)}{\sqrt{(\lambda\pi a)^2 + (p + \mathcal{T}_a)^2}} - \frac{p \exp\Big(\mathcal{H}^\Lambda_0\Big[\underset{[0,\pi]}{\arctan} \frac{|\lambda|\pi\bullet}{p + \mathcal{T}_\bullet}\Big]\Big)}{1 + p} \right) . \end{aligned} \tag{42}$$

This equation is, unfortunately, of little use. The integrals are individually divergent for $\Lambda\to\infty$ so that we have to rely on cancellations on which we have no control.

We compensate this lack by a symmetry argument. Given the boundary function G_{a0}, the Carleman theory computes the full 2-point function G_{ab} via (40). In particular, we get G_{0b} as function of G_{a0}. But the 2-point function is symmetric, $G_{ab} = G_{ba}$, and the special case $b = 0$ leads to the following self-consistency equation:

Proposition 9 *The limit $\theta \to \infty$ of ϕ^4_4-theory on Moyal space is determined by the solution of the fixed point equation $G = TG$,*

$$G_{b0} = \frac{\begin{Bmatrix} 1 & \textit{for } \lambda<0 \\ 1+bF(b) & \textit{for } \lambda>0 \end{Bmatrix}}{1+b} \exp\left(-\lambda \int_0^b dt \int_0^{\Lambda^2} \frac{dp}{(\lambda\pi p)^2 + \left(t+\frac{1+\lambda\pi p\mathcal{H}_p^\Lambda[G_{\bullet 0}]}{G_{p0}}\right)^2} \right) . \quad (43)$$

At this point we can eventually send $\Lambda \to \infty$. Any solution of (43) is automatically smooth and (for $\lambda > 0$ but $F = 0$) monotonously decreasing. Any solution of the true equation (30) (without the difference to $b = 0$) also solves the master equation (43), but not necessarily conversely. In case of a unique solution of (43) it is enough to check one candidate.

Existence of a solution of (43) is established (for $\lambda > 0$ but $F(b) = 0$) by the Schauder fixed point theorem. We consider the following subset of continuously differentiable functions on $\mathbb{R}_+$ vanishing at ∞:

$$\mathcal{K}_\lambda := \Big\{ f \in \mathcal{C}^1_0(\mathbb{R}_+) \;:\; f(0) = 1\,, \quad 0 < f(b) \le \frac{1}{1+b}\,, \\ 0 \le -f'(b) \le \big(\tfrac{1}{1+b} + C_\lambda\big) f(b) \Big\}\,,$$

where C_λ is defined via $2\lambda P_\lambda^2(1+C_\lambda)e^{C_\lambda P_\lambda} = 1$ at $P_\lambda = \frac{\exp(-\frac{1}{\lambda\pi^2})}{\sqrt{1+4\lambda}}$. Then [GW12b]:

1. $\mathcal{K}_\lambda$ is convex,
2. $\overline{T\mathcal{K}_\lambda} \subseteq \mathcal{K}_\lambda$,
3. $(Tf)''(b) \le \big(\frac{23}{4} + \frac{2}{\pi} + \frac{7+8\pi}{2}\frac{1}{(\lambda\pi^2 P_\lambda)^2}\big)(Tf)(b)$ for any $f \in \mathcal{K}_\lambda$,
4. $T : \mathcal{K}_\lambda \to \mathcal{K}_\lambda$ is continuous.

The properties 1.–3. imply that $T\mathcal{K}_\lambda$ is relatively compact in $\mathcal{K}_\lambda$ by a variant of the Arzelá-Ascoli theorem. Together with 4. the Schauder fixed point theorem then guarantees that (43) has a solution $G_{a0} \in \mathcal{K}_\lambda$.

This solution provides G_{ab} via (40) and all higher correlation functions via the universal algebraic recursion formulae (16), (65), (66), etc, or via the linear equations for the basic $(N_1+\ldots+N_B)$-point functions such as (63) and (64). The recursion formula (16) becomes after transition to continuous matrix indices

$$G_{b_0\dots b_{N-1}} = \frac{(-\lambda)}{(1+\mathcal{Y})^2} \sum_{l=1}^{\frac{N-2}{2}} \frac{G_{b_0b_1\dots b_{2l-1}} G_{b_{2l}b_{2l+1}\dots b_{N-1}} - G_{b_{2l}b_1\dots b_{2l-1}} G_{b_0b_{2l+1}\dots b_{N-1}}}{(b_0-b_{2l})(b_1-b_{N-1})} . \tag{44}$$

It involves the finite wavefunction renormalisation $1+\mathcal{Y} = -\frac{dG_{ab}}{db}\big|_{a=b=0}$ given by (41). Of particular interest is the effective coupling constant $\lambda_{\mathit{eff}} = -G_{0000}$. This limit of coinciding indices is not so easy; therefore we directly solve the integral equation for G_{a000} before using the reality condition. We find [GW12b]

$$\lambda_{\mathit{eff}} = \lambda\Big\{1 + \frac{\lambda}{(1{+}\mathcal{Y})} \int_0^\infty dp \, \frac{\Big(\dfrac{1-G_{p0}}{(1+\mathcal{Y})p} - G_{p0}\Big) G_{p0}}{\big(\lambda\pi p G_{p0}\big)^2 + \big(1+\lambda\pi p \mathcal{H}_p^\infty[G_{\bullet 0}]\big)^2}\Big\} . \tag{45}$$

The equation for the basic function $G_{ab|cd}$ arising from (64) is solved in two steps. A first summation over $b \in I$ in (64) yields after passage to the integral representation a Carleman equation for $X_{a|cd} := \int_0^{\Lambda^2} q dq \; G_{aq|cd}$,

$$\begin{aligned} &X_{a|cd}\Big\{1 + \lambda \int_0^\infty dq \; (G_{aq} - G_{0q}) - \lambda \int_0^\infty dq \; \frac{G_{aq} \sin\tau_q(a) \cos\big(\tau_q(a) - \tau_0(a)\big)}{\sin\tau_0(a)}\Big\} \\ &+ \mathcal{H}_a^\infty\Big[\frac{X_{\bullet|cd}}{\pi\bullet} \int_0^\infty q \, dq \; \sin^2\tau_q(\bullet) G_{aq}\Big] \\ &= \lambda \int_0^\infty q \, dq \, (F_{aq|cdcq} + F_{aq|dcdq}) + \frac{\lambda}{(1+\mathcal{Y})^2} (G_{acdc}{+}G_{adcd}) , \end{aligned}$$

where $F_{ab_1|c_1c_2c_3c_4} := \frac{G_{ab_1c_1c_2c_3c_4} G_{b_1c_3} - G_{b_1c_1c_2c_3} G_{ab_1c_3c_4}}{G_{b_1c_1} G_{b_1c_3}}$. Inserted back into (64) gives (after passage to the integral representation) a familiar Carleman equation for $G_{ab|cd}$ with solution

$$\begin{aligned} G_{ab|cd} &= F_{ab|cdcb} + F_{ab|dcdb} \\ &\quad - \frac{\sin\tau_b(a)}{\lambda\pi a} \cos\tau_b(a) G_{ab} X_{a|cd} - G_{ab} \mathcal{H}_a^\infty\Big[\frac{\sin^2\tau_b(\bullet)}{\lambda\pi\bullet} X_{\bullet|cd}\Big] . \end{aligned} \tag{46}$$

The (2+2)-point function $G_{ab|cd}$ turns out to be the most interesting part of the 4-point function in position space (see sec. 4).

