Progress in solving a noncommutative quantum field theory in four dimensions

Harald GROSSE[1] and Raimar WULKENHAAR[2]

[1] *Fakultät für Physik, Universität Wien*
Boltzmanngasse 5, A-1090 Wien, Austria

[2] *Mathematisches Institut der Westfälischen Wilhelms-Universität*
Einsteinstraße 62, D-48149 Münster, Germany

Abstract

We study the noncommutative ϕ^4_4-quantum field theory at the self-duality point. This model is renormalisable to all orders as shown in earlier work of us and does not have a Landau ghost problem. Using the Ward identity of Disertori, Gurau, Magnen and Rivasseau, we obtain from the Schwinger-Dyson equation a non-linear integral equation for the renormalised two-point function alone. The non-trivial renormalised four-point function fulfils a linear integral equation with the inhomogeneity determined by the two-point function. These integral equations are the starting point for a perturbative solution. In this way, the renormalised correlation functions are directly obtained, without Feynman graph computation and further renormalisation steps.

[1]harald.grosse@univie.ac.at
[2]raimar@math.uni-muenster.de

1 Introduction

In order to improve the problems of four-dimensional quantum field theory it was suggested to include "gravity effects" through deforming space-time. The canonical deformation is particularly simple, but the resulting models suffer from the UV/IR-mixing [1].

In our previous work [2] we found a way to handle this problem. We realised that the model defined by the action

$$S = \int d^4x \Big(\frac{1}{2}\phi(-\Delta + \Omega^2 \tilde{x}^2 + \mu^2)\phi + \frac{\lambda}{4}\phi \star \phi \star \phi \star \phi \Big)(x) \tag{1}$$

is renormalisable to all orders of perturbation theory. Here, $\star$ refers to the Moyal product parametrised by the antisymmetric 4×4-matrix Θ, and $\tilde{x} = 2\Theta^{-1}x$. The model is covariant under the Langmann-Szabo duality transformation [3] and becomes self-dual at $\Omega = 1$. Certain variants have also been treated, see [4] for a review.

Evaluation of the β-functions for the coupling constants Ω, λ in first order of perturbation theory leads to a coupled dynamical system which indicates a fixed-point at $\Omega = 1$, while λ remains bounded [5, 6]. The vanishing of the β-function at $\Omega = 1$ was next proven in [7] at three-loop order and finally in [8] to all orders of perturbation theory. It implies that there is no infinite renormalisation of λ, and a non-perturbative construction seems possible [9]. The Landau ghost problem is solved.

The vanishing of the β-function to all orders has been obtained using a Ward identity [8]. We extend this work and derive an integral equation for the two-point function alone by using the Ward identity and Schwinger-Dyson equations. Usually, Schwinger-Dyson equations couple the two-point function to the four-point function. In our model, we show that the Ward identity allows to express the four-point function in terms of the two-point function, resulting in an equation for the two-point function alone. This is achieved in the first step for the bare two-point function. We are able to perform the mass and wavefunction renormalisation directly in the integral equation, giving a *self-consistent non-linear equation for the renormalised two-point function alone.*

Higher n-point functions fulfil a *linear* (inhomogeneous) Schwinger-Dyson equation, with the inhomogeneity given by m-point functions with $m < n$. This means that solving our equation for the two-point function leads to a full non-perturbative construction of this interacting quantum field theory in four dimensions.

So far we treated our equation perturbatively up to third order in λ. The solution shows an interesting number-theoretic structure. It takes values in a polynomial ring with generators

$$\alpha,\ \beta,\ \frac{1-\alpha}{1-\alpha\beta},\ \frac{1-\beta}{1-\alpha\beta},\ \{I_{t(\alpha)}\},\ \{I_{t(\beta)}\} \tag{2}$$

and rational coefficients, where the $I_{t(\alpha)}$ are iterated integrals labelled by rooted trees. Similar structures also appeared in toy models for the Connes-Kreimer Hopf algebra [10]. The $I_{t(\alpha)}$ evaluate to polylogarithms and zeta functions [11].

We hope that a detailed analysis of our model will help for a non-perturbative treatment of more realistic quantum field theories.

2 Action functional and Ward identity

It is convenient to write the action (1) in the matrix base of the Moyal space, see [2, 12]. It simplifies enormously at the self-duality point $\Omega = 1$. We write down the resulting action functionals for the *bare* quantities, which involves the bare mass μ_{bare} and the wave function renormalisation $\phi \mapsto Z^{\frac{1}{2}}\phi$. For simplicity we fix the length scale to $\theta = 4$. This gives

$$S = \sum_{m,n\in\mathbb{N}^2_\Lambda} \frac{1}{2}\phi_{mn}H_{mn}\phi_{nm} + V(\phi) \;, \tag{3}$$

$$H_{mn} = Z\big(\mu^2_{bare} + |m| + |n|\big) \;, \qquad V(\phi) = \frac{Z^2\lambda}{4} \sum_{m,n,k,l\in\mathbb{N}^2_\Lambda} \phi_{mn}\phi_{nk}\phi_{kl}\phi_{lm} \;, \tag{4}$$

It is already used that this model has no renormalisation of the coupling constant [8]. All summation indices $m, n, \dots$ belong to $\mathbb{N}^2$, with $|m| := m_1 + m_2$. The symbol $\mathbb{N}^2_\Lambda$ refers to a cut-off in the matrix size. The scalar field is real, $\phi_{mn} = \overline{\phi_{nm}}$.

We recall the derivation of the Ward identity from [8]. We study a unitary transformation $\phi_{mn} \mapsto \sum_{k,l\in\mathbb{N}^2_\Lambda} U_{mk}\phi_{kl}U^\dagger_{ln}$ and its infinitesimal version

$$\phi_{mn} \mapsto \phi_{mn} + \mathrm{i} \sum_{k\in\mathbb{N}^2_\Lambda} (B_{mk}\phi_{kn} - \phi_{mk}B_{kn}) \;. \tag{5}$$

In contrast to the action functional, the partition function

$$\mathcal{Z}[J] = N \int \mathcal{D}\phi \; e^{-S+\mathrm{tr}(\phi J)} \tag{6}$$

will be invariant under such a transformation. The measure is $\mathcal{D}\phi = \prod_{m,n\in\mathbb{N}^2_\Lambda} d\phi_{mn}$, again with cut-off in the matrix size. The trace is given by $\mathrm{tr}(\phi J) = \sum_{k,l\in\mathbb{N}^2_\Lambda} \phi_{kl}J_{lk}$. We consider the variation of the generating functional $W = \ln \mathcal{Z}$ of connected functions:

$$\begin{aligned} 0 = \frac{\delta W}{\mathrm{i}\delta B_{ab}} &= \frac{1}{\mathcal{Z}} \int \mathcal{D}\phi \,\Big(-\frac{\delta S}{\mathrm{i}\delta B_{ab}} + \frac{\delta}{\mathrm{i}\delta B_{ab}}\big(\mathrm{tr}(\phi J)\big)\Big) e^{-S+\mathrm{tr}(\phi J)} \\ &= \frac{1}{\mathcal{Z}} \int \mathcal{D}\phi \, \sum_n \Big((H_{nb} - H_{an})\phi_{bn}\phi_{na} + (\phi_{bn}J_{na} - J_{bn}\phi_{na})\Big) e^{-S+\mathrm{tr}(\phi J)} \;. \end{aligned} \tag{7}$$

