

# The Moyal Sphere

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We construct a family of constant curvature metrics on the Moyal plane and compute the Gauss–Bonnet term for each of them. They arise from the conformal rescaling of the metric in the orthonormal frame approach. We find a particular solution, which corresponds to the Fubini–Study metric and which equips the Moyal algebra with the geometry of a noncommutative sphere.

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## I. INTRODUCTION

Noncommutative geometry provides a unified framework to describe classical, discrete as well as singular or deformed spaces [3, 10]. Most of the examples studied so far were constructed with a fixed metric, allowing only small perturbations of the gauge type. Only recently a class of models with conformally modified noncommutative metrics was constructed for the noncommutative tori [5] allowing for the computation of the scalar curvature using different approaches [4, 6–8, 16]. Some general approaches to the curvature of noncommutative spaces were also proposed [20–23].

The Moyal deformation of a plane [1, 15] is one of the oldest and best-studied noncommutative spaces. Motivated by the appearance in the string-theory target space [17] it is often used as model of noncommutative space-time (see [18] for a review) and a background for a noncommutative field theory [19]. However, so far the only explicit computations were performed in the case of the flat geometry, which corresponds to the constant metric on the plane, though several authors [20–22] studied formal gravity models on twisted deformations of the Euclidean or Minkowski spacetimes. In this paper we introduce a class of conformally rescaled metrics on the two-dimensional Moyal plane using the orthonormal frame formalism adapted to the noncommutative setting [6]. We compute the scalar curvature and look for the solutions of the constant curvature condition, thus finding the Moyal–Fubini–Study metrics.

The paper is organized as follows: first, we recall the classical (commutative) constant curvature solutions, then we briefly review the noncommutative Moyal plane and compute the scalar curvature using conformally rescaled orthonormal frames. We discuss explicit solutions in the matrix basis for the Moyal algebra as well the first order perturbative correction to the Fubini–Study metric using smooth functions on the plane.

## II. CLASSICAL FUBINI–STUDY METRIC ON THE PLANE

The classical Fubini–Study metric is a standard well-known piece of Riemannian geometry. However, it is rarely approached from the viewpoint of finding all constant curvature invariant under rotations metrics on the plane. To facilitate the comparison of the results for the Moyal-deformed plane and make the article self-contained, we present the standard classical derivation and the properties of the solutions.

Consider the conformally rescaled metric on the plane:

$$k(x, y)^2(dx^2 + dy^2).$$

We assume that  $k = k(r)$ , then the scalar curvature is:

$$R(k) = 2k(r)^{-4} \left( k'(r)k'(r) - \frac{k(r)}{r}k'(r) - k(r)k''(r) \right).$$

Now, let us look for  $k$  such that  $R(k) = C = \text{const.}$  We obtain the differential equation:

$$k(r)k''(r) + \frac{k(r)}{r}k'(r) - k'(r)k'(r) = Ck^4(r),$$

which has a family of nondegenerate solutions for  $A > 0$ ,  $a \geq 1$  and  $b > 0$ :

$$k(r) = \frac{Ar^{a-1}}{b + r^{2a}},$$

so that the scalar curvature is:

$$R(k) = \frac{8a^2b}{A^2},$$

the volume:

$$V(k) = \pi \frac{A^2}{ba}$$

and the Gauss–Bonnet term:

$$\int \sqrt{g}R = 8\pi a.$$

More generally, assume that we have a class of conformally rescaled metrics for which the asymptotics of  $k(r)$  at infinity is given by:

$$k(r) \sim r^{-\alpha} + C_1 r^{-\alpha-1} + C_2 r^{-\alpha-2}.$$

One can easily verify that the scalar curvature is regular, that is:

$$\lim_{r \rightarrow \infty} R(r) < \infty,$$

if  $C_1 = 0$  and  $\alpha \leq 2$ .

We can now compute the Gauss–Bonnet term for such metrics:

$$\begin{aligned} \int_{\mathbb{R}^2} \sqrt{g}R(g) &= 4\pi \int_0^\infty r dr k(r)^{-2} \left( k'(r)k'(r) - \frac{k(r)}{r}k'(r) - k(r)k''(r) \right) \\ &= 4\pi \int_0^\infty dr (rk(r)^{-2}k'(r)k'(r) - k(r)^{-1}k'(r) - rk(r)^{-1}k''(r)) \\ &= 4\pi \int_0^\infty dr (-rk'(r)k(r)^{-1})' = 4\pi \left( (-rk'(r)k(r)^{-1})_\infty - (-rk'(r)k(r)^{-1})_0 \right). \end{aligned}$$

Assuming that  $k(r)$  and its derivatives are regular at  $r = 0$  and that  $k(r)$  behaves like  $r^{-\alpha}$  at  $r \rightarrow \infty$  we obtain:

$$\int_{\mathbb{R}^2} \sqrt{g}R(g) = 4\pi\alpha.$$

Note that the special case, where  $\alpha = 2$ , which is the Fubini–Study metric yields the correct Gauss–Bonnet term for the sphere.