3.4 Perturbation theory

The master equation (43) can, for $F(b) \equiv 0$, be iteratively solved. To lowest order one has $G_{a0} = \frac{1}{1+a} + \mathcal{O}(\lambda)$, from which the next order becomes

$$G_{a0} = \frac{1}{1+a} - \lambda \frac{\log(1+a)}{(1+a)} + \mathcal{O}(\lambda^2) . \tag{47}$$

If we put in $G_{a0} = \frac{1}{(1+a)^{1+\lambda}} + \mathcal{O}(\lambda^2)$ the index $a \mapsto \frac{p^2}{\mu^2}$, see (58), we get

$$\begin{aligned}\int_{\mathbb{R}^4} \frac{dp}{(2\pi\mu)^4}\, e^{\mathrm{i}p(x-y)} G_{\frac{p^2}{\mu^2}0} &= \frac{2^{-\lambda}}{4\pi^2\Gamma(1+\lambda)} \frac{K_{1-\lambda}(\mu\|x-y\|)}{(\mu\|x-y\|)^{1-\lambda}} \\ &\xrightarrow{x-y\to 0} \frac{2^{-2\lambda}\Gamma(1-\lambda)}{4\pi^2\Gamma(1+\lambda)} \frac{1}{(\mu\|x-y\|)^{2-2\lambda}} ,\end{aligned}$$

where $K_\nu(x)$ is the modified Bessel function. We thus conclude that the anomalous dimension is $\eta = -2\lambda$, i.e. negative for the stable sign $\lambda > 0$ of the coupling constant. We shall see in the next section that this result excludes a Wightman theory for $\lambda > 0$. It is worthwhile to mention that this wrong sign is a consequence of renormalisation. The divergent bare 2-point function would lead to the opposite sign. Removing the divergence at $a = 0$ overcompensates for $a > 0$ and gives $\eta = -2\lambda$. In two dimensions, η would be non-negative for $\lambda > 0$.

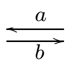
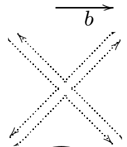
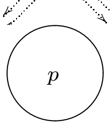
From (47) we get:

- Hilbert transform: $\lambda\pi\mathcal{H}^\infty_a[G_{\bullet 0}] = -\lambda\frac{\log(a)}{1+a} + \mathcal{O}(\lambda^2)$,
- angle function: $\tau_b(a) = \frac{|\lambda|\pi a}{1+a+b}\Big(1 - \lambda\frac{(1+a)\log(1+a) - a\log a}{(1+a+b)}\Big) + \mathcal{O}(\lambda^3)$,
- wavefunction renormalisation: $1 + \mathcal{Y} = -\frac{dG_{a0}}{da}\Big|_{a=0} = 1 + \lambda + \mathcal{O}(\lambda^2)$.

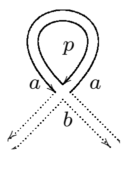
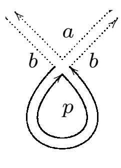
Inserted into (40) one finds

$$G_{ab} = \frac{1}{1+a+b} - \lambda\frac{(1+a)\log(1+a) + (1+b)\log(1+b)}{(1+a+b)^2} + \mathcal{O}(\lambda^2)\,. \tag{48}$$

This result coincides with renormalised 1-loop ribbon graph computation. From the action functional (27) one obtains in the infinite volume limit to continuous matrix indices the following Feynman rules:

-  $= \dfrac{1}{1 + (a+b)(1+\mathcal{Y})}$
-  $= -Z^2\lambda$ (index conserved at every corner)
-  $= (1+\mathcal{Y})^2 \displaystyle\int_0^{\Lambda^2} p\,dp$ for every closed face

To lowest order we have $G_{ab} = \dfrac{1}{1 + (a+b)(1+\mathcal{Y}) - \Gamma^{ren}_{ab}}$, where Γ^{rn}_{ab} is the Taylor remainder of

$\Gamma_{ab} =$  $+$  $+ \mathcal{O}(\lambda^2)$

$$
\begin{aligned}
&= \int_0^{\Lambda^2} p\,dp\, \frac{(-\lambda)}{1+a+p} + \int_0^{\Lambda^2} p\,dp\, \frac{(-\lambda)}{1+b+p} + \mathcal{O}(\lambda^2)\;, \\
&= \Gamma^{ren} + \Bigg(\int_0^{\Lambda^2} p\,dp\, \frac{(-\lambda)}{1+p} + \underbrace{\int_0^{\Lambda^2} p\,dp\, \frac{(+\lambda)a}{(1+p)^2}}_{(Z-1)a} + \big(a \mapsto b\big) + \mathcal{O}(\lambda^2) \Bigg)\;, \qquad \text{(49a)}
\end{aligned}
$$

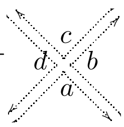
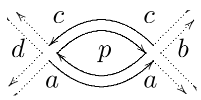
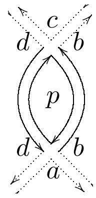
$$
\Gamma^{ren}_{ab} = (-\lambda) \int_0^{\Lambda^2} p\,dp\, \Big(\frac{1}{1+a+p} - \frac{1}{1+p} + \frac{a}{(1+p)^2} \Big) + \big(a \mapsto b\big) + \mathcal{O}(\lambda^2)\;, \qquad \text{(49b)}
$$

in agreement with (48). There is no doubt that the fixed point solution for G_{a0} and the Carleman solution for G_{ab} capture the resummation of infinitely many renormalised Feynman graphs!

From (44) and $\mathcal{Y} = \lambda + \mathcal{O}(\lambda^2)$ we obtain for the 4-point function

$$
\begin{aligned}
G_{abcd} &= \frac{(-\lambda)}{(1+\mathcal{Y})^2} \frac{G_{ab}G_{cd} - G_{ad}G_{cd}}{(a-c)(b-d)} =: G_{ab}G_{bc}G_{cd}G_{da}(-\Gamma_{abcd})\;, \\
\Gamma_{abcd} &= \lambda \Big(1 - \lambda \frac{a - (1+a)\log(1+a) - c + (1+c)\log(1+c)}{a-c} \\
&\qquad - \lambda \frac{b - (1+b)\log(1+b) - d + (1+d)\log(1+d)}{b-d} \Big) + \mathcal{O}(\lambda^3)\;, \qquad \text{(50)}
\end{aligned}
$$

which agrees with

$\Gamma_{abcd} = -$  $-$  $-$  $+ \mathcal{O}(\lambda^3)$

$$
\begin{aligned}
&= -(-\lambda) \underbrace{\Big(1 + 2\lambda \int_0^{\Lambda^2} \frac{p\,dp}{(1+p)^2} \Big)}_{=Z^2 + \mathcal{O}(\lambda^2)} \\
&\quad - (-\lambda)^2 \int_0^{\Lambda^2} \frac{p\,dp}{(1+p+a)(1+p+c)} - (-\lambda)^2 \int_0^{\Lambda^2} \frac{p\,dp}{(1+p+b)(1+p+d)}\;. \qquad \text{(51)}
\end{aligned}
$$

The singularities of Z^2 and of the one-loop 4-point graphs cancel exactly!

3.5 Computer simulations [GW14]

A numerical investigation of (43), for $F(b) \equiv 0$, reveals interesting properties of the ϕ^4_4-theory on Moyal space. We approximate G_{a0} as piecewise linear function on $[0, \Lambda^2]$ sampled according to a geometric progression and view (43) as iteration $G^{n+1}_{a0} = (TG^n)_{a0}$ for some initial function G^0. In this way we find numerically that T satisfies, for any $\lambda \in \mathbb{R}$, the assumptions of the Banach fixed point theorem for Lipschitz functions on $[0, \Lambda^2]$, i.e. T is contractive and (G^n) converges to a fixed point which approximates G_{a0}. Whereas (G^n) converges for any sign of λ

(without discontinuity at $\lambda = 0$), the necessary consistency condition $G_{ab} = G_{ba}$ for (40) turns out to be maximally violated for $\lambda > 0$ (assuming $C = 0 = F(b)$) and satisfied (within numerical error bounds) for $\lambda \leq 0$. The observed relative asymmetry $\sup_{a,b} \left|\frac{G_{ab}-G_{ba}}{G_{ab}+G{ba}}\right|$ of nearly 100 % for $\lambda > 0$ signals that the parameters $C, F(b)$ in (40) which reflect the non-trivial solution of the homogeneous Carleman equation are definitely non-zero. Taking $C, F(b) \neq 0$ for $\lambda > 0$ into account is not feasible at the moment so that our numerical results are reliable only for $\lambda \leq 0$. For $\lambda = 10^7$ and only 2000 sample points in $[0, \Lambda^2]$, the relative asymmetry for $\lambda \leq 0$ is of the order of 5 %.