In the perturbative expansion, the fields in interaction vertices are written as derivatives with respect to the sources, $\phi_{mn} \mapsto \frac{\delta}{\delta J_{nm}}$. After functional integration, we obtain the Ward identity

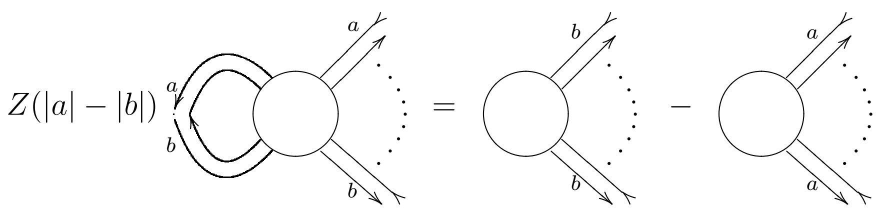
$$\begin{aligned} 0 = \Big\{ \sum_n \Big((H_{nb} - H_{an}) \frac{\delta^2}{\delta J_{nb}\, \delta J_{an}} + \Big(J_{na}\frac{\delta}{\delta J_{nb}} - J_{bn}\frac{\delta}{\delta J_{an}}\Big)\Big) \\ \times \exp\Big(-V\big(\tfrac{\delta}{\delta J}\big)\Big) e^{\frac{1}{2}\sum_{p,q} J_{pq}H^{-1}_{pq}J_{qp}} \Big\}_c \;. \end{aligned} \tag{8}$$

Only the connected functions (symbolised by the subscript c) are generated. The Ward identity (8) tells us that inserting into the connected graphs one special insertion vertex

$$V_{ab}^{ins} := \sum_n (H_{an} - H_{nb})\phi_{bn}\phi_{na} \tag{9}$$

is the same as the difference between the exchanges of external sources $J_{nb} \mapsto J_{na}$ and $J_{an} \mapsto J_{bn}$.

We write Feynman graphs in the Langmann-Szabo self-dual ϕ_4^4-model as ribbon graphs on a genus-g Riemann surface with B external faces. Adding for each external face an external vertex to get a closed surface, the matrix index is constant at every face. Inserting the special vertex V_{ab}^{ins} leads, however, to an index jump from a to b in an external face which meets an external vertex. The corresponding external sources at the jumped face are thus J_{na} and J_{bm} for some other indices m, n. According to the Ward identity, this is the same as the difference between the graphs with face index b and a, respectively:



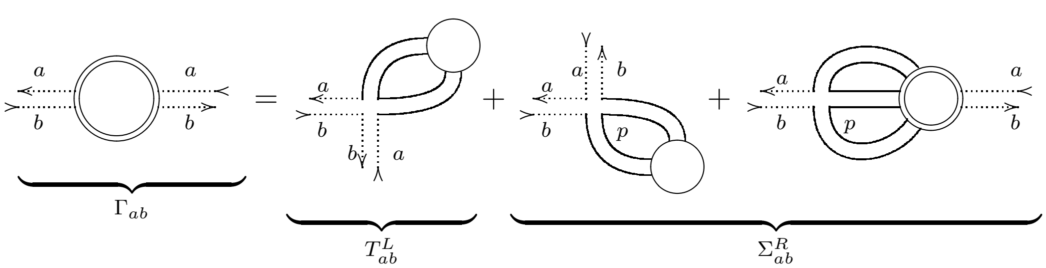
(10)

$$Z(|a| - |b|)G_{[ab]\ldots}^{ins} = G_{b\ldots} - G_{a\ldots}\ . \tag{11}$$

The dots in (11) stand for the remaining face indices. We have used $H_{an} - H_{nb} = Z(|a| - |b|)$.

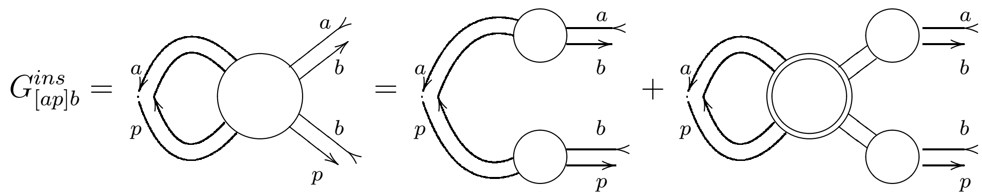
3 Two-point Schwinger-Dyson equation

We consider the Schwinger-Dyson equation for the one-particle irreducible (1PI) *planar* two-point function with respect to the leftmost vertex:



(12)

A double circle in (12) stands for 1PI subgraphs, a single circle for connected graphs. In the graphs contributing to Σ_{ab}^R we open the p-face and compare it with the insertion into the connected two-point function. There are two different places of an insertion: either into a one-particle-*reducible* propagator, or into an 1PI two-point function:



(13)

We amputate the upper G_{ab} two-point function and sum over p. After multiplication by the vertex $Z^2\lambda$, the result is precisely the combination Σ^R_{ab} of graphs:

$$\Sigma^R_{ab} = Z^2\lambda \sum_p (G_{ab})^{-1} G^{ins}_{[ap]b} = -Z\lambda \sum_p (G_{ab})^{-1} \frac{G_{bp} - G_{ba}}{|p| - |a|} \,. \tag{14}$$

The last step follows from (11). The special case $a = b = 0$ and $Z = 1$ of (14) already appeared in [8]. *The fact that we obtained this formula for all $a, b \in \mathbb{N}^2$ allows us to derive a Schwinger-Dyson equation (16) which involves only the two-point function, not the four-point function as usual.* Noting that

$$G^{-1}_{ab} = H_{ab} - \Gamma_{ab} \tag{15}$$

and $T^L_{ab} = Z^2\lambda \sum_q G_{aq}$ in (12), we have for the connected function

$$Z^2\lambda \sum_q G_{aq} - Z\lambda \sum_p (G_{ab})^{-1} \frac{G_{bp} - G_{ba}}{|p| - |a|} = H_{ab} - G^{-1}_{ab} \,. \tag{16}$$

We stress that the two-point function is by definition symmetric, $\Gamma_{ab} = \Gamma_{ba}$, although this is not manifest in (16)!

