

Solvable limits of a 4D noncommutative QFT

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Abstract

In previous work we have shown that the $(\theta \rightarrow \infty)$ -limit of ϕ_4^4 -quantum field theory on noncommutative Moyal space is an exactly solvable matrix model. In this paper we translate these results to position space. We show that the Schwinger functions are symmetric and invariant under the full Euclidean group. The Schwinger functions only depend on matrix correlation functions at coinciding indices per topological sector, and clustering is violated. We prove that Osterwalder-Schrader reflection positivity of the Schwinger two-point function is equivalent to the question whether the diagonal matrix two-point function is a Stieltjes function. Numerical investigations suggest that this can at best be expected for the wrong sign of the coupling constant. The corresponding Wightman functions would describe particles which interact without momentum transfer. The theory differs from a free theory by the presence of non-trivial topological sectors.

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1 Summary of previous work

Years ago we have introduced in [1] a quantum field theory model on four-dimensional Moyal space which is defined by the following action functional for a real scalar field ϕ :

$$S[\phi] = 64\pi^2 \int d^4x \left(\frac{Z}{2} \phi(-\Delta + \Omega_{bare}^2 \|2\Theta^{-1}x\|^2 + \mu_{bare}^2) \phi + \frac{\lambda_{bare} Z^2}{4} \phi \star \phi \star \phi \star \phi \right)(x). \quad (1)$$

Here Θ is a skew-symmetric 4×4 -matrix which defines the Moyal product

$$(f \star g)(x) = \int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{dy dk}{(2\pi)^d} f(x + \frac{1}{2}\Theta k) g(x + y) e^{i(k,y)}, \quad f, g \in \mathcal{S}(\mathbb{R}^d). \quad (2)$$

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We have proved in [1] that the Euclidean quantum field theory arising from (1) is perturbatively renormalisable. This means the following: One introduces a momentum cut-off Λ and normalises four (relevant and marginal) correlation functions to $(\lambda, \Omega, 1, \mu^2)$ independently of Λ . Then one proves that the parameters $(\lambda_{bare}, \Omega_{bare}, Z, \mu_{bare})$ in (1) are functions of $(\lambda, \Omega, \mu, \Lambda)$ in such a way that all correlation functions of the model, considered as functions of $(\lambda, \Omega, \mu, \Lambda)$ and as formal power series in λ , are finite for $\Lambda \rightarrow \infty$ order by order in λ .

A key observation is that $\Omega_{bare} = \Omega = 1$ is a fixed point of (1). At this fixed point a miracle occurs: As shown by Disertori-Gurau-Magnen-Rivasseau [2] order by order in perturbation theory, $\lim_{\Lambda \rightarrow \infty} \lambda_{bare}(\lambda, \mu, \Lambda)$ differs from λ only by a finite ratio, i.e. the β -function vanishes. This is in sharp contrast with usual ϕ_4^4 -model in which $\lambda_{bare}(\lambda, \mu, \Lambda)$ develops a singularity, the Landau pole, already at finite Λ .

Vanishing of the β -function is often a sign of integrability. After initial steps [3] which pointed into the right direction, we have rigorously proved in [4] that a natural scaling limit of the Euclidean field theory associated with (1) is exactly solvable¹. Key ingredients in this proof are the formulation as a matrix model, the use of Ward identities in Schwinger-Dyson equations and the theory of singular integral equations of Carleman type. We have proved that all correlation functions of the model are computable in terms of the solution (which exists by the Schauder fixed point theorem) of a non-linear integral equation for a smooth, positive, monotonously decreasing function on \mathbb{R}_+ that vanishes with all derivatives at ∞ .

The passage from (1) to a matrix model is achieved by the expansion

$$\phi(x) =: \sum_{\underline{m}, \underline{n} \in \mathbb{N}^2} \phi_{\underline{m}\underline{n}} f_{\underline{m}\underline{n}}(x), \quad f_{\underline{m}\underline{n}}(x) := f_{m_1 n_1}(x^0, x^1) f_{m_2 n_2}(x^2, x^3), \quad (3)$$

where $x = (x^0, x^1, x^2, x^3) \in \mathbb{R}^4$, $\underline{m} = (m_1, m_2) \in \mathbb{N}^2$ and

$$f_{\underline{m}\underline{n}}(y^0, y^1) = 2(-1)^m \sqrt{\frac{m!}{n!}} \left(\sqrt{\frac{2}{\theta}} \|y\| e^{in} \right)^{n-m} L_m^{n-m} \left(\frac{2\|y\|^2}{\theta} \right) e^{-\frac{\|y\|^2}{\theta}}, \quad (4)$$

for $y = (y^0, y^1) \equiv \|y\| e^{in} \in \mathbb{R}^2 \equiv \mathbb{C}$. The $L_m^\alpha(t)$ are associated Laguerre polynomials of degree m in t . After an appropriate coordinate transformation in \mathbb{R}^4 , the only non-vanishing components of Θ in (2) are $\Theta_{12} = -\Theta_{21} = \Theta_{34} = -\Theta_{43} =: \theta > 0$. In this situation the matrix basis $f_{\underline{m}\underline{n}}$ satisfies $(f_{\underline{k}\underline{l}} \star f_{\underline{m}\underline{n}})(x) = \delta_{\underline{m}\underline{l}} f_{\underline{k}\underline{n}}(x)$ and $\int_{\mathbb{R}^4} dx f_{\underline{m}\underline{n}}(x) = (2\pi\theta)^2 \delta_{\underline{m}\underline{n}}$. With these identities and with properties of the Laguerre polynomials, (1) takes at $\Omega = 1$, $\lambda_{bare} \equiv \lambda$ and with $|\underline{m}| := m_1 + m_2$ the form

$$S[\phi] = V \left(\sum_{\underline{m}, \underline{n} \in \mathbb{N}^2} E_{\underline{m}} \phi_{\underline{m}\underline{n}} \phi_{\underline{n}\underline{m}} + \frac{S_{int}[\phi]}{V} \right), \quad V := \left(\frac{\theta}{4} \right)^2, \\ E_{\underline{m}} = Z \left(\frac{|\underline{m}|}{\sqrt{V}} + \frac{\mu_{bare}^2}{2} \right), \quad \frac{S_{int}[\phi]}{V} = \frac{Z^2 \lambda}{4} \sum_{\underline{m}, \underline{n}, \underline{k}, \underline{l} \in \mathbb{N}^2} \phi_{\underline{m}\underline{n}} \phi_{\underline{n}\underline{k}} \phi_{\underline{k}\underline{l}} \phi_{\underline{l}\underline{m}}. \quad (5)$$

¹In fact we prove in [4] that general quartic matrix models with action $S = \text{tr}(E\phi^2 + \frac{\lambda}{4}\phi^4)$ are exactly solvable.

In [4] we have studied the matrix representation of the renormalised free energy density²

$$\mathcal{F}[J] := \frac{1}{64\pi^2 V^2 \mu^8} \log \left(\frac{\int \mathcal{D}[\phi] e^{-S[\phi] + V \sum_{\mathbf{a}, \mathbf{b} \in \mathbb{N}^2} \phi_{\mathbf{a}\mathbf{b}} J_{\mathbf{b}\mathbf{a}}}}{\int \mathcal{D}[\phi] e^{-S[\phi]}} \right) \Bigg|_{\substack{Z \mu_{\text{bare}}^2 \mapsto \mu^2 \\ Z \mapsto (1+\mathcal{Y})}} \quad (6)$$

in a natural scaling limit to continuous matrix indices. Here μ^2 is the renormalised squared mass. The unusual wavefunction renormalisation $Z \mapsto (1 + \mathcal{Y})$ for continuous matrix indices simplified the resulting equations enormously, and we keep this convention in the present paper. By (5), the expansion coefficients of \mathcal{F} only depend on the 1-norms $|\underline{m}|$ of the matrix indices so that index summations over \underline{m} restrict to summations over $|\underline{m}|$ with measure $|\underline{m}| + 1$. In [4] we have introduced a cut-off \mathcal{N} in all these summations and coupled it to the volume by

$$\frac{\mathcal{N}}{\sqrt{V}} = \mu^2 (1 + \mathcal{Y}) \Lambda^2, \quad (7)$$

where $\Lambda^2 \in \mathbb{R}_+$ is an integral cut-off which at the end is sent to ∞ . Therefore, the coupled limit $(\mathcal{N}, V) \rightarrow \infty$ converges to a Riemann integral

$$\lim_{(\mathcal{N}, V) \rightarrow \infty} \frac{1}{V} \sum_{\substack{\mathcal{N} \\ \|\underline{m}\| \geq 0}} (|\underline{m}| + 1) f\left(\frac{|\underline{m}|}{\sqrt{V}}\right) = \mu^4 (1 + \mathcal{Y})^2 \int_0^{\Lambda^2} a da f(\mu^2 (1 + \mathcal{Y}) a). \quad (8)$$