On the other hand, if we require that the metric alone remains bounded at  $r = \infty$ , we have, after change of variables  $\rho = \frac{1}{r}$ , that

$$\lim_{\rho \rightarrow 0} k^2(\rho^{-1})\rho^{-4} < \infty.$$

This happens if the asymptotics of  $k(r)$  is  $r^{-\alpha}$  for  $\alpha \geq 2$ . Moreover, if we require that the metric does not vanish at  $r = \infty$  then  $\alpha = 2$  is the only solution for the asymptotic behavior of  $k(r)$ .

### III. THE FLAT GEOMETRY OF THE MOYAL PLANE

We begin by reviewing here shortly basic results on Moyal geometry. We take the algebra of the Moyal plane,  $A_\theta$ , as a vector space  $(\mathcal{S}(\mathbb{R}^2), *)$ , ( $\mathcal{S}$  is the Schwartz space), equipped with the Moyal product  $*$  defined as follows through the oscillatory integrals [15]:

$$(f * g)(x) := (2\pi)^{-2} \int_{\mathbb{R}^2 \times \mathbb{R}^2} e^{i\xi(x-y)} f(x - \frac{1}{2}\Theta\xi) g(y) d^n y d^n \xi, \quad (1)$$

where  $\Theta = \begin{pmatrix} 0 & \theta \\ -\theta & 0 \end{pmatrix}$ ,  $\theta \in \mathbb{R}$ . With the product defined in (1) it is easy to see that

$$f \mapsto \int_{\mathbb{R}^n} f(x) d^n x$$

is a trace on the Moyal algebra and the standard partial derivations  $\partial_{x_1}, \partial_{x_2}$  remain derivations on the deformed algebra.

The algebra can be faithfully represented on the Hilbert space of  $L^2$ -sections of the usual spinor bundle over  $\mathbb{R}^2$  (which is  $\mathcal{H} = L^2(\mathbb{R}^2) \otimes \mathbb{C}^2$ ) acting by Moyal left multiplication  $L^\Theta(a)$ . It was demonstrated first in [9] that the Moyal plane algebra with one of its preferred unitizations yield nonunital, real spectral triples for the standard Dirac operator on the plane arising from the flat Euclidean metric.

### A. Matrix basis for the Moyal algebra

It will be convenient to work with the matrix basis for the Moyal algebra [9]. We define first:

$$f_{0,0} = 2e^{-\frac{1}{\theta}(x_1^2+x_2^2)},$$

then the algebra  $A_\theta$  has a natural basis consisting of:

$$f_{m,n} = \frac{1}{\sqrt{m!n!\theta^{m+n}}} (a^*)^m * f_{00} * (a)^n,$$

or more explicitly:

$$f_{m,n}(r, \phi) = 2(-1)^m \sqrt{\frac{m!}{n!}} e^{i\phi(m-n)} \left( \sqrt{\frac{2}{\theta}} r \right)^{(n-m)} L_m^{n-m} \left( \frac{2r^2}{\theta} \right) e^{-\frac{r^2}{\theta}}. \quad (2)$$

We have for each  $m, n, k, l \geq 0$ :

$$f_{m,n} * f_{k,l} = \delta_{kn} f_{m,l}, \quad f_{m,n}^* = f_{n,m}.$$

In particular for all  $k \geq 0$  all  $f_{k,k}$  are pairwise orthogonal projections of rank one. The natural (not normalized) trace on the Moyal algebra is:

$$\tau(f) = \int_{\mathbb{R}^2} d^2x f(x), \quad \tau(f_{m,n}) = 2\pi\theta\delta_{mn}.$$

It is convenient to work with the linear combinations of derivations:

$$\partial = \frac{1}{\sqrt{2}}(\partial_{x_1} - i\partial_{x_2}), \quad \bar{\partial} = \frac{1}{\sqrt{2}}(\partial_{x_1} + i\partial_{x_2}).$$

Then:

$$\partial f_{m,n} = \sqrt{\frac{n}{\theta}} f_{m,n-1} - \sqrt{\frac{m+1}{\theta}} f_{m+1,n},$$

$$\bar{\partial} f_{m,n} = \sqrt{\frac{m}{\theta}} f_{m-1,n} - \sqrt{\frac{n+1}{\theta}} f_{m,n+1}.$$