We express this Schwinger-Dyson equation in terms of the 1PI function Γ_{ab}, because renormalisation is performed in the 1PI part. After rearranging of $1 = G^{-1}_{ab} G_{ba} = G_{bp} G^{-1}_{pb}$, we have

$$\Gamma_{ab} = Z^2\lambda \sum_p \Big(\frac{1}{H_{bp} - \Gamma_{bp}} + \frac{1}{H_{ap} - \Gamma_{ap}} - \frac{1}{H_{bp} - \Gamma_{bp}} \frac{(\Gamma_{bp} - \Gamma_{ab})}{Z(|p| - |a|)} \Big) \,. \tag{17}$$

To pass to renormalised quantities, we Taylor expand

$$\Gamma_{ab} = Z\mu^2_{bare} - \mu^2 + (Z-1)(|a| + |b|) + \Gamma^{ren}_{ab} \,, \tag{18}$$

$$\Gamma^{ren}_{00} = 0 \qquad (\partial\Gamma^{ren})_{00} = 0 \,, \tag{19}$$

where $\partial\Gamma^{ren}$ is any of the derivatives with respect to a_1, a_2, b_1, b_2. This implies

$$G^{-1}_{ab} = |a| + |b| + \mu^2 - \Gamma^{ren}_{ab} \,. \tag{20}$$

Hence, μ is the renormalised mass, and both G_{ab} and Γ_{ab} should be regular if the cut-off in the matrix indices is removed. The resulting equation is

$$\begin{aligned}
&Z\mu_{bare}^2 - \mu^2 + (Z-1)(|a|+|b|) + \Gamma_{ab}^{ren} \\
&= \lambda \sum_p \Big(\frac{Z}{|b|+|p|+\mu^2-\Gamma_{bp}^{ren}} + \frac{Z^2}{|a|+|p|+\mu^2-\Gamma_{ap}^{ren}} \\
&- \frac{Z}{|b|+|p|+\mu^2-\Gamma_{bp}^{ren}} \frac{\Gamma_{bp}^{ren}-\Gamma_{ab}^{ren}}{(|p|-|a|)} \Big) \,. \qquad (21)
\end{aligned}$$

Notice the difference of the exponent of Z in the two tadpoles! Separating the first Taylor term we obtain

$$Z\mu_{bare}^2 - \mu^2 = \lambda \sum_p \Big(\frac{Z^2+Z}{|p|+\mu^2-\Gamma_{0p}^{ren}} - \frac{Z}{|p|+\mu^2-\Gamma_{0p}^{ren}} \frac{\Gamma_{0p}^{ren}}{|p|} \Big) \qquad (22)$$

and

$$\begin{aligned}
&(Z-1)(|a|+|b|) + \Gamma_{ab}^{ren} \\
&= \lambda \sum_p \Big(\frac{Z}{|b|+|p|+\mu^2-\Gamma_{bp}^{ren}} + \frac{Z^2}{|a|+|p|+\mu^2-\Gamma_{ap}^{ren}} - \frac{Z^2+Z}{|p|+\mu^2-\Gamma_{0p}^{ren}} \\
&- \frac{Z}{|b|+|p|+\mu^2-\Gamma_{bp}^{ren}} \frac{\Gamma_{bp}^{ren}-\Gamma_{ab}^{ren}}{|p|-|a|} + \frac{Z}{p+\mu^2-\Gamma_{0p}^{ren}} \frac{\Gamma_{0p}^{ren}}{|p|} \Big) \,. \qquad (23)
\end{aligned}$$

Deriving (23) at 0 with respect to a_i and b_i leads to a self-consistent system of equations for Z, Γ_{ab}^{ren}. In the next section we analyse this system for continuous indices $a, b \in \mathbb{R}_+ \times \mathbb{R}_+$.

4 Integral representation

For simplicity we replace the indices in $\mathbb{N}$ by continuous variables in $\mathbb{R}_+$. It is crucial that (23) depends only on the sums $|a| = a_1 + a_2$, $|b| = b_1 + b_2$ and $|p| = p_1 + p_2$ of indices. Therefore, also the two-point function Γ_{ab}^{ren} must depend on these sums only. This means that the sum $\sum_{p_1,p_2 \in \mathbb{N}_\Lambda}$ is replaced by the integral $\int_0^\Lambda |p| d|p|$, where we already introduced a cut-off $|p| = p_1 + p_2 \leq \Lambda$. Instead of (23) we thus have

$$\begin{aligned}
&(Z-1)(|a|+|b|) + \Gamma_{ab}^{ren} \\
&= \int_0^\Lambda |p|\, d|p| \Big(\frac{Z}{|b|+|p|+\mu^2-\Gamma_{bp}^{ren}} + \frac{Z^2}{|a|+|p|+\mu^2-\Gamma_{ap}^{ren}} - \frac{Z^2+Z}{|p|+\mu^2-\Gamma_{0p}^{ren}} \\
&- \frac{Z}{|b|+|p|+\mu^2-\Gamma_{bp}^{ren}} \frac{\Gamma_{bp}^{ren}-\Gamma_{ab}^{ren}}{(|p|-|a|)} + \frac{Z}{|p|+\mu^2-\Gamma_{0p}^{ren}} \frac{\Gamma_{0p}^{ren}}{|p|} \Big) \,, \qquad (24)
\end{aligned}$$

with $|a|,|b|,|p| \in \mathbb{R}_+$. We introduce a change of variables

$$
\begin{aligned}
&|a| =: \mu^2 \frac{\alpha}{1-\alpha}\;, \quad |b| =: \mu^2 \frac{\beta}{1-\beta}\;, \quad |p| =: \mu^2 \frac{\rho}{1-\rho}\;, \quad |p|\,d|p| = \mu^4 \frac{\rho\, d\rho}{(1-\rho)^3} \\
&\Gamma^{ren}_{ab} =: \mu^2 \frac{\Gamma_{\alpha\beta}}{(1-\alpha)(1-\beta)}\;, \quad \Lambda =: \mu^2 \frac{\xi}{1-\xi}
\end{aligned}
\tag{25}
$$

and obtain

$$
\begin{aligned}
&(Z-1)\Big(\frac{\alpha}{1-\alpha} + \frac{\beta}{1-\beta}\Big) + \frac{\Gamma_{\alpha\beta}}{(1-\alpha)(1-\beta)} \\
&= \lambda \int_0^\xi \frac{\rho\, d\rho}{(1-\rho)^2} \Big(\frac{Z^2(1-\alpha)}{1-\alpha\rho - \Gamma_{\alpha\rho}} - \frac{Z^2}{1-\Gamma_{0\rho}} \Big) \\
&- \lambda \int_0^\xi \frac{d\rho}{(1-\rho)} \Big(\frac{Z(1-\Gamma_{\beta\alpha})}{1-\beta\rho-\Gamma_{\beta\rho}} + \frac{Z\alpha}{1-\beta\rho-\Gamma_{\beta\rho}} \frac{\Gamma_{\beta\rho}-\Gamma_{\beta\alpha}}{\rho-\alpha} - \frac{Z}{1-\Gamma_{0\rho}} \Big)\;.
\end{aligned}
\tag{26}
$$