Correlation functions in matrix models fall into topological sectors which are distinguished by the genus g of a Riemann surface and the number B of boundary components (punctures, marked points, faces) of the surface. It turns out [4] that in the scaling limit $(\mathcal{N}, V) \rightarrow \infty$ subject to (7), all higher genus contributions with $g \geq 1$ are scaled away, whereas there are reasons to keep the boundary components $B \geq 2$. Every boundary component carries a cycle $J_{\underline{M}^j} := J_{\underline{m}_1 \underline{m}_2} J_{\underline{m}_2 \underline{m}_3} \cdots J_{\underline{m}_{j-1} \underline{m}_j} J_{\underline{m}_j \underline{m}_1}$ of external sources, where $\underline{M}^j = \underline{m}_1 \dots \underline{m}_j$ stands for a collection of j indices in \mathbb{N}^2 . In these notations, the free energy density has a decomposition

$$\mathcal{F}[J] = \frac{1}{64\pi^2} \sum_{K=1}^{\infty} \sum_{\substack{n_1, \dots, n_K=0 \\ n_K \geq 1}}^{\infty} \frac{1}{(V\mu^4)^B} \left(\prod_{j=1}^K \frac{1}{n_j! j^{n_j}} \right) \sum_{\substack{\underline{M}^j \in (\mathbb{N}^2)^j}} \tilde{G}_{|\underline{M}^1| \dots |\underline{M}^1_{n_1}| \dots |\underline{M}^K| \dots |\underline{M}^K_{n_K}|} \prod_{j=1}^K \prod_{i_j=1}^{n_j} \frac{J_{\underline{M}^j_{i_j}}}{\mu^{4j}}. \quad (9)$$

The total number of J -cycles in a function $\tilde{G}_{|\underline{M}^1| \dots |\underline{M}^1_{n_1}| \dots |\underline{M}^K| \dots |\underline{M}^K_{n_K}|}$ is its number $B = n_1 + \dots + n_K$ of boundary components. Defining $N := n_1 + 2n_2 + \dots + Kn_K$ we let $\tilde{G}_{|\underline{M}^1| \dots |\underline{M}^1_{n_1}| \dots |\underline{M}^K| \dots |\underline{M}^K_{n_K}|} =: \mu^N G_{|\underline{M}^1| \dots |\underline{M}^1_{n_1}| \dots |\underline{M}^K| \dots |\underline{M}^K_{n_K}|}$, where G is a dimensionless function as in [4].

We have shown in [4] that in the scaling limit $(V, \mathcal{N}) \rightarrow \infty$ subject to (7), followed by the continuum limit $\Lambda \rightarrow \infty$, the functions \tilde{G} of the matrix indices converge to functions

² \mathcal{F} is related to \mathcal{W} in [4] by $\mathcal{F}[J] = \frac{1}{64\pi^2 V \mu^8} \mathcal{W}[J]$.

of continuous variables³

$$\lim_{\Lambda \rightarrow \infty} \lim_{(V, \mathcal{N}) \rightarrow \infty} \tilde{G}_{|\underline{M}_1^1| \dots |\underline{M}_{n_1}^1| \dots |\underline{M}_1^K| \dots |\underline{M}_{n_K}^K|} =: \tilde{G}(A_1^1 | \dots | A_{n_1}^1 | \dots | A_1^K | \dots | A_{n_K}^K), \quad (10)$$

with $A^j := a_1 a_2 \dots a_j \in \mathbb{R}_+^j$ a cycle of continuous variables $a_i \in \mathbb{R}_+$ related to the discrete cycle $\underline{M}^j \in (\mathbb{N}^2)^n$ by

$$a_i \mu^2 (1 + \mathcal{Y}) := \frac{|\underline{m}_i|}{\sqrt{V}}. \quad (11)$$

That is, write first $\tilde{G}_{|\underline{m}_1| \dots |\dots| \dots |\underline{m}_N|}$ as function of $\frac{|\underline{m}_i|}{\sqrt{V}}$ according to (5) and replace for the limit $\frac{|\underline{m}_i|}{\sqrt{V}} \mapsto a_i \mu^2 (1 + \mathcal{Y})$. The continuous functions $\tilde{G}(A)$ then have a finite non-zero limit $\lim_{\Lambda \rightarrow \infty} \lim_{(V, \mathcal{N}) \rightarrow \infty}$ when expressed in terms of A . Key result of [4] was that all these functions $\tilde{G}(a_1, \dots | \dots | \dots, a_N)$ can be computed explicitly in terms of the solution of the non-linear integral equation for the boundary 2-point function

$$G(a, 0) = \frac{1}{1+a} \exp \left(-\lambda \int_0^a dt \int_0^\infty \frac{dp}{(\lambda \pi p)^2 + \left(t + \frac{1 + \lambda \pi p \mathcal{H}_p[G(\bullet, 0)]}{G(p, 0)} \right)^2} \right). \quad (12)$$

Here, \mathcal{H}_a is the Hilbert transform $\mathcal{H}_a[f(\bullet)] = \frac{1}{\pi} \lim_{\Lambda \rightarrow \infty} \lim_{\epsilon \rightarrow 0} \left(\int_0^{a-\epsilon} + \int_{a+\epsilon}^\Lambda \right) dp \frac{f(p)}{p-a}$. The general 2-point function is then obtained from

$$G(a, b) = \frac{e^{\mathcal{H}_a[\vartheta_b(\bullet)] - \mathcal{H}_0[\vartheta_0(\bullet)]}}{\sqrt{(\lambda \pi a)^2 + \left(b + \frac{1 + \lambda \pi a \mathcal{H}_a[G(\bullet, 0)]}{G(a, 0)} \right)^2}}, \quad \vartheta_b(p) := \arctan_{[0, \pi]} \left(\frac{\lambda \pi p}{b + \frac{1 + \lambda \pi p \mathcal{H}_p[G(\bullet, 0)]}{G(p, 0)}} \right). \quad (13)$$

The interpretation of the prefactor $\frac{1}{\sqrt{V}}$ in (9) remained somewhat obscure in [4]. We argued that at this point the limit $V \rightarrow \infty$ (which would remove everything, or rather would restrict $\mathcal{W} = V\mathcal{F}$ to the sector $B = 1$) should not be taken. In this paper we confirm this by showing that connected Schwinger functions in position space contribute a factor V per boundary component.

2 Schwinger functions

Definition 1 *The connected N -point Schwinger function associated with the action (1) is defined as*

$$\mu^N S_c(\mu x_1, \dots, \mu x_N) := \lim_{V \rightarrow \infty} \sum_{\underline{m}_1, \underline{n}_1, \dots, \underline{m}_N, \underline{n}_N \in \mathbb{N}^2} f_{\underline{m}_1 \underline{n}_1}(x_1) \cdots f_{\underline{m}_N \underline{n}_N}(x_N) \frac{\mu^{4N} \partial^N \mathcal{F}[J]}{\partial J_{\underline{m}_1 \underline{n}_1} \cdots \partial J_{\underline{m}_N \underline{n}_N}} \Big|_{J=0}. \quad (14)$$

³We change the notation of [4]: The functions of continuous variables $G_{a_1 \dots | \dots | \dots a_N}$ in [4] are written as $G(a_1, \dots | \dots | \dots, a_N)$ in the present paper, whereas $G_{|\underline{m}_1| \dots | \dots | \dots |\underline{m}_N|}$ is reserved for discrete matrix indices. Similarly for \tilde{G} .