Furthermore,

$$\partial\bar{\partial} = \frac{1}{2}(\partial_{x_1}^2 + \partial_{x_2}^2).$$

### B. Radial functions in the matrix basis

We define *radial* functions on the Moyal plane as those which in their presentation in the matrix basis have only diagonal elements  $f_{n,n}$ , so:

$$h = \sum_{n=0}^{\infty} h_n f_{n,n}.$$

Each function  $F$  applied to  $h$  is easily computable, since  $f_{n,n}$  are projections, we have:

$$F(h) = \sum_{n=0}^{\infty} F(h_n) f_{n,n}.$$

From the orthogonality of matrix basis  $f_{n,n}$ , the explicit representation (2) and known integrals of Laguerre polynomials one deduces

$$r^a = \sum_{n=0}^{\infty} \theta^{\frac{a}{2}} {}_2F_1(-n, -\frac{a}{2}; 1; 2) f_{n,n}(r).$$

In particular,

$$\sum_{n=0}^{\infty} f_{n,n} = 1, \quad \sum_{n=0}^{\infty} n f_{n,n} = \frac{1}{2} \left( \frac{r^2}{\theta} - 1 \right).$$

Therefore the inverse of the conformal factor for the Fubini–Study metric, which is a linear function of  $r^2$ ,  $k(r)^{-1} = \frac{1}{A}(b + r^2)$  has the following expansion:

$$\frac{1}{A}(b + r^2) = \frac{1}{A} \sum_{n=0}^{\infty} (2\theta n + b + \theta) f_{n,n}.$$

#### IV. CONFORMALLY RESCALED METRIC ON THE MOYAL PLANE

There are several approaches to the conformal rescaling of the metric and computing the curvature for the noncommutative torus and – by analogy – the Moyal plane. Note that each of them uses a different notation and they do not give compatible results in the case of the noncommutative torus.

First of all, one can take a conformal rescaling of the flat Laplace operator:  $\Delta_h = h\Delta h$ , which in the case of the noncommutative torus was studied by Connes–Tretkoff [5]. This corresponds to the rescaling of the metric by  $h^{-2}$

A second approach uses the language of orthonormal frames [6]. It replaces the standard (flat) orthonormal frames, understood as derivations on the algebra, by the conformally rescaled ones,  $\delta_i \rightarrow h\delta_i$ , where  $h$  is from the algebra (or, more generally, from its multiplier). Applying the classical formula for the scalar curvature, which easily adapts to the noncommutative case, one obtains a noncommutative version of the scalar curvature. This corresponds also to the rescaling of the metric by  $h^{-2}$ .

Thirdly, one can use the formalism developed in [20–22] based on a deformed diffeomorphism algebra.

Finally, extending the computations of Jonathan Rosenberg for the Levi-Civita connection [16] on the noncommutative torus one might obtain a similar generalization of the expression for the scalar curvature.

We shall concentrate on the case of orthonormal frames and try to determine whether there exists a radial conformal rescaling for which the scalar of curvature is constant.

##### A. The orthonormal frame approach

Instead of using the metric tensor, the definition of which is somewhat ambiguous in the noncommutative setup, we shall rather use the formalism of orthonormal frames [6]. These are understood as the local choice of basis in the sections of the tangent bundle, identified as vector fields and assumed to be orthonormal. If  $e_i$ ,  $i = 1, \dots, N$  is such a basis, which has the following commutation relations,

$$[e_i, e_j] = c_{ijk} e_k,$$

then the covariant Levi-Civita connection reads

$$\nabla_{e_i}(e_j) = \frac{1}{2}(c_{ijk} + c_{kij} + c_{kji})e_k$$

and its scalar curvature is [6, (2.1)],

$$R = 2 e_i(c_{ijj}) - c_{kii}c_{kjj} - \frac{1}{4}c_{ijk}c_{ijk} - \frac{1}{2}c_{ijk}c_{kji}. \quad (3)$$

Note that the above formula has no ambiguities even if the product is not commutative and therefore allows a straightforward generalization to the setting of noncommutative geometry. An explicit computation of the curvature given by formula (3) for the noncommutative torus was presented in [6].

The usual standard basis of orthonormal frames on the *flat* algebra over the Moyal plane is spanned by two derivations  $\delta_i$ ,  $i = 1, 2$ . For the conformally rescaled metric we shall assume that the new basis of frames be  $e_i = h\delta_i$  for  $h$  in the Moyal algebra or its multiplier [24]. We assume that  $h$  is positive and invertible, by  $h_*^{-1}$  we denote the inverse with respect to the Moyal product, similarly all powers are also Moyal powers.