We have $\frac{\partial}{\partial a_i}\big|_{a=0} = \frac{\partial}{\partial |a|}\big|_{a=0} = (1-\alpha)^2 \frac{\partial}{\partial\alpha}\big|_{\alpha=0} = \frac{\partial}{\partial\alpha}\big|_{\alpha=0}$ so that we obtain with $\Gamma'_{0\rho} := \lim_{\alpha\to 0} \frac{\Gamma_{\alpha\rho}-\Gamma_{0\rho}}{\alpha}$ the following two relations for Z:

$$
Z - 1 = -Z\lambda \int_0^\xi \frac{d\rho}{(1-\rho)} \frac{(\rho + \Gamma'_{0\rho})}{(1-\Gamma_{0\rho})^2}\;,
\tag{27}
$$

$$
Z - 1 = Z^2\lambda \int_0^\xi \frac{\rho\, d\rho}{(1-\rho)^2} \Big(\frac{\rho + \Gamma'_{0\rho}}{(1-\Gamma_{0\rho})^2} - \frac{1}{1-\Gamma_{0\rho}} \Big) - Z\lambda \int_0^\xi \frac{d\rho}{(1-\rho)} \frac{1}{1-\Gamma_{0\rho}} \frac{\Gamma_{0\rho}}{\rho}\;.
\tag{28}
$$

We now express (26) in terms of the connected function $G_{\alpha\beta}$ defined by

$$
1 - \alpha\beta - \Gamma_{\alpha\beta} = \frac{1-\alpha\beta}{G_{\alpha\beta}}\;.
\tag{29}
$$

The result is

$$
\begin{aligned}
&ZG_{\alpha\beta} - 1 - (Z-1)\frac{(1-\alpha)(1-\beta)}{1-\alpha\beta} G_{\alpha\beta} \\
&= \lambda Z^2 G_{\alpha\beta} \frac{(1-\alpha)(1-\beta)}{1-\alpha\beta} \int_0^\xi \frac{\rho\, d\rho}{(1-\rho)^2} \Big(\frac{(1-\alpha) G_{\alpha\rho}}{1-\alpha\rho} - G_{0\rho} \Big) \\
&+ \lambda Z G_{\alpha\beta} \frac{(1-\alpha)(1-\beta)}{1-\alpha\beta} \int_0^\xi \frac{d\rho}{(1-\rho)} G_{0\rho} \\
&- \lambda Z (1-\alpha)(1-\beta) \int_0^\xi \frac{d\rho}{(1-\rho)} \Big(\frac{\rho}{1-\beta\rho} \frac{G_{\beta\rho}}{\rho-\alpha} - \frac{\alpha}{1-\beta\alpha} \frac{G_{\beta\alpha}}{\rho-\alpha} \Big)\;.
\end{aligned}
\tag{30}
$$

Using $\rho + \Gamma'_{0\rho} = \frac{\rho}{G_{0\rho}} + \frac{G'_{0\rho}}{G^2_{0\rho}}$, equation (27) is rewritten as

$$
(Z-1) = -Z\lambda \int_0^\xi \frac{d\rho}{1-\rho} \big(\rho G_{0\rho} + G'_{0\rho} \big)\;, \qquad \text{or}
\tag{31}
$$

$$
Z^{-1} = 1 + \lambda \int_0^\xi d\rho \frac{G_{0\rho}}{1-\rho} - \lambda \int_0^\xi d\rho \Big(G_{0\rho} - \frac{G'_{0\rho}}{1-\rho} \Big)\;.
\tag{32}
$$

We insert (31) into the last term of the first line of (30) and divide by Z:

$$
\begin{aligned}
G_{\alpha\beta} = Z^{-1} &+ \frac{\lambda}{Z^{-1}} G_{\alpha\beta} \frac{(1-\alpha)(1-\beta)}{1-\alpha\beta} \int_0^\xi \frac{\rho\, d\rho}{(1-\rho)^2} \Big(\frac{(1-\alpha) G_{\alpha\rho}}{1-\alpha\rho} - G_{0\rho} \Big) \\
&+ \lambda G_{\alpha\beta} \frac{(1-\alpha)(1-\beta)}{1-\alpha\beta} \int_0^\xi d\rho \Big(G_{0\rho} - \frac{G'_{0\rho}}{1-\rho} \Big) \\
&- \lambda (1-\alpha)(1-\beta) \int_0^\xi \frac{d\rho}{(1-\rho)} \Big(\frac{\rho}{1-\beta\rho} \frac{G_{\beta\rho}}{\rho-\alpha} - \frac{\alpha}{1-\beta\alpha} \frac{G_{\beta\alpha}}{\rho-\alpha} \Big) \,.
\end{aligned} \tag{33}
$$

Insertion of (32) gives

$$
\begin{aligned}
G_{\alpha\beta} = 1 + \lambda \Bigg\{ &\int_0^\xi d\rho \frac{G_{0\rho}}{1-\rho} + \frac{G_{\alpha\beta} \frac{(1-\alpha)(1-\beta)}{1-\alpha\beta} \int_0^\xi \frac{\rho\, d\rho}{(1-\rho)^2} \Big(\frac{(1-\alpha) G_{\alpha\rho}}{1-\alpha\rho} - G_{0\rho} \Big)}{1 + \lambda \int_0^\xi d\rho \frac{G_{0\rho}}{1-\rho} - \lambda \int_0^\xi d\rho \Big(G_{0\rho} - \frac{G'_{0\rho}}{1-\rho} \Big)} \\
&+ \Big(\frac{(1-\alpha)(1-\beta)}{1-\alpha\beta} G_{\alpha\beta} - 1 \Big) \int_0^\xi d\rho \Big(G_{0\rho} - \frac{G'_{0\rho}}{1-\rho} \Big) \\
&- (1-\alpha)(1-\beta) \int_0^\xi \frac{d\rho}{(1-\rho)} \Big(\frac{\rho}{1-\beta\rho} \frac{G_{\beta\rho}}{\rho-\alpha} - \frac{\alpha}{1-\beta\alpha} \frac{G_{\beta\alpha}}{\rho-\alpha} \Big) \Bigg\} \,.
\end{aligned} \tag{34}
$$

