



The Φ_4^3 and Φ_6^3 matrixial QFT models have reflection positive two-point function

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Dedicated to the memory of Wolfhart Zimmermann (1928–2016)

Abstract

We extend our previous work (on $D = 2$) to give an exact solution of the Φ_D^3 large- \mathcal{N} matrix model (or renormalised Kontsevich model) in $D = 4$ and $D = 6$ dimensions. Induction proofs and the difficult combinatorics are unchanged compared with $D = 2$, but the renormalisation – performed according to Zimmermann – is much more involved. As main result we prove that the Schwinger 2-point function resulting from the Φ_D^3 -QFT model on Moyal space satisfies, for real coupling constant, reflection positivity in $D = 4$ and $D = 6$ dimensions. The Källén–Lehmann mass spectrum of the associated Wightman 2-point function describes a scattering part $|p|^2 \geq 2\mu^2$ and an isolated broadened mass shell around $|p|^2 = \mu^2$.

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1. Introduction

The Kontsevich model [1,2] is of paramount importance because it elegantly proves Witten's conjecture [3] about the equivalence of two approaches to quantum gravity in two dimensions:

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the Hermitean one-matrix model [4–6] versus the intersection theory on the moduli space of Riemann surfaces [7–9]. The Kontsevich model is defined by the partition function (we use different notation)

$$\mathcal{Z}[E] := \frac{\int d\Phi \exp\left(-\text{Tr}(E\Phi^2 + \frac{i}{6}\Phi^3)\right)}{\int d\Phi \exp\left(-\text{Tr}(E\Phi^2)\right)}, \tag{1.1}$$

where the integral is over the space of self-adjoint $(\mathcal{N} \times \mathcal{N})$ -matrices with Lebesgue measure $d\Phi$. Perturbative expansion in the ‘coupling constant’ $\frac{i}{6}$ yields an asymptotic expansion of $\log \mathcal{Z}[E]$ in terms of rational functions of the eigenvalues $\{e_i\}$ of the positive self-adjoint matrix E . Kontsevich’s main theorem states that $\log \mathcal{Z}[E]$ is, when viewed as rational function of $\{e_i\}$, the generating function for the intersection numbers.

In a series of papers [10–12], one of us (H.G.) with H. Steinacker turned the Kontsevich model into a quantum field theory on noncommutative Moyal space (inspired by the work of Langmann–Szabo–Zarembo [13]). The action (1.1) was generalised to admit a purely imaginary coupling constant $\frac{i}{6} \mapsto \frac{i}{6}\lambda$, and the interest was turned to correlation functions arising from the formal measure (1.1),

$$\langle \Phi_{k_1 l_1} \cdots \Phi_{k_N l_N} \rangle := \frac{\int d\Phi \Phi_{k_1 l_1} \cdots \Phi_{k_N l_N} \exp\left(-\text{Tr}(E\Phi^2 + A\Phi + \frac{i}{6}\lambda\Phi^3)\right)}{\int d\Phi \exp\left(-\text{Tr}(E\Phi^2 + A\Phi + \frac{i}{6}\lambda\Phi^3)\right)}. \tag{1.2}$$

With the usual trick of generating $\Phi_{kl} = \frac{\partial}{\partial J_{lk}} e^{\text{tr}(\Phi J)} \Big|_{J=0}$ under the integral and after absorbing J, λ in a redefinition of E and Φ , the known solution (in the large- \mathcal{N} limit) of the Kontsevich action (1.1) allows to formally derive all correlation functions (1.2). The term ‘formally’ refers to the fact that the large- \mathcal{N} limit produces the usual divergences of quantum field theory. Their removal by appropriate choice, depending on dimension $\{2, 4, 6\}$, of parameters in E, A, λ was achieved in [10–12]. Explicit formulae for the renormalised functions (1.2) at $N \in \{1, 2, 3\}$ and genus $g = 0$ were given.

In a recent paper [14] we transferred the solution strategy developed by two of us (H.G.+R.W.) for the Φ_4^4 -matricial quantum field theory [15] to the Φ_2^3 -Kontsevich model. This approach uses the Ward–Takahashi identity for the $U(\mathcal{N})$ adjoint action (first employed in [16]) to derive a system of Schwinger–Dyson equations in which the N -point function depend only on N' -point functions with $N' \leq N$. The initial step for the Kontsevich model is, in a special limit of large matrices coupled with an infinitely strong deformation parameter, an integral equation solved by Makeenko and Semenov [17]. From this point of origin we explicitly solved all correlation functions.

The present paper extends [14] to $D = 4$ and $D = 6$ dimensions. Similar as in [11,12] we can recycle almost everything from two dimensions; only the renormalisation is more involved. We follow the renormalisation philosophy advocated by Wolfhart Zimmermann according to which the theory is defined by *normalisation conditions*, at the physical energy scale, of a few relevant and marginal couplings. In its culmination due to Zimmermann [18], this BPHZ renormalisation scheme [19,20,18] completely avoids the use of ill-defined (divergent) quantities. That the BPHZ scheme extends smoothly to non-perturbative renormalisation is our first noticeable message.