We have:

$$[e_1, e_2] = h * \delta_1(h) * h_*^{-1} e_2 - h * \delta_2(h) * h_*^{-1} e_1,$$

so that

$$c_{122} = h * \delta_1(h) * h_*^{-1}, \quad c_{121} = -h * \delta_2(h) * h_*^{-1}$$

and

$$c_{212} = -h * \delta_1(h) * h_*^{-1}, \quad c_{211} = h * \delta_2(h) * h_*^{-1}.$$

We compute the scalar curvature with using formula (3):

$$R = 2h * \delta_i(h * \delta_i(h) * h_*^{-1}) - (h * \delta_i(h) * h_*^{-1})^2 - \frac{1}{2}(h * \delta_i(h) * h_*^{-1})^2 - \frac{1}{2}(h * \delta_i(h) * h_*^{-1})^2,$$

which after simplifications yields:

$$\begin{aligned} R &= 2h^2 * (\delta_{ii}h) * h_*^{-1} + 2h * \delta_i(h) * \delta_i(h) * h_*^{-1} - 2h^2 * \delta_i(h) * h_*^{-1} * \delta_i(h) * h_*^{-1} - 2h\delta_i(h) * \delta_i(h) * h_*^{-1} \\ &= 2h^2 * (\delta_{ii}h) * h_*^{-1} - 2h^2 * \delta_i(h) * h_*^{-1} * \delta_i(h) * h_*^{-1}. \end{aligned} \quad (4)$$

So, to solve the equation  $R(h) = C = \text{const}$  we need to solve:

$$(\Delta h) - \delta_i(h) * h_*^{-1} * \delta_i(h) = Ch_*^{-1},$$

where  $\Delta$  is the standard flat Laplace operator,  $\Delta = \delta_1^2 + \delta_2^2$ .

We shall present the proof of the existence of the solution in the matrix basis as well as compute explicitly the first term of the perturbative expansion for the Fubini–Study metric.

## B. Solution in the matrix basis

**Ansatz:** First we look for a radial solution:

$$h = \sum_{n=0}^{\infty} \phi_n f_{n,n}.$$

Using the action of partial derivatives and the Laplace operator on the basis,

$$\Delta f_{m,n} = \frac{2}{\theta} \left( -(m+n+1)f_{m,n} + \sqrt{(m+1)(n+1)}f_{m+1,n+1} + \sqrt{mn}f_{m-1,n-1} \right),$$

we obtain the following equation:

$$\begin{aligned} \sum_n \left( -(2n+1)\phi_n - n\phi_n^2\phi_{n-1}^{-1} - (n+1)\phi_n^2\phi_{n+1}^{-1} \right) f_{n,n} + \sum_n (n+1)(\phi_{n+1} + \phi_n)f_{n+1,n+1} + \sum_n n(\phi_{n-1} + \phi_n)f_{n-1,n-1} \\ = R \cdot \phi_n^{-1} f_{n,n}, \end{aligned}$$

where we have set  $R = C\theta$ . It yields the following recurrence relation:

$$\frac{n+1}{\phi_{n+1}}(\phi_{n+1}^2 - \phi_n^2) + \frac{n}{\phi_{n-1}}(\phi_{n-1}^2 - \phi_n^2) = R \cdot \phi_n^{-1}, \quad (5)$$

for  $n \geq 1$ . Note that although (5) is of the second order, it has only one degree of freedom, since for  $n = 0$  we have

$$\phi_1^2 - \phi_0^2 = R \frac{\phi_1}{\phi_0}. \quad (6)$$

The recurrence relation is a quadratic one, solving it for  $x = \phi_{n+1}$  we have:

$$(n+1)x^2 + x \left( \frac{n}{\phi_{n-1}} (\phi_{n-1}^2 - \phi_n^2) - \frac{R}{\phi_n} \right) - (n+1)\phi_n^2 = 0,$$

and it is easy to see that it has only one positive root. It can also be easily seen that the sum of roots is positive and since their products is  $-\phi_n^2$  then the positive root must be bigger than  $\phi_n$ . Hence the solution will be an increasing positive sequence. Since  $h$  needs to be a positive operator, we should start with an initial value  $\phi_0 > 0$  and take the positive root at each step to have  $\phi_n > 0$  for every  $n \in \mathbb{N}$ . Note that  $\phi_0 = 0$  is not allowed in (5).

### C. Asymptotic behavior and Gauss–Bonnet term

As we have shown, there exists a family of solutions yielding a positive constant scalar curvature for the Moyal plane. We shall now look for some special solutions and their asymptotic behavior. First of all, observe that in the orthonormal frame formalism we have  $\sqrt{g} = h^{-2}$ , so we can take as the noncommutative volume element  $h_*^{-2}$ . In order to have a finite volume, the growth of the sequence  $\phi_n$  must be faster than  $\sqrt{n}$  so that  $\phi_n^{-2}$  gives a summable series.