Rational fraction expansion yields

$$
\begin{aligned}
G_{\alpha\beta} = 1 + \lambda \Bigg\{ &G_{\alpha\beta} \frac{(1-\beta)}{1-\alpha\beta} \Big(\frac{(1-\alpha)\mathcal{K}^\xi_\alpha - \alpha \mathcal{X}^\xi + \mathcal{M}^\xi_\alpha - \mathcal{L}^\xi_\alpha}{1 + \lambda(\mathcal{X}^\xi - \mathcal{Y}^\xi)} - \alpha \ln(1-\xi) \Big) \\
&+ \Big(\frac{(1-\alpha)(1-\beta)}{1-\alpha\beta} G_{\alpha\beta} - 1 \Big) \mathcal{Y}^\xi \\
&+ \frac{(1-\alpha)}{1-\alpha\beta} \big(\mathcal{M}^\xi_\beta - \mathcal{L}^\xi_\beta \big) - \frac{\alpha(1-\beta)}{1-\alpha\beta} \big(\mathcal{L}^\xi_\beta + \mathcal{N}^\xi_{\alpha\beta} \big) \Bigg\} \,,
\end{aligned} \tag{35}
$$

where

$$
\mathcal{K}^\xi_\alpha := \int_0^\xi d\rho\, \frac{G_{\alpha\rho} - G_{0\rho}}{(1-\rho)^2} \,, \qquad \mathcal{L}^\xi_\alpha := \int_0^\xi d\rho\, \frac{G_{\alpha\rho} - G_{0\rho}}{(1-\rho)} \,, \tag{36}
$$

$$
\mathcal{M}^\xi_\alpha := \int_0^\xi d\rho\, \frac{\alpha\, G_{\alpha\rho}}{(1-\alpha\rho)} \,, \qquad \mathcal{N}^\xi_{\alpha\beta} := \int_0^\xi d\rho\, \frac{G_{\rho\beta} - G_{\alpha\beta}}{(\rho-\alpha)} \,, \tag{37}
$$

$$
\mathcal{X}^\xi := \int_0^\xi d\rho \frac{G_{0\rho}}{(1-\rho)} \,, \qquad \mathcal{Y}^\xi := \int_0^\xi d\rho \Big(G_{0\rho} - \frac{G'_{0\rho}}{(1-\rho)} \Big) \,. \tag{38}
$$

The functions $\mathcal{K}^\xi_\alpha, \mathcal{X}^\xi, \ln(1-\xi)$ are singular for $\xi \to 1$. Fortunately, these singularities cancel. For that we evaluate (35) separately for $\alpha=0$ and $\beta=0$:

$$G_{0\beta}=1+\lambda\Big(((1-\beta)G_{0\beta}-1)\mathcal{Y}^\xi+\mathcal{M}^\xi_\beta-\mathcal{L}^\xi_\beta\Big)\;, \tag{39}$$

$$\begin{aligned}G_{\alpha 0}=1+\lambda\Big(G_{\alpha 0}\Big\{\frac{(1-\alpha)\mathcal{K}^\xi_\alpha-\alpha\mathcal{X}^\xi+\mathcal{M}^\xi_\alpha-\mathcal{L}^\xi_\alpha}{1+\lambda(\mathcal{X}^\xi-\mathcal{Y}^\xi)}-\alpha\ln(1-\xi)\Big\}\\ +((1-\alpha)G_{\alpha 0}-1)\mathcal{Y}^\xi-\alpha\mathcal{N}^\xi_{\alpha 0}\Big)\;.\end{aligned} \tag{40}$$

Taking the symmetry $G_{\alpha 0}=G_{0\alpha}$ into account, the term in braces in (40) must be equal to $\mathcal{M}^\xi_\alpha-\mathcal{L}^\xi_\alpha+\alpha\mathcal{N}^\xi_{\alpha 0}$, so that (35) becomes

$$\begin{aligned}G_{\alpha\beta}=1+\lambda\Bigg(\frac{1-\beta}{1-\alpha\beta}\frac{G_{\alpha\beta}}{G_{0\alpha}}(\mathcal{M}^\xi_\alpha-\mathcal{L}^\xi_\alpha+\alpha\mathcal{N}^\xi_{\alpha 0})+\Big(\frac{(1-\alpha)(1-\beta)}{1-\alpha\beta}G_{\alpha\beta}-1\Big)\mathcal{Y}^\xi\\ +\frac{1-\alpha}{1-\alpha\beta}(\mathcal{M}^\xi_\beta-\mathcal{L}^\xi_\beta)-\frac{\alpha(1-\beta)}{1-\alpha\beta}(\mathcal{L}^\xi_\beta+\mathcal{N}^\xi_{\alpha\beta})\Bigg)\;.\end{aligned} \tag{41}$$

We have checked the equality between (35) and (41) perturbatively up to second order in λ; actually we discovered it in this way.

Since the model is renormalisable [2], the limit $\xi\to 1$ can be taken. We have thus proven:

Theorem 1 *The renormalised planar connected two-point function $G_{\alpha\beta}$ of self-dual non-commutative ϕ^4_4-theory (with continuous indices) satisfies the integral equation*

$$\begin{aligned}G_{\alpha\beta}=1+\lambda\Bigg(\frac{1-\alpha}{1-\alpha\beta}\big(\mathcal{M}_\beta-\mathcal{L}_\beta-\beta\mathcal{Y}\big)+\frac{1-\beta}{1-\alpha\beta}\big(\mathcal{M}_\alpha-\mathcal{L}_\alpha-\alpha\mathcal{Y}\big)\\ +\frac{1-\beta}{1-\alpha\beta}\Big(\frac{G_{\alpha\beta}}{G_{0\alpha}}-1\Big)\big(\mathcal{M}_\alpha-\mathcal{L}_\alpha+\alpha\mathcal{N}_{\alpha 0}\big)-\frac{\alpha(1-\beta)}{1-\alpha\beta}\big(\mathcal{L}_\beta+\mathcal{N}_{\alpha\beta}-\mathcal{N}_{\alpha 0}\big)\\ +\frac{(1-\alpha)(1-\beta)}{1-\alpha\beta}(G_{\alpha\beta}-1)\mathcal{Y}\Bigg)\;,\end{aligned} \tag{42}$$

where $\alpha,\beta\in[0,1)$,

$$\mathcal{L}_\alpha:=\int_0^1 d\rho\,\frac{G_{\alpha\rho}-G_{0\rho}}{1-\rho}\;,\quad \mathcal{M}_\alpha:=\int_0^1 d\rho\,\frac{\alpha\,G_{\alpha\rho}}{1-\alpha\rho}\;,\quad \mathcal{N}_{\alpha\beta}:=\int_0^1 d\rho\,\frac{G_{\rho\beta}-G_{\alpha\beta}}{\rho-\alpha}\;, \tag{43}$$

and $\mathcal{Y}=\lim_{\alpha\to 0}\frac{\mathcal{M}_\alpha-\mathcal{L}_\alpha}{\alpha}$.