This definition requires some explanation:

- Schwinger functions are often introduced as the connected part of

$$\langle \varphi(x_1) \cdots \varphi(x_N) \rangle = \frac{\int \mathcal{D}[\phi] \phi(x_1) \cdots \phi(x_N) e^{-S[\phi]}}{\int \mathcal{D}[\phi] e^{-S[\phi]}}. \quad (15)$$

The connected part can be expressed as functional derivative of $\log \mathcal{Z}[J]$ with respect to sources $J(x_1), \dots, J(x_N)$, for $\mathcal{Z}[J] := \int \mathcal{D}[\phi] e^{-S[\phi] + \int dx \phi(x) J(x)}$. However, this can only make sense in a renormalisation prescription, and renormalisation involves discussion of the infinite volume limit. One typically finds that $\log \mathcal{Z}[J]$ is proportional to the volume so that (15) does not make sense for $V \rightarrow \infty$. Accordingly, (14) does not agree with the connected part of $\langle \varphi(x_1) \cdots \varphi(x_N) \rangle$.

Our point of view is that only the free energy density \mathcal{F} makes sense in the infinite volume limit, and that *Schwinger functions are densities*, too. This implies that all quantities of the renormalised theory must be viewed as dimensionless ratios with an appropriate power of the mass scale μ . In particular, $V \rightarrow \infty$ means $(V\mu^4) \rightarrow \infty$ with μ fixed. Absolute positions x lose their meaning in the limit $V \rightarrow \infty$; only μx is meaningful.

- The definition (14) involves a non-naïve wavefunction renormalisation. From (6) we would expect that a Schwinger function which represents $\langle \varphi(x_1) \cdots \varphi(x_N) \rangle_c$ is the derivative $(\frac{1}{V} \frac{\partial}{\partial J})^N$ applied to $\mathcal{F}[J]$, and not $(\mu^4 \frac{\partial}{\partial J})^N$ as imposed in (14). It is not difficult to see that the removal of the factor $(V\mu^4)^{-1}$ corresponds precisely to a wavefunction renormalisation $\sqrt{Z} \mapsto \frac{\sqrt{Z}}{V\mu^4}$.
- The mass scale is introduced by the normalisation $\tilde{G}_{00} = \mu^2$. For the free theory $\lambda = 0$ in (5) we can compute the free energy density exactly:

$$\begin{aligned} \mathcal{F}[J] \Big|_{\lambda=0} &= \frac{1}{64\pi^2} \cdot \frac{1}{V\mu^4} \cdot \frac{1}{2} \cdot \sum_{\underline{m}, \underline{n} \in \mathbb{N}^2} \left(\frac{\mu^4}{Z \left(\frac{|\underline{m}| + |\underline{n}|}{\sqrt{V}} + \mu_{bare}^2 \right)} \right) \frac{J_{\underline{m}\underline{n}}}{\mu^4} \frac{J_{\underline{n}\underline{m}}}{\mu^4} \\ &= \frac{1}{64\pi^2} \cdot \frac{1}{V\mu^4} \cdot \frac{1}{2} \cdot \sum_{\underline{m}, \underline{n} \in \mathbb{N}^2} \left(\frac{\mu^4}{(V\mu^4)^2 Z \left(\frac{|\underline{m}| + |\underline{n}|}{\sqrt{V}} + \mu_{bare}^2 \right)} \right) (V J_{\underline{m}\underline{n}}) (V J_{\underline{n}\underline{m}}). \end{aligned}$$

This shows again that (14) is compatible with $Z = 1$ and $\mu_{bare}^2 = \mu^2$ for the free theory, and that the naïvely expected $(\frac{1}{V} \frac{\partial}{\partial J})^N$ leads to the same result if we let $\sqrt{Z} = \frac{1}{V\mu^4}$ and $\mu_{bare}^2 = \mu^2$.

- Since the Gaussian in (4) distinguishes the origin and the decomposition $f_{\underline{m}\underline{n}}(x) = f_{m_1 n_1}(x^0, x^1) f_{m_2 n_2}(x^2, x^3)$ distinguishes pairs of coordinate directions, the Schwinger functions (14) are, *a priori*, only invariant under the subgroup $SO(2) \times SO(2)$ of the Euclidean group $\mathbb{R}^4 \rtimes SO(4)$.
- In contrast, the Schwinger functions are *fully symmetric in all its arguments*.

The differentiation of (9) with respect to the J 's in (14) is a standard combinatorial problem. For $\underline{M}^j = \underline{m}_1 \dots \underline{m}_j$ we define

$$f_{\underline{M}^j}(x_1, \dots, x_j) := f_{\underline{m}_1 \underline{m}_2}(x_1) f_{\underline{m}_2 \underline{m}_3}(x_2) \cdots f_{\underline{m}_{j-1} \underline{m}_j}(x_{j-1}) f_{\underline{m}_j \underline{m}_1}(x_j). \quad (16)$$

In terms of dimensionless functions $G_{|\underline{M}_1^1| \dots |\underline{M}_{n_1}^1| \dots |\underline{M}_1^K| \dots |\underline{M}_{n_K}^K|} = \mu^{-N} \tilde{G}_{|\underline{M}_1^1| \dots |\underline{M}_{n_1}^1| \dots |\underline{M}_1^K| \dots |\underline{M}_{n_K}^K|}$ for $N := n_1 + 2n_2 + \dots + Kn_K$ we have

$$\begin{aligned}
& S_c(\mu x_1, \dots, \mu x_N) \\
&= \lim_{V \rightarrow \infty} \frac{1}{64\pi^2} \sum_{n_1+2n_2+\dots+Kn_K=N} \frac{1}{(V\mu^4)^{n_1+n_2+\dots+n_K}} \sum_{\underline{M}_{i_j}^j \in (\mathbb{N}^2)^{j}} G_{|\underline{M}_1^1| \dots |\underline{M}_{n_1}^1| \dots |\underline{M}_1^K| \dots |\underline{M}_{n_K}^K|} \\
&\times \sum_{\sigma \in \mathcal{S}_N} \left(\prod_{j=1}^K \frac{1}{j^{n_j}} \right) (f_{\underline{M}_1^1}(x_{\sigma(1)}) \cdots f_{\underline{M}_{n_1}^1}(x_{\sigma(n_1)})) \\
&\quad \times (f_{\underline{M}_1^2}(x_{\sigma(n_1+1)}, x_{\sigma(n_1+2)}) \cdots f_{\underline{M}_{n_2}^2}(x_{\sigma(n_1+2n_2-1)}, x_{\sigma(n_1+2n_2)})) \\
&\quad \times \cdots \times (f_{\underline{M}_1^K}(x_{\sigma(n_1+\dots+(K-1)n_{K-1}+1)}, \dots, x_{\sigma(n_1+\dots+(K-1)n_{K-1}+K)}) \cdots \\
&\quad \quad \quad \times f_{\underline{M}_{n_K}^K}(x_{\sigma(n_1+\dots+Kn_K-(K-1)}, x_{\sigma(n_1+\dots+Kn_K)})) \cdot \quad (17)
\end{aligned}$$

The summation is over all permutations in the symmetric group \mathcal{S}_N . It is much more convenient to write $N = j_1 + \dots + j_B$, where j_β is the length of the β^{th} cycle in $G_{|\underline{M}_1^1| \dots |\underline{M}_{n_1}^1| \dots |\underline{M}_1^K| \dots |\underline{M}_{n_K}^K|}$. The cycles $(1, 2, \dots, \beta - 1)$ contain $N_\beta^- := j_1 + \dots + j_{\beta-1}$ of the N indices, and the β^{th} cycle adds j_β more. With these conventions (17) can be written equivalently as

$$\begin{aligned}
S_c(\mu x_1, \dots, \mu x_N) &= \lim_{V \rightarrow \infty} \frac{1}{64\pi^2} \sum_{j_1+\dots+j_B=N} \sum_{\{\underline{M}_\beta^{j_\beta} \in (\mathbb{N}^2)^{j_\beta}\}_{\beta=1}^B} G_{|\underline{M}_1^{j_1}| \dots |\underline{M}_B^{j_B}|} \\
&\quad \times \sum_{\sigma \in \mathcal{S}_N} \prod_{\beta=1}^B \frac{1}{V\mu^4 j_\beta} f_{\underline{M}_\beta^{j_\beta}}(x_{\sigma(N_\beta^-+1)}, \dots, x_{\sigma(N_\beta^-+j_\beta)}) \cdot \quad (18)
\end{aligned}$$