For each solution of the recurrence relation (5) we shall compute now the Gauss–Bonnet term:

$$\tau(\sqrt{g} * R) = \int_{\mathbb{R}^2} \sqrt{g} R,$$

where, again,  $\sqrt{g}$  is the noncommutative Moyal volume element. Note that the expression has no ambiguity because of the trace property of  $\tau$  and we need to compute:

$$\begin{aligned} \int_{\mathbb{R}^2} \sqrt{g} R &= \tau \left( 2 \left( (\Delta h) - \delta_i(h) * h_*^{-1} * \delta_i(h) \right) * h^{-1} \right) \\ &= \frac{2\pi}{\theta} \int_0^\infty r dr \sum_{n=0}^\infty \left\{ \left[ -(2n+1) - n\phi_n\phi_{n-1}^{-1} - (n+1)\phi_n\phi_{n+1}^{-1} \right] f_{n,n} \right. \\ &\quad \left. + (n+1)(\phi_{n+1}\phi_n^{-1} + 1)f_{n+1,n+1} + n(\phi_{n-1}\phi_n^{-1} + 1)f_{n-1,n-1} \right\}. \end{aligned}$$

Now recall that  $\int_0^\infty r dr f_{n,n}(r) = \theta$  for all  $n \in \mathbb{N}$ . However, to compute the integral with need to introduce a cut-off in the series. We thus have

$$\begin{aligned} \sum_{n=0}^N \left[ -(2n+1) - n\phi_n\phi_{n-1}^{-1} - (n+1)\phi_n\phi_{n+1}^{-1} \right] + \sum_{n=0}^{N+1} n(\phi_n\phi_{n-1}^{-1} + 1) + \sum_{n=0}^{N-1} (n+1)(\phi_n\phi_{n+1}^{-1} + 1) \\ = (N+1)(\phi_{N+1}\phi_N^{-1} - \phi_N\phi_{N+1}^{-1}). \end{aligned}$$

The expression above has a finite and nonvanishing limit as  $N \rightarrow \infty$  if the sequence  $h_N = \phi_N\phi_{N-1}^{-1}$  has a limit 1 and  $N(h_N - 1)$  has a finite limit. This requirement alone is not sufficient to determine the asymptotic form of the solution. As the recurrence relation is highly nonlinear we can only check that some asymptotics are compatible with the relations as well as the above requirement for the Gauss–Bonnet term. In particular, possible asymptotics include the power-growing sequences  $\phi_n \sim An^a$  as well as power-growing sequences modified by logarithms.

To have an insight into the entire family of possible solutions we have carried out a numerical study of the solutions (see fig. 1), which confirms that the asymptotics is of that type and gives a glimpse of the relation  $a = a(\phi_0)$ .

Assuming that asymptotically  $\phi_n \sim An^a$  as  $n$  tends to  $\infty$  for some  $A, a > 0$ , then

$$\lim_{N \rightarrow \infty} (N+1)(\phi_{N+1}\phi_N^{-1} - \phi_N\phi_{N+1}^{-1}) = 2a,$$

and

$$\int_{\mathbb{R}^2} \sqrt{g} R = 8\pi a,$$

which is, in fact, quite similar to the classical case.

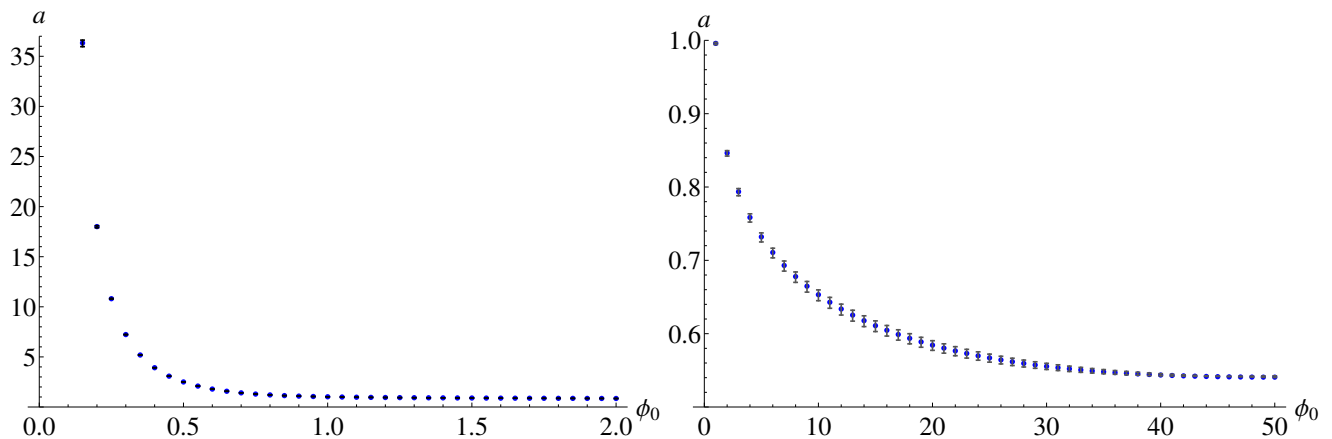


FIG. 1. The above plots were obtained by a numerical computation of  $\phi_n$  up to  $n = 5000$  starting from a given value of  $\phi_0$ . The asymptotic exponent  $a$  is then computed as  $\log \phi_N / \log N$  at  $N = 5000$ . The error bars are obtained by performing the numerical computation for  $N = 6000$  (upper) and  $N = 4000$  (lower). They show the stability of the numerical algorithm.