5 Perturbative solution

The integral equation (42) is the starting point of a perturbative solution $G_{\alpha\beta}=\sum_{n=0}^\infty\lambda^n G^{(n)}_{\alpha\beta}$. This gives directly the renormalised planar two-point function, without need of Feynman graph computation and further renormalisation steps. In particular, all integrals in $\mathcal{L}_\alpha, \mathcal{M}_{\alpha\beta}, \mathcal{N}_{\alpha\beta}$ are regular (explicitly verified to $\mathcal{O}(\lambda^4)$). The solution is conveniently expressed in terms of *iterated integrals* labelled by *rooted trees*:

$$
\begin{aligned}
I_\alpha &:= \int_0^1 dx\, \frac{\alpha}{1-\alpha x} = -\ln(1-\alpha)\;, \\
I_{\underset{\bullet}{\alpha}} &:= \int_0^1 dx\, \frac{\alpha\, I_x}{1-\alpha x} = \mathrm{Li}_2(\alpha) + \frac{1}{2}\big(\ln(1-\alpha)\big)^2 \\
I_{\underset{\bullet\;\bullet}{\alpha}} &= \int_0^1 dx\, \frac{\alpha\, I_x \cdot I_x}{1-\alpha x} = -2\,\mathrm{Li}_3\Big(-\frac{\alpha}{1-\alpha}\Big)\;, \\
I_{\underset{\underset{\bullet}{\bullet}}{\alpha}} &= \int_0^1 dx\, \frac{\alpha\, I_{\underset{\bullet}{x}}}{1-\alpha x} = -2\mathrm{Li}_3\Big(-\frac{\alpha}{1-\alpha}\Big) - 2\mathrm{Li}_3(\alpha) - \ln(1-\alpha)\zeta(2) \\
&\qquad\qquad + \ln(1-\alpha)\mathrm{Li}_2(\alpha) + \frac{1}{6}\big(\ln(1-\alpha)\big)^3\;.
\end{aligned} \tag{44}
$$

Similar iterated integrals appeared in toy models for the Hopf algebra of Connes-Kreimer [10] (where the root is above). We find up to third order

$$
\begin{aligned}
G_{\alpha\beta} &= 1 + \lambda\Big\{A(I_\beta-\beta) + B(I_\alpha-\alpha)\Big\} \\
&\quad + \lambda^2\Big\{A\big(\beta I_{\underset{\bullet}{\beta}} - \beta I_\beta\big) - \alpha AB\big((I_\beta)^2 - 2\beta I_\beta + I_\beta\big) \\
&\qquad + B\big(\alpha I_{\underset{\bullet}{\alpha}} - \alpha I_\alpha\big) - \beta BA\big((I_\alpha)^2 - 2\alpha I_\alpha + I_\alpha\big) \\
&\qquad + AB\big((I_{\underset{\bullet}{\alpha}} - \alpha) + (I_{\underset{\bullet}{\beta}} - \beta) + (I_\alpha-\alpha)(I_\beta-\beta) + \alpha\beta(\zeta(2)+1)\big)\Big\} \\
&\quad + \lambda^3\Big\{A\mathcal{W}_\beta + \alpha AB\big(-\mathcal{U}_\beta + I_\alpha I_\beta + I_{\underset{\bullet}{\alpha}} I_\beta\big) + \alpha A^2B(\mathcal{V}_\beta) \\
&\qquad + B\mathcal{W}_\alpha + \beta BA\big(-\mathcal{U}_\alpha + I_\beta I_\alpha + I_{\underset{\bullet}{\beta}} I_\alpha\big) + \beta B^2A(\mathcal{V}_\alpha) \\
&\qquad + AB\big(\mathcal{T}_\beta + \mathcal{T}_\alpha - I_\beta(I_\alpha)^2 - I_\alpha(I_\beta)^2 - 6I_\alpha I_\beta\big) \\
&\qquad + AB^2\big((1-\alpha)(I_{\underset{\bullet}{\alpha}} - \alpha) + 3I_\alpha I_\beta + I_{\underset{\bullet}{\beta}} I_\alpha + I_\beta(I_\alpha)^2\big) \\
&\qquad + BA^2\big((1-\beta)(I_{\underset{\bullet}{\beta}} - \beta) + 3I_\alpha I_\beta + I_{\underset{\bullet}{\alpha}} I_\beta + I_\alpha(I_\beta)^2\big)\Big\} + \mathcal{O}(\lambda^4)\;,
\end{aligned} \tag{45}
$$

where we have defined

$$
\begin{aligned}
A &:= \frac{1-\alpha}{1-\alpha\beta}\;, \qquad B := \frac{1-\beta}{1-\alpha\beta}\;, \\
\mathcal{T}_\beta &:= \beta I_{\underset{\bullet}{\beta}} - \beta I_\beta + (I_\beta - \beta)\;, \\
\mathcal{U}_\beta &:= -\beta I_{\underset{\bullet\;\bullet}{\beta}} - (I_\beta)^3 + \beta I_{\underset{\bullet}{\beta}} I_\beta + 2I_{\underset{\bullet}{\beta}} I_\beta + \beta\zeta(2) I_\beta - 2\beta\zeta(3) \\
&\quad - 2(I_\beta)^2 + \beta(I_\beta)^2 + I_{\underset{\bullet}{\beta}} + \beta I_{\underset{\bullet}{\beta}} + 2I_\beta - \beta^2\;, \\
\mathcal{V}_\beta &:= \beta I_{\underset{\underset{\bullet}{\bullet}}{\beta}} - \beta^2 I_{\underset{\bullet\;\bullet}{\beta}} - 2\beta^2\zeta(3) + 2\beta I_{\underset{\bullet}{\beta}} I_\beta - I_\beta^3 + 2\beta I_\beta\zeta(2) - 3\beta^2\zeta(2) \\
&\quad + (1-\beta)\big(2\beta I_{\underset{\bullet}{\beta}} - 3I_\beta^2 + 3\beta I_\beta - 3I_\beta + \beta\big)\;, \\
\mathcal{W}_\beta &:= (I_{\underset{\underset{\bullet}{\bullet}}{\beta}} - \beta\zeta(2)) - \frac{1}{2}I_\beta \frac{I_\beta - \beta}{\beta} + \frac{1}{2}(I_\beta)^2 - (I_{\underset{\bullet}{\beta}} - \beta) - \frac{1}{2}(I_\beta - \beta) - \frac{1}{2}\beta^2\;.
\end{aligned} \tag{46}
$$

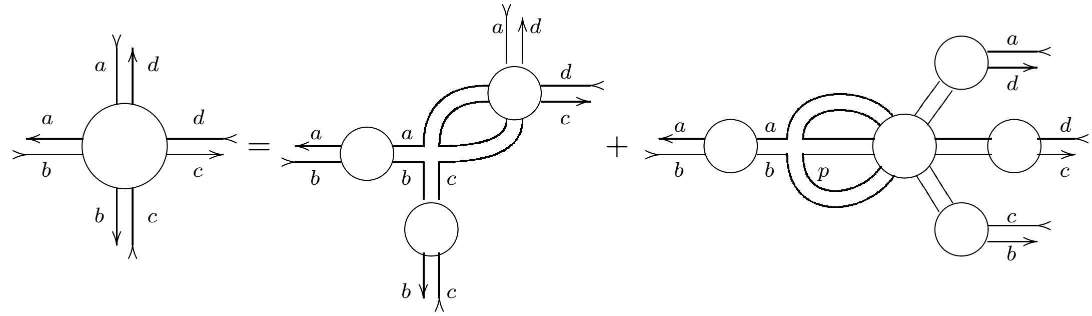
We notice that up to third order, the solution $G_{\alpha\beta}$ is a polynomial with rational coefficients in α, β, A, B, $\zeta(2)$, $\zeta(3)$ and the iterated integrals[1] (44). It is remarkable how the non-symmetric equation (42) leads to the symmetric solution for $G_{\alpha\beta}$!