To proceed we assume that $G_{|\underline{M}_1^1| \dots |\underline{M}_B^{j_B}|}$ has, for each cycle $\underline{M}_\beta^{j_\beta} = \underline{m}_1^\beta \dots \underline{m}_{j_\beta}^\beta$, a representation as Laplace transform in the total norm $\|\underline{M}_\beta^{j_\beta}\| := |\underline{m}_1^\beta| + \dots + |\underline{m}_{j_\beta}^\beta|$ and Fourier transform in neighbored differences. For $\underline{\omega}^j = (\omega_1^j, \dots, \omega_{j-1}^j) \in \mathbb{R}^{j-1}$ let $\langle \underline{\omega}^j, \underline{M}^j \rangle := \sum_{i=1}^{j-1} \omega_i (|\underline{m}_i| - |\underline{m}_{i+1}|)$. We assume

$$\begin{aligned}
G_{|\underline{M}_1^{j_1}| \dots |\underline{M}_B^{j_B}|} &= \int_{\mathbb{R}_+^B} d(t_1, \dots, t_B) \int_{\mathbb{R}^{N-B}} d(\underline{\omega}_1^{j_1}, \dots, \underline{\omega}_B^{j_B}) \mathcal{G}(t_1, \underline{\omega}_1^{j_1} | \dots | t_B, \underline{\omega}_B^{j_B}) \\
&\quad \times \prod_{\beta=1}^B e^{-\frac{t_\beta}{\sqrt{V}} \|\underline{M}_\beta^{j_\beta}\| + \frac{i}{\sqrt{V}} \langle \underline{\omega}_\beta^{j_\beta}, \underline{M}_\beta^{j_\beta} \rangle} \cdot \quad (19)
\end{aligned}$$

The existence assumption of the inverse Laplace transform \mathcal{G} is only a technical trick which in the end will be reverted. Insertion into (18) gives

$$\begin{aligned}
& S_c(\mu x_1, \dots, \mu x_N) \\
&= \lim_{V \rightarrow \infty} \frac{1}{64\pi^2} \sum_{j_2+\dots+j_B=N} \int_{\mathbb{R}_+^B} d(t_1, \dots, t_B) \int_{\mathbb{R}^{N-B}} d(\underline{\omega}_1^{j_1}, \dots, \underline{\omega}_B^{j_B}) \mathcal{G}(t_1, \underline{\omega}_1^{j_1} | \dots | t_B, \underline{\omega}_B^{j_B}) \\
&\quad \times \sum_{\sigma \in \mathcal{S}_N} \prod_{\beta=1}^B \left(\frac{1}{V\mu^4 j_\beta} \sum_{\underline{M}_\beta^{j_\beta} \in (\mathbb{N}^2)^{j_\beta}} f_{\underline{M}_\beta^{j_\beta}}(x_{\sigma(N_\beta^-+1)}, \dots, x_{\sigma(N_\beta^-+j_\beta)}) e^{-\frac{t_\beta}{\sqrt{V}} \|\underline{M}_\beta^{j_\beta}\| + \frac{i}{\sqrt{V}} \langle \underline{\omega}_\beta^{j_\beta}, \underline{M}_\beta^{j_\beta} \rangle} \right) \cdot \quad (20)
\end{aligned}$$

The index sum is achieved by Corollary 5 which we prove in the Appendix. According to (3) the $f_{\underline{mn}}$ that we need in (20) are products of each two $f_{m_i m_{i+1}}$, one for (x_i^0, x_i^1) , the other for (x_i^0, x_i^1) . Both have in the notation of (38) common factors

$$z_1 = e^{-\frac{t_\beta}{\sqrt{V}} + \frac{i}{\sqrt{V}} \omega_{\beta,1}}, \quad z_i = e^{-\frac{t_\beta}{\sqrt{V}} + \frac{i}{\sqrt{V}} (\omega_{\beta,i} - \omega_{\beta,i-1})} \text{ for } 2 \leq i \leq j_\beta - 1, \quad z_{j_\beta} = e^{-\frac{t_\beta}{\sqrt{V}} - \frac{i}{\sqrt{V}} \omega_{\beta,j_\beta-1}}.$$

For $V \rightarrow \infty$ all z_i converge to 1. The denominator $1 - \prod_{i=1}^{j_\beta} (-z_i) = 1 - (-1)^{j_\beta} e^{-j_\beta \frac{t_\beta}{\sqrt{V}}}$ in (38) converges to 2 for j_β odd but behaves as $j_\beta \frac{t_\beta}{\sqrt{V}}$ for j_β even. The scalar and vector products in (38) receive common factors for $V \rightarrow \infty$, so that we conclude with $\theta = 4\sqrt{V}$:

$$\begin{aligned} & \lim_{V \rightarrow \infty} \frac{1}{V \mu^4 j_\beta} \sum_{\underline{M}_\beta^{j_\beta} \in (\mathbb{N}^2)^{j_\beta}} f_{\underline{M}_\beta^{j_\beta}}(x_{\sigma(N_\beta^-+1)}, \dots, x_{\sigma(N_\beta^-+j_\beta)}) e^{-\frac{t_\beta}{\sqrt{V}} \|\underline{M}_\beta^{j_\beta}\| + \frac{i}{\sqrt{V}} \langle \underline{\omega}_\beta^{j_\beta}, \underline{M}_\beta^{j_\beta} \rangle} \\ &= \begin{cases} \frac{4^{j_\beta}}{\mu^4 j_\beta^3 t^2} \exp\left(-\frac{\left\| \sum_{i=1}^{j_\beta} (-1)^{i-1} x_{\sigma(N_\beta^-+i)} \right\|^2}{2j_\beta t}\right) & \text{for } j_\beta \text{ even,} \\ 0 & \text{for } j_\beta \text{ odd.} \end{cases} \end{aligned} \quad (21)$$

The surviving factor for j_β even can be written as

$$\frac{4^{j_\beta}}{\mu^4 j_\beta^3 t^2} e\left(-\frac{\left\| \sum_{i=1}^{j_\beta} (-1)^{i-1} x_{\sigma(N_\beta^-+i)} \right\|^2}{2j_\beta t}\right) = \frac{4^{j_\beta}}{4\pi^2 j_\beta} \int_{\mathbb{R}^4} \frac{d^4 p_\beta}{\mu^4} e^{i \langle p_\beta, \sum_{i=1}^{j_\beta} (-1)^{i-1} x_{\sigma(N_\beta^-+i)} \rangle} e^{-\frac{j_\beta}{2} \|p_\beta\|^2 t}. \quad (22)$$

This result is inserted back into (20). Comparing the resulting $(t, \underline{\omega})$ -integral with (19), the $\underline{\omega}$ -independence of (22) forces all of the j_β matrix indices $m_{\beta,i}$ in $\underline{M}_\beta^{j_\beta}$ to be equal. Then $\|\underline{M}_\beta^{j_\beta}\|$ is j_β times the norm $\|m_{\beta,i}\|$ of any of these indices. The result is precisely the Laplace transform of $\mathcal{G}(\dots, t_\beta, \dots)$ to $G(\dots, \frac{\|m_{\beta,i}\|}{\sqrt{V}} \mapsto \frac{\|p_\beta\|^2}{2}, \dots)$. The remaining limit $V \rightarrow \infty$ applies to the reconstructed G , but as noted in (11), this limit sends $\frac{\|m_{\beta,i}\|}{\sqrt{V}} = \frac{\|p_\beta\|^2}{2}$ into the continuous variable $a_i \mu^2 (1 + \mathcal{Y})$, hence a_i into $\frac{\|p_\beta\|^2}{2\mu^2(1+\mathcal{Y})}$. We have thus proved:

Proposition 2 *The connected Schwinger functions (14) take the form*

$$\begin{aligned} S_c(\mu x_1, \dots, \mu x_N) &= \frac{1}{64\pi^2} \sum_{\substack{j_1 + \dots + j_B = N \\ j_\beta \text{ even}}} \sum_{\sigma \in \mathcal{S}_N} \left(\prod_{\beta=1}^B \frac{4^{j_\beta}}{j_\beta} \int_{\mathbb{R}^4} \frac{d^4 p_\beta}{4\pi^2 \mu^4} e^{i \langle \frac{p_\beta}{\mu}, \sum_{i=1}^{j_\beta} (-1)^{i-1} \mu x_{\sigma(N_\beta^-+i)} \rangle} \right) \\ &\quad \times G\left(\underbrace{\frac{\|p_1\|^2}{2\mu^2(1+\mathcal{Y})}, \dots, \frac{\|p_1\|^2}{2\mu^2(1+\mathcal{Y})}}_{j_1} \mid \dots \mid \underbrace{\frac{\|p_B\|^2}{2\mu^2(1+\mathcal{Y})}, \dots, \frac{\|p_B\|^2}{2\mu^2(1+\mathcal{Y})}}_{j_B}\right). \end{aligned} \quad (23)$$

Consequently, Schwinger functions are invariant under the full Euclidean group. The Schwinger functions only detect the restricted sector of the underlying matrix model where all matrix indices of a boundary component coincide. \square

In particular, the Schwinger two-point function reads

$$S_c(\mu x, \mu y) = \int_{\mathbb{R}^4} \frac{d^4 p}{(2\pi\mu)^4} e^{i \langle \frac{p}{\mu}, (\mu x - \mu y) \rangle} G\left(\frac{\|p\|^2}{2\mu^2(1+\mathcal{Y})}, \frac{\|p\|^2}{2\mu^2(1+\mathcal{Y})}\right). \quad (24)$$

The perturbative result $G(a, b) = \frac{1}{1+(1+\mathcal{Y})(a+b)} + \mathcal{O}(\lambda)$ obtained in [4] agrees with the expectation $\mu^2 S_c(\mu x, \mu y) = \int_{\mathbb{R}^4} \frac{d^4 p}{(2\pi)^4} \frac{e^{i\langle p, (x-y) \rangle}}{\|p\|^2 + \mu^2} + \mathcal{O}(\lambda)$ for the Euclidean ϕ_4^4 -model.

3 Analytic continuation to Minkowski space

Under a set of conditions established by Osterwalder-Schrader [5, 6], Schwinger functions of a Euclidean quantum field theory have an analytic continuation to Wightman functions [7] of a relativistic quantum field theory. Whether this is the case for the Schwinger functions (23) is of great interest, because non-trivial four-dimensional examples are rare.

The relation between Euclidean and Minkowskian Moyal-deformed field theories has already been addressed in literature. In joint work of one of us (HG) with Lechner, Ludwig and Verch [8] it was proved for degenerate deformations where time remains commutative that the Osterwalder-Schrader correspondence commutes (up to isomorphism) with Moyal deformation. This result was achieved in an algebraic approach to the Osterwalder-Schrader reconstruction theorem which is due to Schlingemann [9]. For deformations of full rank such a correspondence cannot be expected. As shown by Bahns [10], Wightman functions in a Minkowskian Moyal-deformed field theory admit an analytic continuation to imaginary time, but this continuation does not agree with the Schwinger functions of Euclidean Moyal-deformed field theory, at least in a framework close to perturbation theory. Adding the harmonic oscillator potential (1) to the Moyal deformation is also problematic in Minkowski space [11].

We will show in this section that the limit $\theta \rightarrow \infty$ cures all problems [10, 11] arising in full-rank Minkowskian Moyal-deformed theories. The question whether or not the Schwinger functions (23) define a Wightman quantum field theory is, in principle, decidable in view of their exact solution established in [4]. The lack of better knowledge of the properties of the fixed point solution (12) forces us to postpone the answer. We will extract that a necessary condition for (23) defining a Wightman theory is that $a \mapsto G(a, a)$ is a Stieltjes function [12]. First numerical investigations [13] of (12) suggest that this can only be expected for the wrong sign $\lambda \leq 0$ of the coupling constant. A related consequence of the numerical behaviour is the negative anomalous dimension $\eta = -2\lambda$. This is a surprising result which in view of [9] shows that the Euclidean operator algebra generated by the Schwinger function is highly sensitive to the sign of λ . In particular, the Osterwalder-Schrader correspondence of Moyal ϕ_4^4 -theory is inaccessible in perturbation theory.

Prior to analytic continuation is the expression of the Schwinger functions in terms of position differences $\xi_{(k)} := x_{k+1} - x_k$ [6]. Fixing a permutation σ and the number B of boundary components, the Schwinger functions (23) only depend on the B sums $\xi_{\beta(\sigma)} := \sum_{l=1}^{\frac{j_\beta}{2}} \xi_{(\sigma(N_\beta^- + 2l - 1))}$. We distinguish temporal and spatial directions, $\xi_{\beta(\sigma)} = (\xi_{\beta(\sigma)}^0, \vec{\xi}_{\beta(\sigma)})$, $q_\beta = (q_\beta^0, \vec{q}_\beta)$ and collect these by bold symbols $\boldsymbol{\xi}(\sigma) = (\xi_{1(\sigma)}, \dots, \xi_{B(\sigma)})$, $\mathbf{q} = (q_1, \dots, q_B)$, $\boldsymbol{\xi}^0(\sigma) = (\xi_{1(\sigma)}^0, \dots, \xi_{B(\sigma)}^0)$, $\mathbf{q}^0 = (q_1^0, \dots, q_B^0)$, $\vec{\boldsymbol{\xi}}(\sigma) = (\vec{\xi}_{1(\sigma)}, \dots, \vec{\xi}_{B(\sigma)})$, $\vec{\mathbf{q}} = (\vec{q}_1, \dots, \vec{q}_B)$. We also let $\boldsymbol{\xi}^0(\sigma) \cdot \mathbf{q}^0 = \sum_{\beta=1}^B \xi_{\beta(\sigma)}^0 q_\beta^0$ and $\vec{\boldsymbol{\xi}}(\sigma) \cdot \vec{\mathbf{q}} = \sum_{\beta=1}^B \langle \vec{\xi}_{\beta(\sigma)}, \vec{q}_\beta \rangle$. Under appropriate

analyticity conditions [6], the Schwinger functions are Fourier-Laplace transforms

$$S_c(\mu x_1, \dots, \mu x_N) = \sum_{\sigma \in \mathcal{S}_N} \sum_{B=1}^{\frac{N}{2}} S_N^{\sigma, B}(\mu \boldsymbol{\xi}(\sigma)),$$

$$S_N^{\sigma, B}(\mu \boldsymbol{\xi}(\sigma)) \Big|_{\xi_{\beta(\sigma)}^0 > 0} = \frac{1}{\mu^{4B}} \int_{\mathbb{R}_+^B} d^B \mathbf{q}^0 \int_{\mathbb{R}^{3B}} d^{3B} \vec{\mathbf{q}} \hat{W}_N^B(\frac{\mathbf{q}}{\mu}) e^{-\boldsymbol{\xi}(\sigma) \cdot \mathbf{q}^0 + i \vec{\boldsymbol{\xi}}(\sigma) \cdot \vec{\mathbf{q}}}. \quad (25)$$

The functions \hat{W}_N^B on \mathbb{R}^{4B} are candidates for the Fourier transform of Wightman N -point functions with B independent position differences. We remark that this restricted position dependence is for $B = \frac{N}{2}$ identical with a free field theory where the Osterwalder-Schrader reconstruction theorem is established.