The most striking outcome of our computer simulations concerns the finite wavefunction renormalisation $(1 + \mathcal{Y})$ given by (41). Figure 1 shows both $\mathcal{Y}$ and the effective coupling constant λ_{eff} given by (45) as functions of λ. We find clear

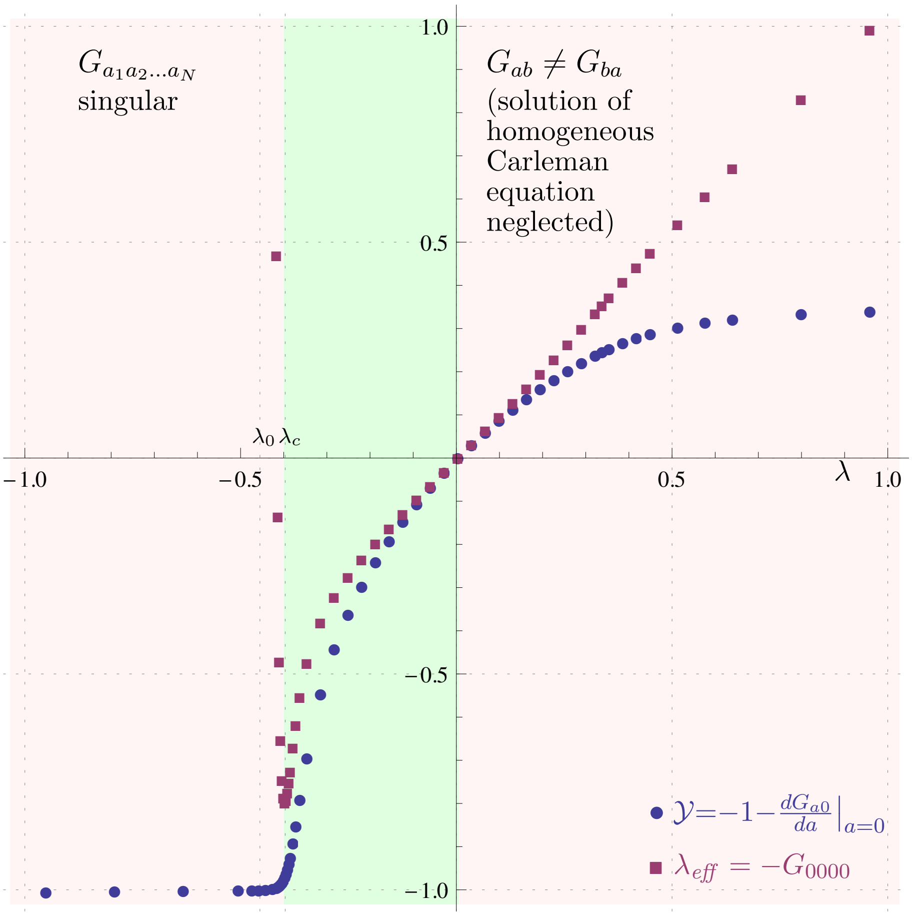


Figure 1: $\mathcal{Y}, \lambda_{eff}$ based on G_{a0} for $\Lambda^2{=}10^7$ with 2000 sample points.

evidence for a second-order phase transition: $\mathcal{Y}'$ is discontinuous at $\lambda_c = -0.396$, and we have in reasonable approximation a critical behaviour

$$1 + \mathcal{Y} = \begin{cases} A(\lambda - \lambda_c)^{\alpha} & \text{for } \lambda \geq \lambda_c\,, \\ 0 & \text{for } \lambda < \lambda_c\,, \end{cases} \tag{52}$$

for some $A, \alpha > 0$. To be precise, we find $1+\mathcal{Y} = 0$ only at $\lambda_0 = -0.455$, but this seems to be due to the discretisation. Of course, there cannot be a discontinuity in $\mathcal{Y}'$ for finite Λ, but Figure 1 is strong support for a critical behaviour (52) in the limit $\Lambda^2 \to \infty$. It is worthwhile to mention that nothing particular happens at the expected pole $\lambda_b = -\frac{1}{72} = 0.014$ of Borel resummation! Since $1 + \mathcal{Y} = 0$ (within numerical error bounds) in the phase $\lambda < \lambda_c$, we see from (44) that higher N-point functions will not exist for $\lambda < \lambda_c$. Most surprisingly, as we discuss at the end of section 4.2, a key property of the Schwinger 2-point function $S_c(x, y)$ in position space is precisely realised in $[\lambda_c, 0]$, not outside! In fact, as shown in Figure 2, one has in reasonable approximation $G_{ab} = 0$ for $0 \le a, b \le \Lambda_0^2$, where

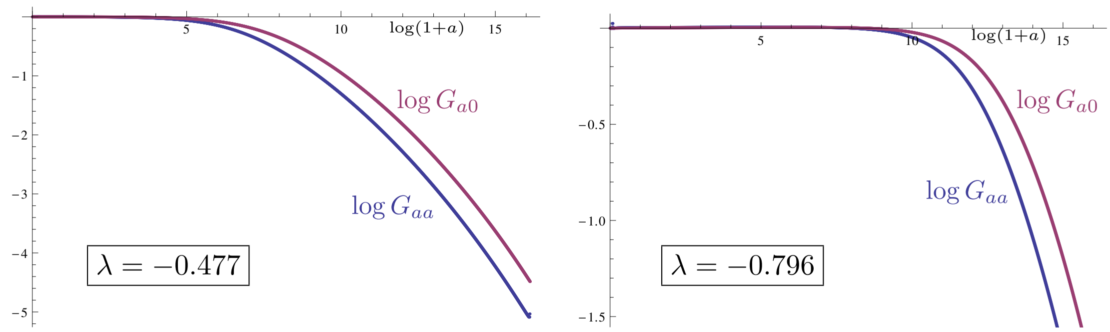


Figure 2: Plots of $\log G_{a0}$ and $\log G_{aa}$ over $\log(1+a)$ for $\lambda < \lambda_c$.

Λ_0^2 increases with $\lambda_c - \lambda > 0$. This could leave the possibility of meaningful higher functions (44) for matrix indices $0 \le a_i \le \Lambda_0^2$, but not for larger indices. Such a picture could have the interpretation of a maximal momentum cut-off of the Euclidean particles in the phase $\lambda < \lambda_c$.

4 Schwinger functions and reflection positivity

In the previous section we have constructed the connected matrix correlation functions $G_{|\underline{q}_1^1 \dots \underline{q}_{N_1}^1| \dots |\underline{q}_1^B \dots \underline{q}_{N_B}^B|}$ of the $(\theta \to \infty)$-limit of ϕ_4^4-theory on Moyal space. These functions arise from the topological expansion (6) of the free energy

$$\log \frac{\mathcal{Z}[J]}{\mathcal{Z}[0]} = \sum_{B=1}^{\infty} \sum_{1 \le N_1 \le \dots \le N_B}^{\infty} \frac{(V\mu^4)^{2-B}}{S_{N_1 \dots N_B}} \sum_{\underline{q}_i^\beta \in \mathbb{N}^2} G_{|\underline{q}_1^1 \dots \underline{q}_{N_1}^1| \dots |\underline{q}_1^B \dots \underline{q}_{N_B}^B|} \prod_{\beta=1}^{B} \frac{1}{N_\beta} \Big(\frac{J_{\underline{q}_1^\beta \underline{q}_2^\beta}}{\mu^3} \cdots \frac{J_{\underline{q}_{N_\beta}^\beta \underline{q}_1^\beta}}{\mu^3} \Big). \tag{53}$$

Since $\lim_{V\mu^4 \to \infty} G_{|\underline{q}_1^1 \dots \underline{q}_{N_1}^1| \dots |\underline{q}_1^B \dots \underline{q}_{N_B}^B|}$ is finite, the limit $\lim_{V \to \infty} \frac{1}{V\mu^4} \log \frac{\mathcal{Z}[J]}{\mathcal{Z}[0]}$ of the naturally expected free energy density removes (in addition to the removal of higher-genus contributions) all contributions from $B \ge 2$. As shown in previous sections, this planar limit is an exactly solvable and (without any doubt) non-trivial matrix model.