It is tempting to conjecture that $G_{\alpha\beta}$ is at *any order* n a polynomial with rational coefficients in α, β, A, B, (multiple) zeta values [11] and iterated integrals labelled by rooted trees with at most n vertices. Proving this conjecture is a main step to prove Borel summability of the two-point function. Note that there are $n!$ (not necessarily connected) rooted trees (with multiplicities) with n vertices, which means that at order n in the perturbation series there would be only $\mathcal{O}(n!)$ independent contributions.

We show in the next section for $n = 4$ that the corresponding Schwinger-Dyson equation for an $(n > 2)$-point function is *linear* and inhomogeneous, with the inhomogeneity given by m-point functions with $m < n$. Such equations are straightforward to estimate if the two-point function is known. After all, this would be the very first construction of an interacting quantum field theory in four dimensions.

6 Four-point Schwinger-Dyson equation

Here we demonstrate for the planar four-point function that the knowledge of the two-point function permits a successive construction of the whole theory. Starting point is the Schwinger-Dyson equation for the planar connected four-point function G_{abcd}. Following the a-face in direction of the arrow, there is a distinguished vertex at which the first ab-line starts. For this vertex there are two possibilities for the matrix index of the diagonally opposite corner to the a-face: either c or a summation vertex p:



(47)

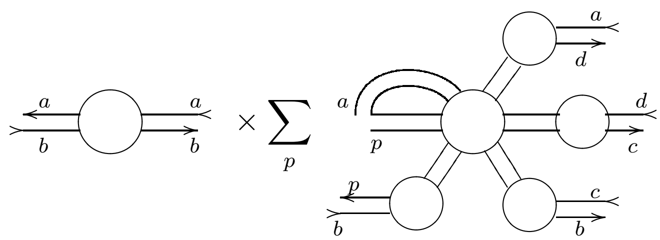
We let $G^{(1)}_{abcd}$ and $G^{(2)}_{abcd}$ be the corresponding two graphs on the rhs. We write $G^{(1)}_{abcd}$ as a product of the vertex $Z^2\lambda$, the left connected two-point function, the downward two-point function and an insertion, which is reexpressed by means of the Ward-identity:

$$\begin{aligned} G^{(1)}_{abcd} &= Z^2\lambda G_{ab}G_{bc}G^{ins}_{[ac]d} = Z\lambda G_{ab}G_{bc}\frac{1}{(|a|-|c|)}\big(G_{cd}-G_{ad}\big) \\ &= Z\lambda G_{ab}G_{bc}G_{cd}G_{da}\frac{1}{(|a|-|c|)}\Big(\frac{1}{G_{ad}}-\frac{1}{G_{cd}}\Big)\,. \end{aligned} \tag{48}$$

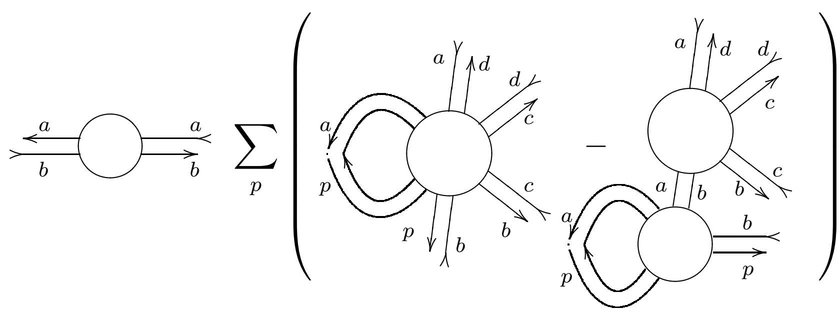
[1]There appears the integral $\dfrac{I_\alpha-\alpha}{\alpha} = \displaystyle\int_0^1 d\rho\;\frac{\alpha\rho}{1-\alpha\rho}$, which seems to be more appropriate than I_α itself.

In the last graph in (47) we open the p-face to get an insertion. However, this insertion is not into the full connected four-point function! The connected four-point function G_{abcd} contains at least one ab-line, which is not present in the subgraph under consideration. Therefore, we have to subtract from the general four-point insertion the insertion into the G_{ab} two-point function:

$G^{(2)}_{abcd} = Z^2\lambda$



$= Z^2\lambda$



(49)

In the very last graph, the whole ab-line is considered as part of the lower bubble, giving the insertion $G^{ins}_{[ap]b}$. The remaining upper bubble has the two-point function G_{ab} amputated, but together with the G_{ab} prefactor in front of the sum we obtain the full connected four-point function. In summary, we have

$$\begin{aligned} G^{(2)}_{abcd} &= Z^2\lambda\Big(\sum_p G_{ab}G^{ins}_{[ap]bcd} - G^{ins}_{[ap]b}G_{abcd}\Big) \\ &= Z\lambda\sum_p G_{ab}\frac{1}{|a|-|p|}\big(G_{pbcd} - G_{abcd}\big) \\ &\quad - Z\lambda\sum_p \frac{1}{|a|-|p|}\big(G_{pb} - G_{ab}\big)G_{abcd} \\ &= Z\lambda\sum_p \frac{1}{|a|-|p|}\big(G_{ab}G_{pbcd} - G_{pb}G_{abcd}\big)\;. \end{aligned} \tag{50}$$

After amputation of the external two-point functions we obtain the Schwinger-Dyson equation for the *renormalised* 1PI four-point function $G_{abcd} = G_{ab}G_{bc}G_{cd}G_{da}\Gamma^{ren}_{abcd}$ as follows:

$$\Gamma^{ren}_{abcd} = Z\lambda\frac{1}{|a|-|c|}\Big(\frac{1}{G_{ad}} - \frac{1}{G_{cd}}\Big) + Z\lambda\sum_p \frac{1}{|a|-|p|}G_{pb}\Big(\frac{G_{dp}}{G_{ad}}\Gamma^{ren}_{pbcd} - \Gamma^{ren}_{abcd}\Big)\;. \tag{51}$$