Proving the Osterwalder-Schrader axioms [6] for the Schwinger functions (23), which would imply (25) with the correct properties of \hat{W}_N^B , is an open problem which we only address partly. We restrict ourselves to the 2-point function (24) which has the usual number $B = 1$ of independent position difference vectors. We prove:

Proposition 3 *Necessary and sufficient for the Schwinger 2-point function $S_2(\mu \xi) := \sum_{\sigma \in \mathcal{S}_2} S_2^{\sigma, 1}(\mu \xi)$ being the Fourier-Laplace transform of a positive Wightman function $\hat{W}_2^1(\frac{\mathbf{q}}{\mu})$ is that $a \mapsto G(a, a)$ is a Stieltjes function,*

$$G(a, a) = \int_0^\infty \frac{d\rho(\frac{M^2}{\mu^2})}{(2(1 + \mathcal{Y})a + \frac{M^2}{\mu^2})}, \quad (26)$$

where $\rho(\frac{M^2}{\mu^2})$ is a positive measure.

Stieltjes functions were thoroughly studied by Widder [12]: A function $\mathbb{R}_+ \ni x \mapsto f(x) \in \mathbb{R}$ is Stieltjes iff f is smooth and

$$(S1) \quad f(x) \geq 0 \text{ for all } x \in \mathbb{R}_+$$

$$(S2) \quad (-1)^n \frac{d^{2n+1}}{dx^{2n+1}}(x^{n+1} f(x)) \geq 0 \text{ for all } x \in \mathbb{R}_+ \text{ and } n \in \mathbb{N}.$$

Proof of Prop. 3. This is a consequence of the Källén-Lehmann spectral representation [14, 15]. Inserting (26) into (24) we have

$$S_2(\mu \xi) := \sum_{\sigma \in \mathcal{S}_2} S_2^{\sigma, 1}(\mu \xi_\sigma) = \int_0^\infty d\rho(\frac{M^2}{\mu^2}) \int_{\mathbb{R}^4} \frac{dp^0 d^3 \vec{p}}{(2\pi\mu)^4} \frac{\mu^2 e^{ip^0 \xi^0 + i \vec{p} \cdot \vec{\xi}}}{((p^0)^2 + \vec{p} \cdot \vec{p} + M^2)}. \quad (27)$$

For $\xi^0 > 0$ the p^0 -integral is evaluated by the residue theorem:

$$S_2(\mu \xi) \Big|_{\xi^0 > 0} = \int_0^\infty d\rho(\frac{M^2}{\mu^2}) \int_{\mathbb{R}^3} \frac{d^3 \vec{p}}{(2\pi\mu)^3} \frac{\mu}{2\omega_{\vec{p}, M}} e^{-\omega_{\vec{p}, M} \xi^0 + i \vec{p} \cdot \vec{\xi}}, \quad \omega_{\vec{p}, M} := \sqrt{\vec{p} \cdot \vec{p} + M^2}. \quad (28)$$

This gives the desired representation $S_2(\mu\xi)\Big|_{\xi^0>0} = \frac{1}{\mu^4} \int_0^\infty dq^0 \int_{\mathbb{R}^3} d\vec{q} \hat{W}_2^1\left(\frac{q}{\mu}\right) e^{-q^0\xi^0+i\vec{q}\cdot\vec{\xi}}$ as Fourier-Laplace transform with

$$\hat{W}_2^1(q) = \frac{\theta(q^0)}{(2\pi)^3} \int_0^\infty d\rho\left(\frac{M^2}{\mu^2}\right) \delta\left(\frac{(q^0)^2 - \vec{q}\cdot\vec{q} - M^2}{\mu^2}\right), \quad (29)$$

where θ and δ are the Heaviside and Dirac distributions. The final formula (29) is recognised as the Källén-Lehmann spectral representation [14, 15] of a two-point function in a general Wightman quantum field theory. The converse steps starting with (29) show that the Stieltjes property (26) is necessary. \square

We are currently unable to determine whether the matrix 2-point function $a \mapsto G(a, a)$ is Stieltjes, i.e. satisfies Widder's conditions (S1)+(S2). Every Stieltjes function is completely monotonic, i.e.

$$(CM) \quad (-1)^n f^{(n)}(x) \geq 0 \text{ for all } x \in \mathbb{R}_+.$$

Complete monotonicity (CM) for the function $a \mapsto G(a, 0)$ might be in reach. We recall from [4] that $G(a, 0)$ is the solution (12) of a fixed point problem $f = Tf$ where the non-linear map T preserves the space of positive, monotonously decreasing functions. With some effort it seems possible to prove that T preserves the space of completely monotonic functions. It is of course not obvious that complete monotonicity of $a \mapsto G(a, 0)$ is transferred to $a \mapsto G(a, a)$ given by (13).

In between the Stieltjes functions and the completely monotonous functions lies the class of generalised Stieltjes functions of order $\kappa > 0$ which admit a representation

$$f(x) = c + \int_0^\infty \frac{d\rho_\kappa(M^2)}{(x + M^2)^\kappa}, \quad (30)$$

where $c \geq 0$ and $\rho_\kappa(M^2)$ is a positive measure. For a review on generalised Stieltjes functions we refer to a recent article of Sokal [16] which identifies the precise conditions under which a real function f is a generalised Stieltjes function. The identity [16, eq. (7)]

$$\frac{1}{(x+t)^\kappa} = \frac{\Gamma(\kappa')}{\Gamma(\kappa)\Gamma(\kappa'-\kappa)} \int_0^\infty du u^{\kappa'-\kappa-1} \frac{1}{(x+t+u)^{\kappa'}} \quad (31)$$

implies that a Stieltjes function of order κ is also a Stieltjes function of order $\kappa' > \kappa$. Moreover, any generalised Stieltjes function is completely monotonic.

If $a \mapsto G(a, a)$ happens to be a generalised Stieltjes function of order $2 \leq k \in \mathbb{N}$ (integer-order suffices by (31)), then the same steps as in the proof of Proposition 3 yield an analytic continuation of the Schwinger function S_2 to Minkowski space. The big difference is that the corresponding Wightman function $\hat{W}_2^1(q)$ involves the derivative $\delta^{(k-1)}\left(\frac{(q^0)^2 - \vec{q}\cdot\vec{q} - M^2}{\mu^2}\right)$ of the Dirac distribution which is not positive for $k \geq 2$. This means that *Osterwalder-Schrader reflection positivity cannot be expected for $a \mapsto G(a, a)$ being a generalised Stieltjes function of order $\kappa > 1$.*

Numerical investigations [13] and also the perturbative solution of the 2-point function $G(a, a)$ tend to suggest the asymptotic behaviour

$$G(a, a) \stackrel{a \rightarrow \infty}{\propto} \frac{1}{(1 + 2(1 + \mathcal{Y})a)^{1+\lambda}}. \quad (32)$$

This would imply that for the physical coupling constant $\lambda > 0$ the matrix 2-point function $a \mapsto G(a, a)$ is not Stieltjes and as such does not permit an analytical continuation to a positive Wightman quantum field theory. A related consequence is the negative anomalous dimension for $\lambda > 0$: If (32) holds exactly, i.e. $G(\frac{\|p\|^2}{2\mu^2(1+\mathcal{Y})}, \frac{\|p\|^2}{2\mu^2(1+\mathcal{Y})}) = (\frac{\|p\|^2}{\mu^2} + 1)^{-(1+\lambda)}$, then (24) becomes $S_c(\mu x, \mu y) = \frac{2^{-\lambda}}{4\pi^2\Gamma(1+\lambda)} \frac{K_{1-\lambda}(\mu\|x-y\|)}{(\mu\|x-y\|)^{1-\lambda}}$. To obtain this result one expresses $(\frac{\|p\|^2}{\mu^2} + 1)^{-(1+\lambda)}$ by the Γ -function integral, evaluates the resulting Gaussian integral over $p \in \mathbb{R}^4$ and uses the integral representation [17, §8.432.6] of the modified Bessel function $K_\nu(z)$. From $K_\nu(z) \stackrel{z \rightarrow 0}{\propto} \frac{\Gamma(\nu)}{2} (\frac{z}{2})^\nu$ [18, §9.6.9] it follows $S_c(\mu x, \mu y) \stackrel{x-y \rightarrow 0}{\propto} \frac{2^{-2\lambda}\Gamma(1-\lambda)}{4\pi\Gamma(1+\lambda)} \frac{1}{(\mu\|x-y\|)^{2-2\lambda}}$, which means that the anomalous dimension would be $\eta = -2\lambda$.