4.1 Schwinger functions

We are interested here in another limit to Schwinger functions [Sch59] in position space. For this end we revert the matrix representation (24) and take the infinite volume limit $V\mu^4 \to \infty$, where we carefully have to pass to densities. Absolute position $x \in \mathbb{R}^4$ have no meaning, only μx can be used. This means that we consider (up to a factor discussed below)

$$\big\langle \phi(\mu x_1)\dots\phi(\mu x_N)\big\rangle \equiv \sum_{\underline{m}_1,\underline{n}_1,\dots,\underline{m}_N,\underline{n}_N\in\mathbb{N}^2} f_{\underline{m}_1\underline{m}_2}(\mu x_1)\cdots f_{\underline{m}_N\underline{m}_N}(\mu x_N)\big\langle \Phi_{\underline{m}_1\underline{n}_1}\dots\Phi_{\underline{m}_N\underline{n}_N}\big\rangle,$$

where the matrix correlation functions $\big\langle \Phi_{\underline{m}_1\underline{n}_1}\dots\Phi_{\underline{m}_N\underline{n}_N}\big\rangle$ are obtained by derivatives of (53) with respect to $J_{\underline{m}_1\underline{n}_1},\dots,J_{\underline{m}_N\underline{n}_N}$. We shall see in this section that the additional index summation over $\underline{m}_i,\underline{n}_i\in\mathbb{N}^2$ gives a meaningful limit only if we redefine the volume factor in the free energy density to $\mathcal{F}=\frac{1}{(V\mu^4)^2}\log\frac{\mathcal{Z}[J]}{\mathcal{Z}[0]}$. The occurrence of V^2 as the volume has its origin in the spectral geometry of the Moyal plane with harmonic propagation [GW13a, GW12a] which has a finite volume $(\frac{V}{\Omega})^2$.

Definition 10 *The connected Schwinger functions associated with the action (26) are*

$$\begin{aligned}
&\mu^N S_c(\mu x_1,\dots,\mu x_N)\\
&:=\lim_{V\mu^4\to\infty}\sum_{\underline{m}_1,\underline{n}_1,\dots,\underline{m}_N,\underline{n}_N\in\mathbb{N}^2} f_{\underline{m}_1\underline{n}_1}(\mu x_1)\cdots f_{\underline{m}_N\underline{n}_N}(\mu x_N)\frac{\mu^{4N}\partial^N\mathcal{F}[J]}{\partial J_{\underline{m}_1\underline{n}_1}\dots\partial J_{\underline{m}_N\underline{n}_N}}\Big|_{J=0}, \qquad (54)\\
&\mathcal{F}[J]:=\frac{1}{64\pi^2V^2\mu^8}\log\left(\frac{\int\mathcal{D}[\Phi]\;e^{-S[\Phi]+V\sum_{\underline{a},\underline{b}\in\mathbb{N}^2}\Phi_{\underline{a}\underline{b}}J_{\underline{b}\underline{a}}}}{\int\mathcal{D}[\Phi]\;e^{-S[\Phi]}}\right)_{\substack{Z\mu^2_{bare}\mapsto\mu^2\\ Z\mapsto(1+\mathcal{Y})}},
\end{aligned}$$

where $S[\Phi]$ is given by (27) and $f_{\underline{m}\underline{n}}$ by (24). By $(\quad)_{\substack{Z\mu^2_{bare}\mapsto\mu^2\\ Z\mapsto(1+\mathcal{Y})}}$ we symbolise the renormalisation of sec. 3.2.

Note that by construction the J-derivatives, and hence the Schwinger functions, are fully symmetric in $\mu x_1,\dots,\mu x_N$. Applying the J-derivatives the the topological expansion (53) into J-cycles produces an $f_{\underline{m}\underline{n}}$-cycle for each of the B boundary components:

$$\begin{aligned}
S_c(\mu x_1,\dots,\mu x_N) &= \lim_{V\mu^4\to\infty}\frac{1}{64\pi^2}\sum_{N_1+\dots+N_B=N}\sum_{\underline{q}_i^\beta\in\mathbb{N}^2} G_{|\underline{q}_1^1\dots\underline{q}_{N_1}^1|\dots|\underline{q}_1^B\dots\underline{q}_{N_B}^B|}\\
&\quad\times\sum_{\sigma\in\mathcal{S}_N}\prod_{\beta=1}^B\frac{f_{\underline{q}_1\underline{q}_2}(\mu x_{\sigma(N_1+\dots+N_{\beta-1}+1)})\cdots f_{\underline{q}_{N_\beta}\underline{q}_1}(\mu x_{\sigma(N_1+\dots+N_\beta)})}{V\mu^4N_\beta}.
\end{aligned}$$

We compute the sum over the indices $\underline{q}_i^\beta \in \mathbb{N}^2$ by Laplace-Fourier transform of G. For that we temporarily assume that G has, for every boundary component, a representation as Laplace transform in the total sum of index norms and Fourier transform in differences of index norms. This transform will be reverted in the end so that the analyticity assumption is not necessary (future analytic continuation to Minkowski space would imply representation as Laplace transform):

$$\begin{aligned}
&G_{|\underline{q}_1^1 \dots \underline{q}_{N_1}^1|\dots|\underline{q}_1^B \dots \underline{q}_{N_B}^B|} \\
&= \int_{\mathbb{R}_+^B} d(t^1,\dots,t^B) \int_{\mathbb{R}^{N-B}} d(\omega_1^1,\dots,\omega_{N_1-1}^1,\dots,\omega_1^B,\dots,\omega_{N_B-1}^B) \\
&\qquad \times \mathcal{G}(t^1,\omega_1^1,\dots,\omega_{N_1-1}^1|\dots|t^B,\omega_1^B,\dots,\omega_{N_B-1}^B) \\
&\qquad \times \prod_{\beta=1}^B \exp\Big(-\frac{t^\beta}{\sqrt{V\mu^4}} \sum_{i=1}^{N_\beta} |\underline{q}_i^\beta| + \frac{\mathrm{i}}{\sqrt{V\mu^4}} \sum_{i=1}^{N_\beta-1} \omega_i^\beta (|\underline{q}_i^\beta| - |\underline{q}_{i+1}^\beta|)\Big) \;. \qquad (55)
\end{aligned}$$

Note that the 1-norms $|\underline{q}_i^\beta| = q_{i,1}^\beta + q_{i,2}^\beta$ imply a factorisation of the exponential, $\exp(\dots) = \prod_i \big(z_i^\beta(t^\beta,\vec{\omega}^\beta)\big)^{q_{i,1}^\beta}\big(z_i^\beta(t^\beta,\vec{\omega}^\beta)\big)^{q_{i,2}^\beta}$.