#### D. The Moyal–Fubini–Study metric

It is obvious from the above computations that in the case of linear asymptotics  $\phi_n \sim n$ , that is  $a = 1$ , we obtain the same Gauss–Bonnet term as for the classical sphere and therefore the solution to (5) with  $\phi_n \sim n$  could be understood as the Moyal–Fubini–Study solution.

Although we cannot solve exactly the recurrence relation even in this particular case, one can systematically find the asymptotics of  $\phi_n$ . Explicit computations give up to  $o(\frac{1}{n^3})$ :

$$\phi_n = n + \frac{1}{2}(R+1) + \frac{1}{8}\frac{1}{n} - \frac{13R+9}{144}\frac{1}{n^2} + \frac{1}{32}\left(-\frac{1}{4} + \frac{26}{9}R + \frac{29}{18}R^2\right)\frac{1}{n^3} + \dots$$

We remark as well that the case  $a = 1$  corresponds exactly to the asymptotic behavior of the coefficients of the classical Fubini–Study metric. Therefore the Moyal–Fubini–Study is, in fact, a perturbation of the classical Fubini–Study metric.

#### E. Curvature à la Rosenberg

In this section we would like to compare the above computation to the curvature à la Rosenberg [16]. To recall briefly, in two dimensions we have (classically)

$$R = g^{\mu\nu}R_{\mu\nu} = g^{\mu\nu}g^{\kappa\lambda}R_{\kappa\mu\lambda\nu} = 2g^{11}g^{22}R_{1212}.$$

In Rosenberg’s convention  $g^{11} = g^{22} = e^{-h} =: H^{-1}$  and (see [16, (4.3)])

$$R_{1212} = -\frac{1}{2}(\Delta(H) - \delta_i(H)H^{-1}\delta_i(H)).$$

Note, that since  $[H, \delta(H)] \neq 0$ , the scalar curvature is not uniquely defined in this framework as we can always choose  $g^{11}R_{1212}g^{22} \neq g^{11}g^{22}R_{1212}$ . However, if we are interested in the case of the constant curvature it does not affect the equation:

$$\Delta(H) - \delta_i(H)H^{-1}\delta_i(H) = -CH^{-2}. \quad (7)$$

If one takes a radial ansatz for  $H$  as we did in Section IV B, one discovers that (7) yields a recursion relation very similar to (5),

$$\frac{n+1}{\phi_{n+1}}(\phi_{n+1}^2 - \phi_n^2) + \frac{n}{\phi_{n-1}}(\phi_{n-1}^2 - \phi_n^2) = -c\phi_n^{-2}.$$



Let us note that both formulae (4) and (7) have correct classical (i.e.  $[H, \delta(H)] = 0$ ) limits since

$$\begin{aligned} R(g_{ij} = H\delta_{ij}) &= H^{-3} (\delta_i(H)\delta_j(H) - H\Delta H) \\ R(g_{ij} = h^{-2}\delta_{ij}) &= 2(h\Delta h - \delta_i(h)\delta_i(h)). \end{aligned}$$

This suggests that the curvature à la Rosenberg is the same as the one obtained in the orthonormal frame formalism with  $H = h^{-2}$ . However, although this is true in the commutative limit, the formulae do not match exactly in the non-commutative framework. Indeed,

$$R_R(H = h^{-2}) = h(\Delta h) + (\Delta h)h - h\delta_i(h)h^{-1}\delta_i(h) - \delta_i(h)h^{-1}\delta_i(h)h + \delta_i(h)\delta_i(h) - h\delta_i(h)h^{-2}\delta_i(h)h.$$

*Remark IV.1.* It is also possible to compare our result with the outcomes of the formalism based on a deformed diffeomorphism algebra [20–22]. If one sets  $G_{11} = G_{22} = H$  and uses formulae (8.23), (9.3), (9.4) and (9.5) from [20], then one obtains (unambiguously)  $R = -H_*^{-1} * (\partial_\mu \partial_\mu H - (\partial_\mu H) * H^{-1} * (\partial_\mu H)) * H_*^{-1}$ . This leads to the same equation for the scalar curvature as the one obtained in Rosenberg's formalism (7).

## V. PERTURBATIVE SOLUTION FOR THE MOYAL–FUBINI–STUDY METRIC

In the physics literature the deformation parameter  $\theta$  is frequently treated as a small perturbation parameter and all possible physical quantities and fields are computed up to certain order in  $\theta$ . We can follow this principle and attempt to solve the equation (IV A) up to the next leading order around the classical Fubini–Study solution. We begin with some technical computations.