In terms to the 1PI function (20) we have

$$
\begin{aligned}
Z^{-1}\Gamma^{ren}_{abcd} &= \lambda\Big(1 - \frac{\Gamma^{ren}_{ad} - \Gamma^{ren}_{cd}}{|a| - |c|}\Big) \\
&+ \lambda \sum_p \frac{|a| + |d| + \mu^2 - \Gamma^{ren}_{ad}}{|p| + |b| + \mu^2 - \Gamma^{ren}_{pb}} \frac{\dfrac{\Gamma^{ren}_{pbcd} - \Gamma^{ren}_{abcd}}{|p| - |a|}}{|p| + |d| + \mu^2 - \Gamma^{ren}_{pd}} \\
&+ \lambda\Gamma^{ren}_{abcd} \sum_p \frac{1 - \dfrac{\Gamma^{ren}_{ad} - \Gamma^{ren}_{pd}}{|a| - |p|}}{(|p| + |b| + \mu^2 - \Gamma^{ren}_{pb})(|p| + |d| + \mu^2 - \Gamma^{ren}_{pd})}\;.
\end{aligned}
\tag{52}
$$

Passing to the integral representation and the variables (25), we find for $\Gamma_{\alpha\beta\gamma\delta} := \Gamma^{ren}_{abcd}$

$$
\begin{aligned}
Z^{-1}\Gamma_{\alpha\beta\gamma\delta} &= \lambda\Big(1 - \frac{(1-\gamma)\Gamma_{\alpha\delta} - (1-\alpha)\Gamma_{\gamma\delta}}{(1-\delta)(\alpha-\gamma)}\Big) \\
&+ \lambda \int_0^\xi \rho\, d\rho \frac{(1-\beta)(1-\alpha\delta - \Gamma_{\alpha\delta})}{(1-\beta\rho - \Gamma_{\beta\rho})} \frac{\dfrac{\Gamma_{\rho\beta\gamma\delta} - \Gamma_{\alpha\beta\gamma\delta}}{\rho - \alpha}}{1 - \delta\rho - \Gamma_{\delta\rho}} \\
&+ \lambda\Gamma_{\alpha\beta\gamma\delta} \int_0^\xi \frac{\rho\, d\rho}{(1-\rho)} \frac{(1-\beta)\Big((1-\delta) - \dfrac{(1-\rho)\Gamma_{\alpha\delta} - (1-\alpha)\Gamma_{\rho\delta}}{(\alpha-\rho)}\Big)}{(1-\beta\rho - \Gamma_{\beta\rho})(1-\delta\rho - \Gamma_{\delta\rho})} \\
&= \lambda\Big(\frac{1}{G_{\alpha\delta}} - \frac{(1-\alpha)(1-\gamma\delta)(G_{\alpha\delta} - G_{\gamma\delta})}{G_{\alpha\delta}G_{\gamma\delta}(1-\delta)(\alpha-\gamma)}\Big) \\
&+ \lambda \int_0^\xi \rho\, d\rho \frac{(1-\beta)(1-\alpha\delta)G_{\beta\rho}G_{\delta\rho}}{G_{\alpha\delta}(1-\beta\rho)(1-\delta\rho)} \frac{\Gamma_{\rho\beta\gamma\delta} - \Gamma_{\alpha\beta\gamma\delta}}{\rho - \alpha} \\
&- \lambda\Gamma_{\alpha\beta\gamma\delta} \int_0^\xi \rho\, d\rho \frac{(1-\beta)(1-\alpha\delta)G_{\beta\rho}}{G_{\alpha\delta}(1-\beta\rho)(1-\delta\rho)} \frac{(G_{\rho\delta} - G_{\alpha\delta})}{(\rho-\alpha)} \\
&+ \lambda\Gamma_{\alpha\beta\gamma\delta} \int_0^\xi d\rho \Big(\frac{G_{\beta\rho}}{1-\rho} - \frac{\beta G_{\beta\rho}}{1-\beta\rho} - \frac{G_{\beta\rho}(1-\beta)}{(1-\delta\rho)(1-\beta\rho)}\Big)\;.
\end{aligned}
\tag{53}
$$

Now we insert (32) for Z^{-1} and bring the last two lines to the lhs. It arises a combination where the limit $\xi \to 1$ exists:

Theorem 2 *The renormalised planar 1PI four-point function $\Gamma_{\alpha\beta\gamma\delta}$ of self-dual noncommutative ϕ_4^4-theory (with continuous indices $\alpha, \beta, \gamma, \delta \in [0, 1)$) satisfies the integral equation*

$$\Gamma_{\alpha\beta\gamma\delta} = \lambda \cdot \frac{\begin{array}{c}\Big(1 - \dfrac{(1-\alpha)(1-\gamma\delta)(G_{\alpha\delta} - G_{\gamma\delta})}{G_{\gamma\delta}(1-\delta)(\alpha-\gamma)} \\ \qquad + \displaystyle\int_0^1 \rho\, d\rho \frac{(1-\beta)(1-\alpha\delta)G_{\beta\rho}G_{\delta\rho}}{(1-\beta\rho)(1-\delta\rho)} \frac{\Gamma_{\rho\beta\gamma\delta} - \Gamma_{\alpha\beta\gamma\delta}}{\rho - \alpha}\Big)\end{array}}{\begin{array}{c}G_{\alpha\delta} + \lambda\Big((\mathcal{M}_\beta - \mathcal{L}_\beta - \mathcal{Y})G_{\alpha\delta} + \displaystyle\int_0^1 d\rho \frac{G_{\alpha\delta}G_{\beta\rho}(1-\beta)}{(1-\delta\rho)(1-\beta\rho)} \\ \qquad + \displaystyle\int_0^1 \rho\, d\rho \frac{(1-\beta)(1-\alpha\delta)G_{\beta\rho}}{(1-\beta\rho)(1-\delta\rho)} \frac{(G_{\rho\delta} - G_{\alpha\delta})}{(\rho - \alpha)}\Big)\end{array}} \tag{54}$$

In lowest order we find

$$\begin{aligned}\Gamma_{\alpha\beta\gamma\delta} = \lambda - \lambda^2 \Big(&\frac{(1-\gamma)(I_\alpha - \alpha) - (1-\alpha)(I_\gamma - \gamma)}{\alpha - \gamma} \\ &+ \frac{(1-\delta)(I_\beta - \beta) - (1-\beta)(I_\delta - \delta)}{\beta - \delta}\Big) + \mathcal{O}(\lambda^3)\;.\end{aligned} \tag{55}$$

Note that $\Gamma_{\alpha\beta\gamma\delta}$ is cyclic in the four indices, and that $\Gamma_{0000} = \lambda + \mathcal{O}(\lambda^3)$.

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