For every boundary component $\beta = 1,\dots,B$, we thus need to compute

$$\begin{aligned}
&\sum_{q_1,\dots,q_{N'}=0}^\infty \frac{f_{q_1 q_2}(\mu\vec{y}_1)\cdots f_{q_{N'} q_1}(\mu\vec{y}_{N'})}{\sqrt{V\mu^4 N'}} z_1^{q_1}\cdots z_{N'}^{q_{N'}} \\
&= 2^{N'} \sum_{q_1,\dots,q_{N'}=0}^\infty e^{-\frac{1}{2}(r_1+\dots+r_{N'})} \frac{L_{q_1}^{q_2-q_1}(r_1)\cdots L_{q_{N'}}^{q_1-q_{N'}}(r_{N'})}{\sqrt{V\mu^4 N'}} (-\tilde{z}_1)^{q_1}\cdots(-\tilde{z})_{N'}^{q_{N'}} \;, \qquad (56)
\end{aligned}$$

where $r_i = \frac{\mu^2|\vec{y}_i|^2}{\sqrt{4V\mu^4}}$ and $\tilde{z}_j = \frac{\vec{y}_{j-1}}{\vec{y}_j}\exp(-\frac{t-\mathrm{i}(\omega_j-\omega_{j-1})}{\sqrt{V\mu^4}})$, with $\vec{y}_i \in \mathbb{C}$, $\vec{y}_0 \equiv \vec{y}_{N'}$ and $\omega_0 = \omega_{N'} \equiv 0$. One has

Lemma 11 ([GW13b]) *For $|\tilde{z}_j| < 1$, a cyclic product of Laguerre polynomials (i.e. $N' + j \equiv j$) is summed to*

$$\sum_{q_1,\dots,q_{N'}=0}^\infty \prod_{j=1}^{N'} (-\tilde{z}_j)^{q_j} L_{q_j}^{q_{j+1}-q_j}(r_j) = \frac{\exp\left(-\frac{\sum_{j,k=1}^{N'} r_j(-\tilde{z}_{k+j})\cdots(-\tilde{z}_{N'+j})}{1-(-\tilde{z}_1)\cdots(-\tilde{z}_{N'})}\right)}{1-(-\tilde{z}_1)\cdots(-\tilde{z}_{N'})} \;. \qquad (57)$$

The denominators in (57) become

$$1-(-\tilde{z}_1)\cdots(-\tilde{z}_{N'}) = 1-(-1)^{N'}\exp\Big(-\frac{N't}{\sqrt{V\mu^4}}\Big) \xrightarrow{V\mu^4\to\infty} \begin{cases} \frac{N't}{\sqrt{V\mu^4}} & \text{for } N' \text{ even}\;, \\ 2 & \text{for } N' \text{ odd}\;. \end{cases}$$

Together with the prefactor $\frac{1}{\sqrt{V\mu^4 N'}}$, the sum (56) converges for $V\mu^4 \to \infty$ to zero if N' is odd, whereas if N' is even the limit is non-zero and finite, depending only on t but no longer on ω_j. Recombining the two $\mathbb{N}^2$-components we produce factors $\dfrac{\exp\big(-\frac{\|\mu X\|^2}{2N't}\big)}{(N't)^2} = \displaystyle\int_{\mathbb{R}^4} \frac{dp}{4\pi^2\mu^4} e^{-\mathrm{i}\langle \frac{p}{\mu},\mu X\rangle} \exp\Big(-\frac{N't\|p\|^2}{2\mu^2}\Big)$ for every even N'. Altogether we arrive at

$$\begin{aligned}
&\lim_{V\mu^4\to\infty} \sum_{\underline{q}_1,\dots,\underline{q}_{N'}\in\mathbb{N}^2} \frac{f_{\underline{q}_1\underline{q}_2}(\mu x_1)\cdots f_{\underline{q}_{N'}\underline{q}_1}(\mu x_{N'})}{V\mu^4 N'} z_1^{q_{1,1}+q_{1,2}}\cdots z_{N'}^{q_{N',1}+q_{N',2}} \\
&= \begin{cases} \dfrac{4^{N'}}{N'} \displaystyle\int_{\mathbb{R}^4} \frac{dp}{4\pi^2\mu^4} e^{-\mathrm{i}\langle \frac{p}{\mu},\mu(x_1-x_2+\dots+x_{N'-1}-x_{N'})\rangle} \exp\Big(-\frac{N't\|p\|^2}{2\mu^2}\Big) & \text{for } N' \text{ even },\\ 0 & \text{for } N' \text{ odd }. \end{cases}
\end{aligned}$$

Integration of $\mathcal{G}(t^1,\omega_1^1,\dots,\omega_{N_1-1}^1|\dots|t^B,\omega_1^B,\dots,\omega_{N_B-1}^B)$ against $\exp(-\frac{N_\beta t^\beta\|p^\beta\|^2}{2\mu^2})$ in (55) returns to the original function $G_{|\underline{q}_1^1\dots\underline{q}_{N_1}^1|\dots|\underline{q}_1^B\dots\underline{q}_{N_B}^B|}$, but with

1. for each β, all $|\underline{q}_i^\beta|$ coincide (no ω-dependence),
2. $\frac{\sum_{i=1}^{N_\beta}|\underline{q}_i^\beta|}{\sqrt{V\mu^4}} = \frac{N_\beta}{2\mu^2}\|p\|^2$, hence $\frac{|\underline{q}_i^\beta|}{\sqrt{V\mu^4}} \stackrel{V\mu^4\to\infty}{\longrightarrow} (1+\mathcal{Y})q = \frac{|p|^2}{2\mu^2}$ in the limit to the integral representation.

We have thus proved [GW13b]:

Theorem 12 *The connected N-point Schwinger functions of the ϕ_4^4-model on extreme Moyal space $\theta\to\infty$ are given by*

$$\begin{aligned}
&S_c(\mu x_1,\dots,\mu x_N) \\
&= \frac{1}{64\pi^2} \sum_{\substack{N_1+\dots+N_B=N\\ N_\beta \text{ even}}} \sum_{\sigma\in\mathcal{S}_N} \Bigg(\prod_{\beta=1}^B \frac{4^{N_\beta}}{N_\beta} \int_{\mathbb{R}^4} \frac{dp_\beta}{4\pi^2\mu^4} e^{\mathrm{i}\big\langle \frac{p_\beta}{\mu}, \sum_{i=1}^{N_\beta}(-1)^{i-1}\mu x_{\sigma(N_1+\dots+N_{\beta-1}+i)}\big\rangle} \Bigg) \\
&\qquad\times G_{\underbrace{\frac{\|p_1\|^2}{2\mu^2(1+\mathcal{Y})},\dots,\frac{\|p_1\|^2}{2\mu^2(1+\mathcal{Y})}}_{N_1}\Big|\dots\Big|\underbrace{\frac{\|p_B\|^2}{2\mu^2(1+\mathcal{Y})},\dots,\frac{\|p_B\|^2}{2\mu^2(1+\mathcal{Y})}}_{N_B}}\;. \qquad (58)
\end{aligned}$$

Some comments:

- Only a restricted sector of the underlying matrix model contributes to position space: All strands of the same boundary component carry the same matrix index.
- Schwinger functions are symmetric and invariant under the full Euclidean group. This comes truly surprising since $\theta\neq 0$ breaks both translation invariance and manifest rotation invariance. The limit $\theta\to\infty$ was expected to make this symmetry violation even worse!
- The most interesting sector is the case where every boundary component has $N_\beta = 2$ indices. It is described by the $(2+\dots+2)$-point functions $G_{\frac{\|p_1\|^2}{2\mu^2(1+\mathcal{Y})}\frac{\|p_1\|^2}{2\mu^2(1+\mathcal{Y})}\big|\dots\big|\frac{\|p_B\|^2}{2\mu^2(1+\mathcal{Y})}\frac{\|p_B\|^2}{2\mu^2(1+\mathcal{Y})}}$.

- This sector describes the propagation and interaction of B particles without any momentum exchange. This is acceptable for a 2D-model. In four dimensions, absence of momentum transfer is a sign of *triviality*.
- However, typical triviality proofs rely on clustering, analyticity in Mandelstam representation or absence of bound states. All this needs verification.

It is already clear that clustering is maximally violated. Looking for instance at the (2+2)-sector, we have

$$\lim_{\mu a \to \infty} S_c^{2+2}(\mu x_1, \mu x_2, \mu(x_3 + a), \mu(x_4 + a)) \\ = \int \frac{dp\, dq}{4\pi^6 \mu^4} \, G_{\frac{\|p\|^2}{2\mu^2(1+\mathcal{Y})} \frac{\|p\|^2}{2\mu^2(1+\mathcal{Y})} \big| \frac{\|q\|^2}{2\mu^2(1+\mathcal{Y})} \frac{\|q\|^2}{2\mu^2(1+\mathcal{Y})}} e^{\mathrm{i}\langle p, x_1 - x_2\rangle + \mathrm{i}\langle q, x_3 - x_4\rangle} \tag{59}$$

independent of the distance between $\{x_1, x_2\}$ on one hand and $\{x_3, x_4\}$ on the other hand. Absence of clustering means that the vacuum state (of a hypothetical continuation to a Wightman theory) is not a pure state. Non-pure states can be decomposed into pure states which describe different topological sectors.