### A. Perturbative expansion of the Moyal product

Using the formal expression for the Moyal product of two functions on  $\mathbb{R}^2$ :

$$(f * g)(p) = \left( e^{\frac{i\theta}{2} \partial_p \partial_q} f(p)g(q) \right)_{|p=q},$$

we obtain the following perturbative expansion of the Moyal product of two radial functions:

$$f * g = fg - \frac{\theta^2}{8r} (f''g' + f'g''),$$

where derivatives are with respect to the argument  $r$ .

As a consequence the following gives the perturbative formula for the Moyal inverse of the radial function:

$$f_*^{-1} = f^{-1} + \frac{\theta^2}{4r} ((f')^3 f^{-4} - f''f'f^{-3}).$$

We shall also need the formula for the perturbative expansion of the following product:

$$\delta_i(f) * f_*^{-1} * \delta_i(f).$$

Explicit computations give:

$$\begin{aligned} \delta_i(f) * f_*^{-1} * \delta_i(f) &= (f')^2 f^{-1} + \theta^2 \left( \frac{1}{4} \frac{1}{r^4} (f')^2 f^{-1} + \frac{1}{2} \frac{1}{r^3} (f')^3 f^{-2} - \frac{1}{2} \frac{1}{r^3} f' f'' f^{-1} + \frac{1}{r^2} (f')^4 f^{-3} \right. \\ &\quad - \frac{3}{2} \frac{1}{r^2} (f')^2 f'' f^{-2} + \frac{1}{4} \frac{1}{r} (f')^5 f^{-4} + \frac{1}{4} \frac{1}{r^2} (f') f''' f^{-1} + \frac{1}{4} \frac{1}{r^2} (f'')^2 f^{-1} \\ &\quad \left. - \frac{3}{4} \frac{1}{r} (f')^3 f'' f^{-3} + \frac{1}{4} \frac{1}{r} (f')^2 f''' f^{-2} + \frac{1}{2} \frac{1}{r} f' (f'')^2 f^{-2} - \frac{1}{4} \frac{1}{r} f'' f''' f^{-1} \right). \end{aligned}$$