Conversely, (12) and (13) might⁴ define an analytical continuation of the model to $\lambda < 0$. This wrong-sign noncommutative $\lambda\phi_4^4$ -model could then have an analytical continuation to a Wightman quantum field theory. In a certain sense this parallels the construction of the commutative planar wrong-sign $\lambda\phi_4^4$ -model by t'Hooft [19] and Rivasseau [20].

The negative anomalous dimension resulting from the faster decay of $G(a, a)$ in a for $\lambda > 0$ in comparison with the free theory $\lambda = 0$ where $G(a, a) = \frac{1}{1+2a}$ exactly is the result of the renormalisation. The two-dimensional model which does not require a wavefunction renormalisation has, at least perturbatively, the opposite behaviour $G(a, a)^{(D=2)} = \frac{1}{1+2a} + \frac{\lambda}{\mu^2} \frac{2\log(1+a)}{(1+2a)^2} + \mathcal{O}(\lambda^2)$ than the 4-dimensional case $G(a, a)^{(D=4)} = \frac{1}{1+2a} - \lambda \frac{(2+2a)\log(1+a)}{(1+2a)^2} + \mathcal{O}(\lambda^2)$. The perturbative result also suggests that in $D = 2$ the difference between free and interacting theory is subleading to a power law, $G(a, a)^{(D=2)} \stackrel{a \rightarrow \infty}{\propto} \frac{1}{1+2a}$ independent of the coupling constant. This favours the conjecture that the two-dimensional model can define a Wightman theory for any sign of the coupling constant.

4 Interpretation

In this paper we have translated the matrix model correlation functions of Moyal-deformed ϕ_4^4 -theory solved in [4] to position space. This involves a different infinite volume limit as pointed out below:

1. *Matrix model limit.* This limit arises directly in matrix formulation [4] from the free energy density $\mathcal{W} = \lim_{V \rightarrow \infty} \frac{1}{V} \log \mathcal{Z}$ with its usual volume dependence. As seen from [4, Prop. 3.5], this limit eliminates all non-trivial topology of the matrix model, i.e. both the non-planar sector with genus $g \geq 1$ and the sector with $B \geq 2$ boundary components. The restriction to trivial topology agrees with other large- \mathcal{N} limits of matrix models. There remain the planar regular N -point functions

⁴Existence of a solution of (12) is, so far, only established for $\lambda > 0$.

$G(b_0, \dots, b_{N-1})$ with $b_i \in \mathbb{R}_+$ for which there is an exact recursion formula [4, eq. (4.50)] that provides $G(b_0, \dots, b_{N-1})$ as weighted difference quotients of products of two-point functions $G(a, b)$. The combinatorics involves non-crossing partitions counted by the Catalan numbers. The two-point function is given as a function (13) of its boundary $G(a, 0)$, which itself is the solution of a non-linear integral equation (12).

2. *Statistical physics limit.* This is the limit which gives the Schwinger functions of Definition 1. There are two non-naïve volume scalings involved, a procedure that is common in statistical physics. We first define the free energy density as $\mathcal{F} = \frac{1}{(V\mu^4)^2} \log \mathcal{Z}$. The additional volume factor is due to the fact that the spectral geometry [21, 22] behind the noncommutative quantum field theory under consideration has a finite volume V^2 , not V . According to the second and third remark after Definition 1, there is also a wavefunction renormalisation $\sqrt{\mathcal{Z}}$ to $\frac{\sqrt{\mathcal{Z}}}{V\mu^4}$ involved.

According to [4, Prop. 3.5], \mathcal{F} has an expansion into planar topological sectors with B boundary components and prefactor $\frac{1}{(V\mu^4)^B}$. The next step is to notice that individual matrix element correlation functions give according to (3) and (4) the amplitude of a Gaussian wave packet in position space. The assembly of plane waves from Gauß packets involves sums over the matrix indices. As proved by Corollary 5, this index summation produces a factor $V\mu^4$ per boundary component with even length, whereas no such factor arises for a boundary component of odd length. This means that all sectors with $B \geq 1$ and an even number of sources/fields per boundary component contribute to \mathcal{F} in *position space*. This makes the statistical physics limit the topologically richest one.

The resulting Schwinger functions $S(\mu x_1, \dots, \mu x_N)$ given in (23) define a Euclidean quantum field theory on \mathbb{R}^4 . These Schwinger functions have the full Euclidean invariance, they are symmetric and (as discussed in section 3) they might possess an analytic continuation to Wightman functions of a four-dimensional relativistic quantum field theory, possibly only for $\lambda \leq 0$. The resulting field theory limit is close to a free theory, but there are differences that we describe below.

The richest sector of the model is the case $B = \frac{N}{2}$ made of the $(2+2+\dots+2)$ -point functions. This sector describes the propagation and interaction of $B = \frac{N}{2}$ particles with momenta $p_1, \dots, p_{\frac{N}{2}}$. These particles interact, but in a way that the *momentum is unchanged*, precisely as with free fields. If two or more of these $\frac{N}{2}$ particles have coinciding momenta, then another interaction channel is opened which is described by the sectors with $B < \frac{N}{2}$. It would be interesting to extend this model to scalar fields of several components. In this case momentum could be exchanged between the components.

The key difference to a free theory is the (maximal) violation of the cluster property. Any connected $(N \geq 4)$ -point Schwinger function (23) contains contributions which do not decay to zero if a subset of positions x_i is shifted infinitely away. Let us consider the case

$N = 4$ in (23). For $0 \neq x \in \mathbb{R}^4$ one has

$$\begin{aligned}
& \lim_{\tau \rightarrow \infty} S_4(\mu x_1, \mu x_2, \mu(x_3 + \tau x), \mu(x_4 + \tau x)) \\
&= \int_{\mathbb{R}^4 \times \mathbb{R}^4} \frac{d^4 p d^4 q}{2\pi^6 \mu^8} e^{i\langle p, x_1 - x_2 \rangle + i\langle q, x_3 - x_4 \rangle} G\left(\frac{\|p\|^2}{2\mu^2(1+\mathcal{Y})}, \frac{\|p\|^2}{2\mu^2(1+\mathcal{Y})} \middle| \frac{\|q\|^2}{2\mu^2(1+\mathcal{Y})}, \frac{\|q\|^2}{2\mu^2(1+\mathcal{Y})}\right) \\
&+ \int_{\mathbb{R}^4} \frac{d^4 p}{(2\pi\mu)^4} \frac{e^{i\langle p, x_1 - x_2 + x_3 - x_4 \rangle} + e^{i\langle p, x_1 - x_2 + x_4 - x_3 \rangle}}{2} G\left(\frac{\|p\|^2}{2\mu^2(1+\mathcal{Y})}, \frac{\|p\|^2}{2\mu^2(1+\mathcal{Y})}, \frac{\|p\|^2}{2\mu^2(1+\mathcal{Y})}, \frac{\|p\|^2}{2\mu^2(1+\mathcal{Y})}\right),
\end{aligned} \tag{33}$$

because all other permutations vanish almost everywhere by the Riemann-Lebesgue lemma.

Assuming validity of the other Osterwalder-Schrader axioms [5, 6], the corresponding Wightman quantum field theory would also lack the clustering property. Wightman's reconstruction theorem [7] then implies that the vacuum would be a mixed state. Its decomposition into pure states corresponds to a decomposition into different topological sectors. It would be very interesting to study this non-trivial topology of the ϕ_4^4 -model on Moyal space in the limit $\theta \rightarrow \infty$. Unfortunately, the lack of more detailed knowledge about the diagonal matrix 2-point function $G(a, a)$ moves this investigation into the future.