Let us give an intuitive explanation why the limit $\theta \to \infty$ of extreme noncommutativity is so close to an ordinary field theory expected for $\theta \to 0$. The interaction term in momentum space

$$\frac{\lambda}{4} \int_{(\mathbb{R}^4)^4} \Big(\prod_{i=1}^{4} \frac{dp_i}{(2\pi)^4} \Big) \delta(p_1 + \cdots + p_4) \exp\Big(\mathrm{i} \sum_{i<j} \langle p_i, \Theta p_j \rangle\Big) \prod_{i=1}^{4} \hat{\phi}(p_i)$$

leads to the Feynman rule $\lambda \exp\big(\mathrm{i} \sum_{i<j} \langle p_i, \Theta p_j \rangle\big)$, plus momentum conservation. For $\theta \to \infty$, this converges to zero almost everywhere by the Riemann-Lebesgue lemma, *unless* p_i, p_j *are linearly dependent*. This case of linearly dependent momenta might be protected for topological reasons, and these are precisely the boundary components $B > 1$ which guarantee full Lebesgue measure!

4.2 Reflection positivity

Under conditions identified by Osterwalder-Schrader [OS73, OS75], Schwinger functions [Sch59] of a Eulidean quantum field theory permit an analytical continuation to Wightman functions [Wig56, SW64] of a true relativistic quantum field theory. In simplified terms, the reconstruction theorem of Osterwalder-Schrader for a field theory on $\mathbb{R}^d$ says:

Theorem 13 ([OS73, OS75]) *Assume the Schwinger functions* $S(x_1, \ldots, x_N)$ *satisfy*

0. *growth conditions,*
1. *Euclidean covariance,*

2. reflection positivity: for each tuple $(f_0,\dots,f_K)$ of test functions $f_N\in\mathcal{S}(\mathbb{R}^{Nd})$,

$$\sum_{M,N=0}^{K}\int dx\,dy\;S(x_1,\dots,x_N,y_1,\dots,y_M)\overline{f_N(x_1^r,\dots,x_N^r)}f_M(y_1,\dots,y_M)\geq 0\;,$$

where $(x^0,x^1,\dots x^{d-1})^r:=(-x^0,x^1,\dots x^{d-1})$,

3. permutation symmetry.

Then the $S(\xi_1,\dots\xi_{N-1})\big|_{\xi_i^0>0}$, with $\xi_i=x_i-x_{i+1}$, are Laplace-Fourier transforms of Wightman functions in a relativistic quantum field theory. If in addition the $S(x_1,\dots,x_N)$ satisfy

4. clustering

then the Wightman functions satisfy clustering, too.

Representation as Laplace transform in ξ^0 requires analyticity in $\mathrm{Re}(\xi^0)>0$. For the Schwinger 2-point function (58), such analyticity in ξ^0 is a corollary of analyticity of the function $a\mapsto G_{aa}$ in $\mathbb{C}\setminus]-\infty,0]$. We will show that analyticity and reflection positivity boil down to *Stieltjes functions*, i.e. functions $f:\mathbb{R}_+\to\mathbb{R}$ which have a representation as a Stieltjes transform (see [Wid38])

$$f(x)=c+\int_0^\infty\frac{d(\rho(t))}{x+t}\;,\qquad c=f(\infty)\geq 0\;,\tag{60}$$

where ρ is non-negative and non-decreasing. We prove:

Proposition 14 *The Schwinger function $S_c(\mu\xi)=\displaystyle\int_{\mathbb{R}^4}\frac{dp}{(2\pi\mu)^4}e^{\mathrm{i}p\xi}G_{\frac{\|p\|^2}{2\mu^2(1+\mathcal{Y})}\frac{\|p\|^2}{2\mu^2(1+\mathcal{Y})}}$ identified in (58) is the analytic continuation of a Wightman 2-point function if and only if $a\mapsto G_{aa}$ is Stieltjes.*

Proof. This is verified by explicit calculation. If $a\mapsto G_{aa}$ is Stieltjes, we have in terms of $\omega_{\vec p}(t):=\sqrt{\vec p^2+2\mu^2(1+\mathcal{Y})t}$

$$\begin{aligned}
S_c(\mu\xi)\big|_{\xi^0>0}&=\int_{\mathbb{R}^3}\frac{d\vec p}{(2\pi\mu)^3}\int_{-\infty}^\infty\frac{dp^0}{2\pi\mu}e^{\mathrm{i}p^0\xi^0+\mathrm{i}\vec p\cdot\vec\xi}\int_0^\infty\frac{d\rho(t)}{t+\frac{(p^0)^2+\vec p^2}{2\mu^2(1+\mathcal{Y})}}\\
&=2\mu(1+\mathcal{Y})\int_{\mathbb{R}^3}\frac{d\vec p\,e^{\mathrm{i}\vec p\cdot\vec\xi}}{(2\pi\mu)^3}\int_0^\infty\frac{d\rho(t)}{2\omega_{\vec p}(t)}\int_{-\infty}^\infty\frac{dp^0}{2\pi\mathrm{i}}\Big(\frac{e^{\mathrm{i}p^0\xi^0}}{p^0-\mathrm{i}\omega_{\vec p}(t)}-\frac{e^{\mathrm{i}p^0\xi^0}}{p^0+\mathrm{i}\omega_{\vec p}(t)}\Big)\\
&=2\mu(1+\mathcal{Y})\int_0^\infty\frac{d\rho(t)}{2\omega_{\vec p}(t)}\int_{\mathbb{R}^3}\frac{d\vec p\,e^{-\xi^0\omega_{\vec p}(t)+\mathrm{i}\vec p\cdot\vec\xi}}{(2\pi\mu)^3}\\
&=\int_0^\infty\frac{2(1+\mathcal{Y})\,d\rho(t)}{\mu^4}\int_0^\infty dq^0\int_{\mathbb{R}^3}d\vec q\,\hat W_t(q)e^{-q^0\xi^0+\mathrm{i}\vec q\cdot\vec x}\;,
\end{aligned}\tag{61a}$$

$$\hat W_t(q):=\frac{\theta(q^0)}{(2\pi)^3}\delta\Big(\frac{(q^0)^2-\vec q^2-2\mu^2(1+\mathcal{Y})t}{\mu^2}\Big)\;.\tag{61b}$$

The step from the second to third line is the residue theorem. We observe that $\hat{W}_t(q)$ is precisely the Källén-Lehmann spectral representation [Käl52, Leh54] of a Wightman 2-point function. $\square$

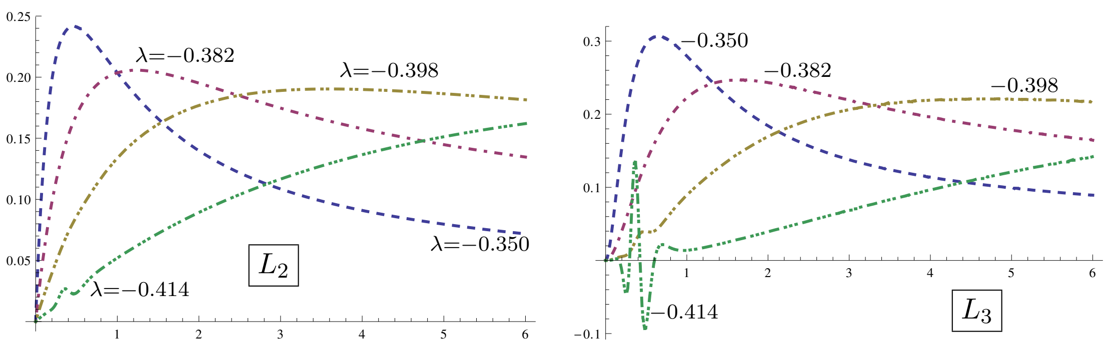
Remarkably, the Stieltjes property can be tested by purely real conditions:

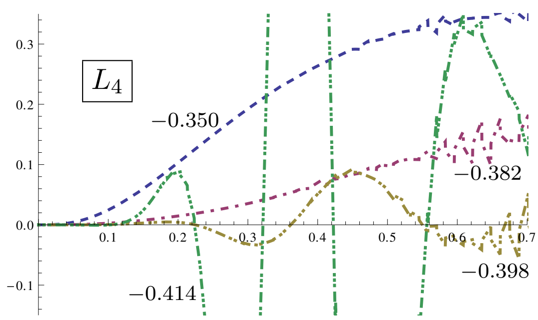
Theorem 15 (Widder [Wid38]) *A function $f : \mathbb{R}_+ \to \mathbb{R}$ is Stieltjes iff it is smooth, non-negative and satisfies $L_{k,t}[f(\bullet)] \geq 0$, where*

$$L_{k,t}[f(\bullet)] := \frac{(-t)^{k-1}}{c_k} \frac{d^{2k-1}}{dt^{2k-1}} \big(t^k f(t)\big)\;, \qquad c_1 = 1,\; c_{k>1} = k!(k-2)!\;.$$

In that case, the measure is recovered by $\rho'(t) = \lim_{k\to\infty} L_{k,t}[f(\bullet)]$ (weakly and almost everywhere).

The perturbatively established anomalous dimension $\eta = -2\lambda$ implies that $a \mapsto G_{aa}$ cannot be Stieltjes for $\lambda > 0$. The restriction to negative coupling constant is reminiscent of the planar wrong-sign $\lambda\phi^4_4$-model [tHo82, Riv83]. Recall that our matrix model also reduces to the planar sector, but as result of the infinite volume limit and not by hand. We nonetheless keep a non-trivial topology in form of $B \geq 1$ boundary components. Moreover, we have an exact solution for $S(x_1, \ldots, x_N)$, not only an existence proof.





- based on interpolation of discrete data, noisy for $k \geq 4$
- Stieltjes property clearly violated for $\lambda < \lambda_c$

Figure 3: Widder's criteria $L_{k,a}[G_{\bullet\bullet}] := \frac{(-a)^{k-1}}{k!(k-2)!} \frac{d^{2k-1}}{da^{2k-1}}(a^k G_{aa}) \geq 0$ for $\lambda \approx \lambda_c$.

Whether or not $a \mapsto G_{aa}$ is a Stieltjes function for $\lambda < 0$ is a highly interesting question. A first idea can be obtained by computer simulations, see sec. 3.5. We show in Figure 3 interpolation results for λ near the critical coupling constant. We find clear evidence that $a \mapsto G_{aa}$ is not a Stieltjes function for $\lambda < \lambda_c$, where $\lambda_c \approx -0.396$ locates the discontinuity of $\mathcal{Y}'(\lambda)$. For $\lambda \in [\lambda_c, 0]$ the results are not conclusive (as k is too small). Since G_{aa} and G_{a0} show a very similar behaviour (see e.g. Fig. 2), the functions $L_{k,t}[G_{\bullet 0}]$ (which are easy to compute) give some indication about $L_{k,t}[G_{\bullet\bullet}]$ (which we are interested in). From (43) one can prove the following identity [GW14]:

$$\frac{(\log G_{a0})^{(\ell)}}{(\ell-1)!} = \frac{(-1)^\ell}{(1+a)^\ell} + (-1)^\ell \operatorname{sign}(\lambda)\, \mathcal{H}_0^\Lambda\Big[\sin\big(\ell\tau_a(\bullet)\big)\Big(\frac{\sin\tau_a(\bullet)}{|\lambda|\pi\bullet}\Big)^\ell\Big]\,. \qquad (62)$$

The resulting integrated 'mass densities' $\tilde{\rho}_k(m^2) = \int_0^{m^2} dt\ L_{k,t}[G_{\bullet 0}]$ are shown in Figure 4. We find clear evidence for a mass gap, $\lim_{k\to\infty} \tilde{\rho}_k(\mu^2) = 0$ for

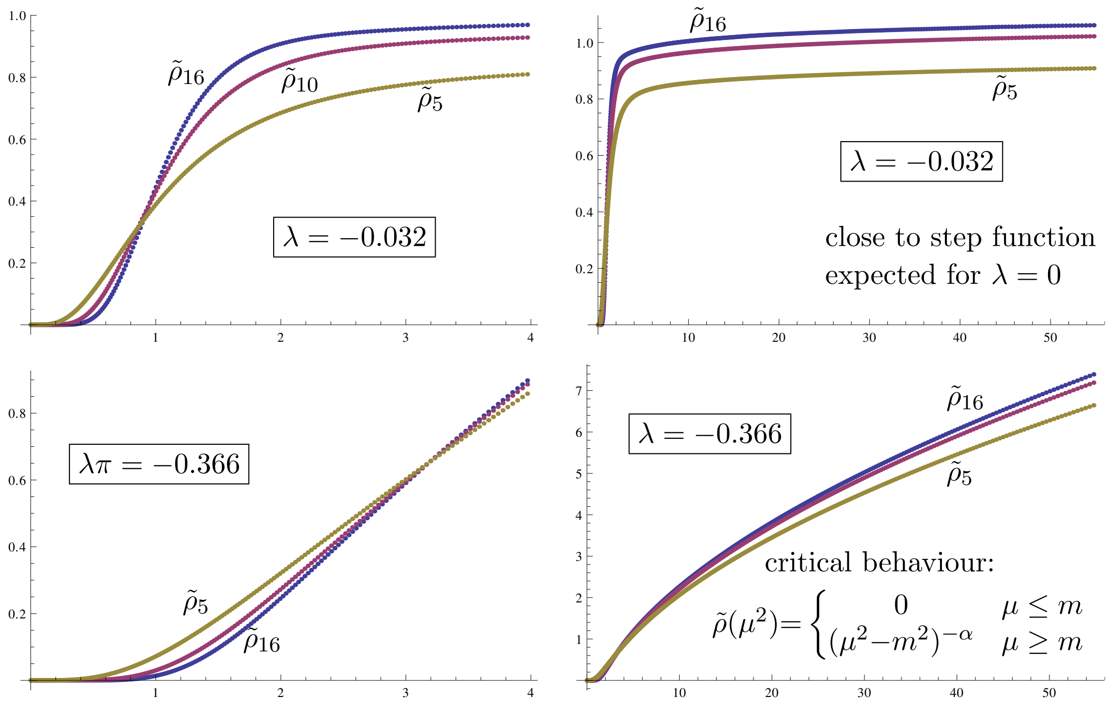


Figure 4: $\tilde{\rho}_k(m^2) = \int_0^{m^2} dt\ L_{k,t}[G_{\bullet 0}]$ as approximation for the mass density of $a \mapsto G_{aa}$. In each row, the left picture is zoomed into small μ, showing evidence for a mass gap. The right pictures show the global behaviour, close to a step function for $\lambda \nearrow 0$, close to criticality for $\lambda \searrow \lambda_c$.

$0 \leq \mu^2 \leq m^2$. For $\lambda \nearrow 0$ the integrated mass density approaches (as expected) a step function, whereas for $\lambda \searrow \lambda_c$ we notice a power-law behaviour typical for critical phenomena. In particular, for $\lambda_c < \lambda < 0$ there is no further gap in the support of $\tilde{\rho}'$, which signals scattering right away from m^2 (not only from the two-particle threshold on). We interpret this as scattering of a massive particle with an infrared cloud. This scattering would be a remnant of the underlying non-trivial matrix model before the projection to diagonal matrices.

4.3 Summary

We have shown that the ϕ^4_4-model on noncommutative Moyal space, considered in the limit $\theta \to \infty$ of extreme noncommutativity, is an exactly solvable and non-trivial matrix model. Euclidean symmetry is violated in the beginning, but we identified a limit which projects to diagonal matrices where Euclidean symmetry is restored. One would not expect that such a brutal projection can respect any quantum field theory axioms. Surprisingly, the first consistency checks, positivity of the lowest Widder criteria $L_{k,t}[G_{\bullet\bullet}]$, are passed for the only interesting interval $[\lambda_c, 0]$ of the coupling constant!