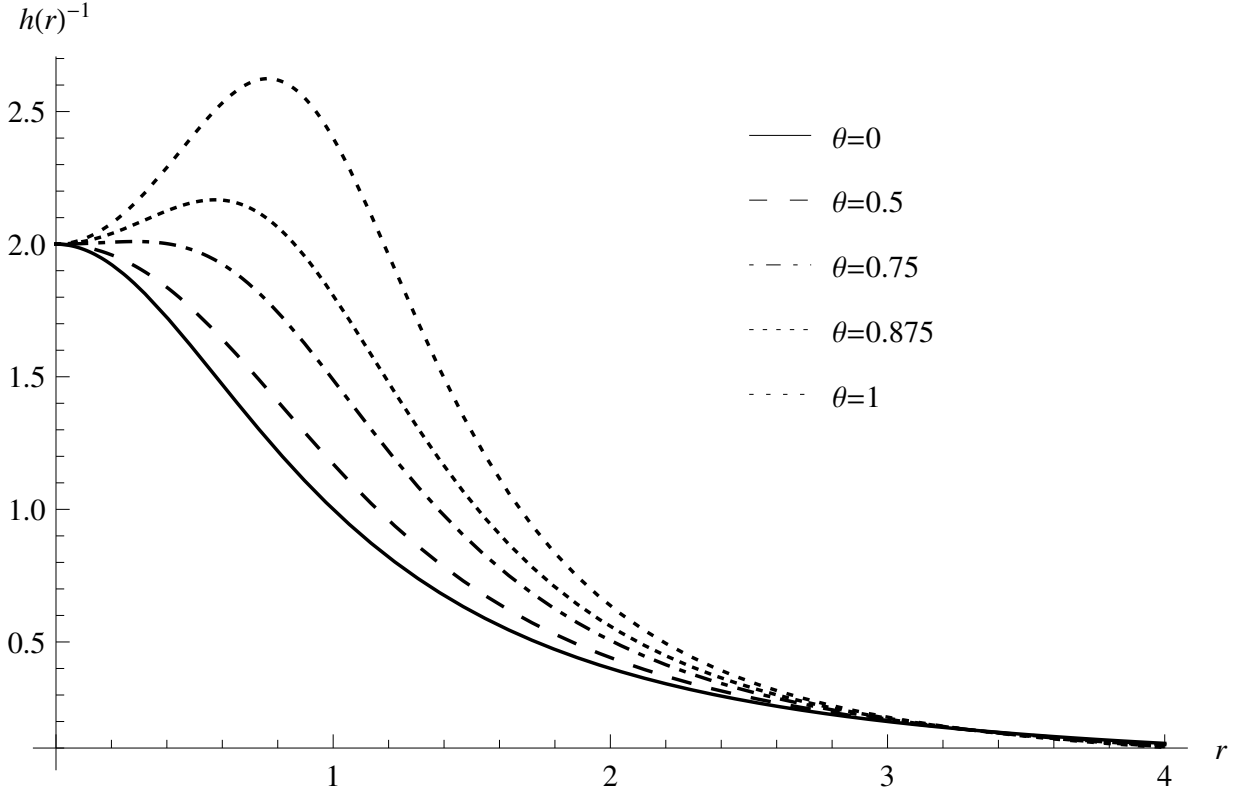


FIG. 2. The graph of the Moyal-Fubini-Study conformal factor  $h(r)^{-1}$  up to the order  $o(\theta^2)$ , for different values of  $\theta$ . The parameters are chosen  $C_1 = -\frac{2}{3}\eta$  (which yields  $\epsilon(0) = 0$ ) and  $\eta = \frac{1}{2}$  (which corresponds to  $R = 1$ ).

### B. The Moyal-Fubini-Study metric up to order $\theta^2$

We look for the solution of the constant curvature equation in the form:

$$h(r) = h_{FS}(r) + \theta^2 \epsilon(r) + o(\theta^2).$$

The classical Fubini-Study solution is  $h_{FS}(r) = \eta(1+r^2)$ , using the perturbative expansion derived above one obtains the following differential equation in order  $\theta^2$ :

$$\epsilon''(r) + \left( \frac{1}{r} - \frac{4r}{1+r^2} \right) \epsilon'(r) + \frac{4}{(1+r^2)} \epsilon(r) + 8\eta \frac{(1-r^2)}{(1+r^2)^3} = 0.$$

The most general solution is of the form:

$$\epsilon(r) = C_1(r^2 - 1) + C_2((r^2 - 1)\log(r) - 2) - \frac{\eta}{3} \frac{1}{1+r^2} \left( (1-r^4)\log(1+r^2) + 2(r^4 - 1)\log(r) + r^2 - 2 \right),$$

which, however, might have a logarithmic singularity at  $r=0$ . However, setting one of the integration constants,  $C_2 = \frac{2}{3}\eta$ , fixes the problem. The second integration constant determines the value of  $\epsilon$  at  $r=0$  as  $\epsilon(0) = -C_1 - \frac{2}{3}\eta$ .

In fig. 2 we present the plot showing how the Fubini-Study conformal factor gets altered by the deformation parameter  $\theta$ . Note that we have displayed the graph of  $1/h$ , which is equal to the conformal factor  $k$  from Section 2, in the case when  $\theta = 0$ .

## VI. CONCLUSIONS AND DISCUSSION

We have demonstrated that on the Moyal plane there exists a family of metrics which resemble the classical Fubini-Study metric and for which the scalar curvature computed in the orthonormal frame formalism is constant.

Unlike in the situation of the noncommutative torus some explicit computation and explicit solutions for the conformally rescaled metric are possible both in the matrix basis formalism and as a perturbative solution in  $\theta$ . Since the  $C^*$ -completion of the Moyal algebra is the algebra of compact operators, it is natural to consider its simplest unitization as a model for the quantum sphere like the standard Podleś sphere [14]. As the  $C^*$ -algebras describe only topology this alone cannot distinguish between different geometries, which could be constructed over these noncommutative topological spaces. Therefore only the examples of metrics in fact provide relevant noncommutative geometries.

The metric we have found would be a natural candidate for the “Moyal sphere”, a spherical noncommutative geometry with a metric that has constant scalar curvature. This is a basis to further studies of such geometries and their extensions. First of all, it would be interesting to see whether the constructed metric leads to a finitely summable spectral triple of dimension 2 over the unitized Moyal algebra (note that the unitization is different from the one in [9]) and to verify whether the distance function between the states on the Moyal algebra, in particular vector states and coherent states (see [2, 12]) is bounded from above.

A further question concerns the existence and description of a three-dimensional noncommutative sphere with an action of  $U(1)$  group so that the “Moyal sphere” is a fixed-point subalgebra (homogeneous space). This would lead to the construction and studies of Moyal magnetic monopole solutions.

As Moyal deformation is used as a model for noncommutative space-time (in 4 dimensions) one can check whether similar Fubini–Study type solutions and geometries exist in higher dimensions.

The sphere-like metric could also provide a method for the regularization of the quantum field theory over the Moyal space. The usual approach has a severe infrared problem [13] as result of unbounded distances. The metric regularization might provide an alternative solution to the harmonic oscillator approach [11].

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- [24] Note that the rescaled frames are no longer derivations, however, if  $h$  is taken from the commutant of the algebra (or its multiplier),  $e_i$  will be derivations from the Moyal algebra into the algebra of bounded operators on the Hilbert space [6]. Since this does not change anything in the computations, for the sake of simplicity we work with  $h$  from the algebra itself.