At first glance it seems surprising that the limit $\theta \rightarrow \infty$ of infinite noncommutativity is so close to a traditional quantum field theory expected for $\theta \rightarrow 0$. An intuitive explanation is the following. The Feynman rule in momentum space for the quartic interaction vertex with momenta p_1, \dots, p_4 reads $\mathcal{V}(p_1, \dots, p_4) = \lambda e^{i\sum_{i<j} p_i^\mu \theta_{\mu\nu} p_j^\nu} \delta(p_1 + \dots + p_4)$. If $f(p_1, \dots, p_4)$ is any L^1 -function of the momenta, then $\lim_{\theta \rightarrow \infty} f(p_1, \dots, p_4) \mathcal{V}(p_1, \dots, p_4) = 0$ almost everywhere by the Riemann-Lebesgue lemma. This shows that up to measure zero, the $\theta \rightarrow \infty$ limit of a quantum field theory on Moyal space is a free theory. To the exceptional points where the limit is non-zero belong linearly dependent momenta p_i where the phase vanishes. Now comes the crucial point: *These subspaces of total measure zero where the theory is not free are possibly protected for topological reasons, and this is the case for $B > 1$ boundary components.* The subspace where the momenta of fields attached to each boundary add up to zero has full Lebesgue measure. In connection with the correct volume-dependent field renormalisation this establishes qualitatively our result that *the $\theta \rightarrow \infty$ limit of noncommutative ϕ_4^4 -theory differs from a free theory by the presence of non-trivial topological sectors.*

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A Sum over products of Laguerre polynomials

Lemma 4 For $t_i \in \mathbb{R}_+$, $z_i \in \mathbb{C}$ with $|z_i| < 1$ and cyclic identification $J + i \equiv i$ of indices one has

$$\sum_{m_1, \dots, m_J=0}^{\infty} \prod_{i=1}^J z_i^{m_i} L_{m_i}^{m_{i+1}-m_i}(t_i) = \frac{1}{1 - (z_1 \cdots z_J)} \exp\left(-\frac{\sum_{i,j=1}^J t_i(z_{j+i} \cdots z_{J+i})}{1 - (z_1 \cdots z_J)}\right). \quad (34)$$

Proof. We use the generating function [17, §8.975.3] of Laguerre polynomials

$$\sum_{n=0}^{\infty} L_n^{\alpha-n}(t) z^n = e^{-zt}(1+z)^\alpha \quad (35)$$

to split up the index chain:

$$\begin{aligned} & \sum_{m_1, \dots, m_J=0}^{\infty} \prod_{i=1}^J z_i^{m_i} L_{m_i}^{m_{i+1}-m_i}(t_i) \\ &= \sum_{m_1, \dots, m_J=0}^{\infty} \left(\prod_{i=1}^{J-1} z_i^{m_i} L_{m_i}^{m_{i+1}-m_i}(t_i) \right) \cdot \frac{1}{m_J!} \frac{d^{m_J}}{du^{m_J}} \sum_{n=0}^{\infty} u^n z_J^n L_n^{m_1-n}(t_J) \Big|_{u=0} \\ &= \sum_{m_1, \dots, m_J=0}^{\infty} \left(\prod_{i=1}^{J-1} z_i^{m_i} L_{m_i}^{m_{i+1}-m_i}(t_i) \right) \cdot \frac{1}{m_J!} \frac{d^{m_J}}{du^{m_J}} e^{-uz_J t_J} (1 + uz_J)^{m_1} \Big|_{u=0}. \end{aligned} \quad (36)$$

Now we successively sum over m_1, m_2, \dots, m_{J-1} where each sum is of the form (35). There remains a final sum over $m_J \equiv m$:

$$\begin{aligned} \sum_{m_1, \dots, m_J=0}^{\infty} \prod_{i=1}^J z_i^{m_i} L_{m_i}^{m_{i+1}-m_i}(t_i) &= \sum_{m=0}^{\infty} \frac{1}{m!} \frac{d^m}{du^m} \left(\left(A + u \prod_{i=1}^J z_i \right)^m e^{-uB-C} \right) \Big|_{u=0}, \\ A &:= 1 + \sum_{i=1}^{J-1} z_i \cdots z_{J-1} \quad B := \sum_{i=0}^{J-1} (z_0 \cdots z_i) t_i, \quad C := \sum_{i=1}^{J-1} \sum_{j=1}^i (z_j \cdots z_i) t_i. \end{aligned}$$

This leads to

$$\begin{aligned} \sum_{m_1, \dots, m_J=0}^{\infty} \prod_{i=1}^J z_i^{m_i} L_{m_i}^{m_{i+1}-m_i}(t_i) &= e^{-C} \sum_{m=0}^{\infty} \sum_{k=0}^m \frac{1}{m!} \binom{m}{k} \binom{m}{k} (m-k)! (-BA)^k \left(\prod_{i=1}^J z_i \right)^{m-k} \\ &= e^{-C} \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \frac{(m+k)!}{k! k! m!} (-BA)^k \left(\prod_{i=1}^J z_i \right)^m \\ &= \frac{1}{1 - \prod_{i=1}^J z_i} \exp\left(-C - \frac{BA}{1 - \prod_{i=1}^J z_i}\right). \end{aligned} \quad (37)$$

It is straightforward to check $C + \frac{BA}{1 - \prod_{i=1}^J z_i} = \frac{\sum_{i,j=1}^J t_i(z_{j+i} \cdots z_{J+i})}{1 - (z_1 \cdots z_J)}$, which yields the assertion (34). \square

Corollary 5 Let $\langle x, y \rangle$, $\|x\|$ and $x \times y = \det(x, y)$ be scalar product, norm and (third component of) vector product of $x, y \in \mathbb{R}^2$. Then for $x_i \in \mathbb{R}^2$ and $z_i \in \mathbb{C}$ with $|z_i| < 1$, the $f_{mn}(x_i)$ defined in (4) satisfy (with cyclic identification $J + i \equiv i$ of indices where necessary)

$$\begin{aligned} & \sum_{m_1, \dots, m_J=0}^{\infty} \prod_{i=1}^J f_{m_i m_{i+1}}(x_i) z_i^{m_i} \\ &= \frac{2^J}{1 - \prod_{i=1}^J (-z_i)} \exp \left(- \frac{\sum_{i=1}^J \|x_i\|^2}{\theta} \frac{1 + \prod_{i=1}^J (-z_i)}{1 - \prod_{i=1}^J (-z_i)} \right) \\ & \times \exp \left(- \frac{2}{\theta} \sum_{1 \leq k < l \leq J} \left((\langle x_k, x_l \rangle - i x_k \times x_l) \frac{\prod_{j=k+1}^l (-z_j)}{1 - \prod_{i=1}^J (-z_i)} + (\langle x_k, x_l \rangle + i x_k \times x_l) \frac{\prod_{j=l+1}^{J+k} (-z_j)}{1 - \prod_{i=1}^J (-z_i)} \right) \right). \end{aligned} \quad (38)$$

Proof. From (4) we get for $0 \neq x_i \in \mathbb{R}^2 \equiv \mathbb{C}$ and $|z_i|$ sufficiently small

$$\prod_{i=1}^J f_{m_i m_{i+1}}(x_i) z_i^{m_i} = 2^J e^{-\frac{1}{\theta} \sum_{i=1}^J x_i \bar{x}_i} \prod_{i=1}^J \tilde{z}_i^{m_i} L_{m_i}^{m_{i+1} - m_i}(t_i) \Big|_{t_i = \frac{2}{\theta} x_i \bar{x}_i, \tilde{z}_i = -z_i \frac{x_{i-1}}{x_i}}. \quad (39)$$

This yields

$$\sum_{i,j=1}^J t_i (\tilde{z}_{j+i} \cdots \tilde{z}_{J+i}) = \frac{2}{\theta} \sum_{i,j=1}^J x_{i+j-1} \bar{x}_i (-z_{j+i}) \cdots (-z_{J+i})$$

for the sum (34) of Laguerre polynomials. Splitting the sum into the cases $j = 1, j = 2, \dots, J - i$ and $j = J - i + 1 \dots J$ we confirm (38) with $x_i \bar{x}_j = \langle x_i, x_j \rangle - i x_i \times x_j$. \square

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