If these miracles continue and all Osterwalder-Schrader axioms (except for clustering) hold, we would get a relativistic quantum field theory in four dimensions. This theory is somewhat strange as 'particles' keep their momenta in interaction processes. Nevertheless, the theory is not completely trivial. We find scattering remnants from the noncommutative geometrical (i.e. matricial) substructure. Only the external matrix indices are put 'on-shell', internally all degrees of freedom contribute.

We have seen that clustering is maximally violated. The interaction is insensitive to positions in different boundary components. In particular, 'particles' are never asymptotically free.

A Schwinger-Dyson equations for $B = 2$

We find for the (1+1)- and (2+2)-point functions

$$
\begin{aligned}
G_{|a|c|} &= -\frac{\lambda}{E_a+E_a}\Big(\frac{1}{V}\sum_{p\in I}\Big(G_{|ap|}G_{|a|c|} - \frac{G_{|p|c|}-G_{|a|c|}}{E_p-E_a}\Big) - \frac{G_{|cc|}-G_{|ac|}}{E_c-E_a}\Big) && \Big\}\,(63a)\\
&\quad - \frac{\lambda}{V^2(E_a+E_a)}\Big(3G_{|a|a|}G_{|a|c|} + G_{|a|cac|} + G_{|c|aaa|} + \frac{1}{V}\sum_{n\in I} G_{|a|c|an|}\Big) && \Big\}\,(63b)\\
&\quad - \frac{\lambda}{V^4(E_a+E_a)}G_{|a|a|a|c|}\;, && \Big\}\,(63c)
\end{aligned}
$$

$$
\begin{aligned}
&G_{|ab|cd|}\\
&= -\frac{\lambda}{E_a+E_b}\Big(\frac{1}{V}\sum_{p\in I}\Big(\big(G_{|ap|}G_{|ab|cd|}+G_{|ab|}G_{|ap|cd|}\big) - \frac{G_{|pb|cd|}-G_{|ab|cd|}}{E_p-E_a}\Big) \\
&\qquad +G_{|ab|}\big(G_{|cacd|}+G_{|dadc|}\big) - \frac{G_{|cbcd|}-G_{|cbad|}}{E_c-E_a} - \frac{G_{|dbdc|}-G_{|dbac|}}{E_d-E_a}\Big) && \Big\}\,(64a)\\
&\quad -\frac{\lambda}{V^2(E_a+E_b)}\Big(G_{|a|a|}G_{|ab|cd|} + G_{ab}G_{|a|a|cd|} + \frac{1}{V}\sum_{n\in I} G_{|an|ab|cd|} \\
&\qquad +G_{|cd|aaab|}+G_{|cd|baba|}+G_{|ab|cacd|}+G_{|ab|cddad|} - \frac{G_{|b|a|cd|}-G_{|b|b|cd|}}{E_b-E_a}\Big) && \Big\}\,(64b)\\
&\quad - \frac{\lambda}{V^4(E_a+E_b)}G_{|a|a|ab|cd|}\;. && \Big\}\,(64c)
\end{aligned}
$$

These are basic functions which are not simplified by reality. As before, (63a) and (64a) preserve the genus, whereas $g \mapsto g{+}1$ in (63b)+(64b) and $g \mapsto g{+}2$ in (63c)+(64c). The higher $(N_1{+}N_2)$-point functions with one $N_i \geq 3$ simplify by reality to universal recursion formulae. For N_i odd we have

$$
\begin{aligned}
&G_{|b_0\dots b_{2l}|c_1\dots c_{N-2l-1}|} \\
&= -\lambda \sum_{k=1}^{N-2l-1} \frac{G_{|c_1\dots c_{k-1}b_0b_1\dots b_{2l}c_kc_{k+1}\dots c_{N-2l-1}|} - G_{|c_1\dots c_{k-1}c_kb_1\dots b_{2l}b_0c_{k+1}\dots c_{N-2l-1}|}}{(E_{b_1}-E_{b_{2l}})(E_{b_0}-E_{c_k})} \\
&- \lambda \sum_{j=1}^{l} \frac{G_{|b_0b_1\dots b_{2j-2}|c_1\dots c_{N-2l-1}|}G_{|b_{2j-1}b_{2j}\dots b_{2l}|} - G_{|b_{2j-1}b_1\dots b_{2j-2}|c_1\dots c_{N-2l-1}|}G_{|b_0b_{2j}\dots b_{2l}|}}{(E_{b_1}-E_{b_{2l}})(E_{b_0}-E_{b_{2j-1}})} \\
&- \lambda \sum_{j=1}^{l} \frac{G_{|b_0b_1\dots b_{2j-1}|}G_{|b_{2j}b_{2j+1}\dots b_{2l}|c_1\dots c_{N-2l-1}|} - G_{|b_{2j}b_1\dots b_{2j-1}|}G_{|b_0b_{2j+1}\dots b_{2l}|c_1\dots c_{N-2l-1}|}}{(E_{b_1}-E_{b_{2l}})(E_{b_0}-E_{b_{2j}})} \\
&- \frac{\lambda}{V^2} \sum_{k=1}^{2l} \frac{G_{|b_0b_1\dots b_{k-1}|b_kb_{k+1}\dots b_{2l}|c_1\dots c_{N-2l-1}|} - G_{|b_kb_1\dots b_{k-1}|b_0b_{k+1}\dots b_{2l}|c_1\dots c_{N-2l-1}|}}{(E_{b_1}-E_{b_{2l}})(E_{b_0}-E_{b_k})}\;.
\end{aligned}
\tag{65}
$$

The last line increases the genus and is absent in $G^{(0)}_{|b_0b_1\dots b_{2l}|c_1\dots c_{N-2l-1}|}$. For N_i even one finds

$$
\begin{aligned}
&G_{|ab_1\dots b_{2l-1}|c_1\dots c_{N-2l}|} \\
&= -\lambda \sum_{j=1}^{l-1} \frac{G_{|b_1\dots b_{2j-1}a|c_1\dots c_{N-2l}|}G_{|b_{2j}b_{2j+1}\dots b_{2l-1}|} - G_{|b_1\dots b_{2j-1}b_{2j}|c_1\dots c_{N-2l}|}G_{|ab_{2j+1}\dots b_{2l-1}|}}{(E_{b_1}-E_{b_{2l-1}})(E_a-E_{b_{2j}})} \\
&- \lambda \sum_{j=1}^{l-1} \frac{G_{|b_1\dots b_{2j-1}a|}G_{|b_{2j}b_{2j+1}\dots b_{2l-1}|c_1\dots c_{N-2l}|} - G_{|b_1\dots b_{2j-1}b_{2j}|}G_{|ab_{2j+1}\dots b_{2l-1}|c_1\dots c_{N-2l}|}}{(E_{b_1}-E_{b_{2l-1}})(E_a-E_{b_{2j}})} \\
&- \lambda \sum_{k=1}^{N-2l} \frac{G_{|c_1\dots c_{k-1}ab_1\dots b_{2l-1}c_kc_{k+1}\dots c_{N-2l}|} - G_{|c_1\dots c_{k-1}c_kb_1\dots b_{2l-1}ac_{k+1}\dots c_{N-2l}|}}{(E_{b_1}-E_{b_{2l-1}})(E_a-E_{c_k})} \\
&- \frac{\lambda}{V^2} \sum_{k=1}^{2l-1} \frac{G_{|b_1\dots b_{k-1}a|b_kb_{k+1}\dots b_{2l-1}|c_1\dots c_{N-2l}|} - G_{|b_1\dots b_{k-1}b_k|ab_{k+1}\dots b_{2l-1}|c_1\dots c_{N-2l}|}}{(E_{b_1}-E_{b_{2l-1}})(E_a-E_{b_k})}\;.
\end{aligned}
\tag{66}
$$

Again, the last line increases the genus and is absent in $G^{(0)}_{|b_0b_1\dots b_{2l-1}|c_1\dots C_{N-2l}|}$.

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