For reasons explained below we are particularly interested in *real* Φ^3 coupling constant, i.e. $i\lambda \mapsto \lambda$ in (1.2). This is a drastic change! The partition function does not make any sense for real

coupling constants. Our point of view is to *define* quantum field theory by quantum equations of motion, i.e. Schwinger–Dyson equations. These equations can formally be derived from the partition function, but then we forget the partition function, declare the equations as exact and construct exact solutions. Whereas the Φ_6^3 -Kontsevich model with imaginary coupling constant is asymptotically free [12], our real Φ_6^3 -model has positive β -function. But this is not a problem; there is no Landau ghost, and the theory remains well-defined at any scale! In other words, the real Φ_6^3 -Kontsevich model could avoid triviality.

It is instructive to compare our exact results with a perturbative BPHZ renormalisation of the model. In $D = 6$ dimensions the full machinery of Zimmermann’s forest formula [18] is required. We provide in sec. 5 the BPHZ-renormalisation of the 1-point function up to two-loop order. One of the contributing graphs has an overlapping divergence with already 6 different forests. Individual graphs show the full number-theoretic richness of quantum field theory: up to two loops we encounter logarithms, polylogarithms Li_2 and $\zeta(2) = \frac{\pi^2}{6}$. The amplitudes of the graphs perfectly sum up to the Taylor expansion of the exact result.

The original BPHZ scheme with normalisation conditions at a single scale leads in just-renormalisable models to the renormalon problem which prevents Borel resummation of the perturbation series. We also provide in sec. 5 an example of a graph which shows the renormalon problem. But all these problems cancel as our exact correlation functions are analytic(!) in the coupling constant. Exact BPHZ renormalisation is fully consistent (for the model under consideration)!

The most significant result of this paper is derived in sec. 6. Matrix models such as the Kontsevich model Φ_D^3 arise from QFT-models on noncommutative geometry. The prominent Moyal space gives rise to an external matrix E having linearly spaced eigenvalues with multiplicity reflecting the dimension D . In [21] two of us (H.G.+R.W.) have shown that translating the type of scaling limit considered for the matrix model correlation functions back to the position space formulation of the Moyal algebra leads to Schwinger functions of an ordinary quantum field theory on \mathbb{R}^D . Euclidean symmetry and invariance under permutations are automatic. The most decisive Osterwalder–Schrader axiom [22,23], *reflection positivity*, amounts for the Schwinger 2-point function to the verification that the diagonal matrix model 2-point function is a Stieltjes function. We proved in [14] that for the $D = 2$ -dimensional Kontsevich model this is *not* the case. To our big surprise and exaltation, we are able to prove:

Theorem 1.1. *The Schwinger 2-point function resulting from the scaling limit of the Φ_D^3 -QFT model on Moyal space with real coupling constant satisfies reflection positivity in $D = 4$ and $D = 6$ dimensions. As such it is the Laplace–Fourier transform of the Wightman 2-point function*

$$\hat{W}_2(p_0, p_1, \dots, p_{D-1}) = \frac{\theta(p_0)}{(2\pi)^{D-1}} \int_0^\infty dM^2 \varrho\left(\frac{M^2}{\mu^2}\right) \delta(p_0^2 - p_1^2 - \dots - p_{D-1}^2 - M^2) \quad (1.3)$$

of a true relativistic quantum field theory [24] (θ, δ are the Heaviside and Dirac distributions). Its Källén–Lehmann mass spectrum $\varrho\left(\frac{M^2}{\mu^2}\right)$ [25,26] is explicitly known and has support on a scattering part with $M^2 \geq 2\mu^2$ and an isolated broadened mass shell around $M^2 = \mu^2$ of non-zero width.

The original Kontsevich model with purely imaginary coupling constant cannot be positive. This property is shared with the $\lambda\Phi_4^4$ matrixial quantum field theory [15] in which the stable phase

with $\lambda > 0$, where the partition functions has a chance to exist, cannot be reflection positive [21]. For $\lambda < 0$, where the partition function is meaningless, a lot of evidence has been given [27,28] for reflection positivity of the 2-point function.

The exciting question whether also the higher Schwinger functions of Φ_D^3 are reflection positive in $D \in \{4, 6\}$ is left for future investigation.

We dedicate this paper to the memory of Wolfhart Zimmermann. The scientific community owes him, amongst others, the LSZ formalism, momentum space Taylor subtraction under the integral, the forest formula to handle overlapping divergences, operator product expansion and the reduction-of-couplings scenario. Our contribution applies several of these concepts to a non-perturbative setting. We are convinced that Wolfhart Zimmermann would have enjoyed these results.

One of us (H.G.) was several times invited (mostly by Julius Wess) to the MPI in Munich at Föhringer Ring. But during these visits and after seminars we got little by little good contacts to Wolfhart, too. It was an honour to give a summary [29] of our earlier work with R.W. at the Ringberg Meeting celebrating the 80th birthday of Wolfhart Zimmermann. And afterwards I enjoyed phone calls from him showing his interests in our work. We will remember Wolfhart Zimmermann as a deep thinker and very kind person.

2. The setup

Let the ‘field’ $\Phi = \Phi^*$ be a self-adjoint operator, of finite rank $\leq \mathcal{N}$, on some infinite-dimensional Hilbert space \mathcal{H} . In the end we are interested in a limit $\mathcal{N} \rightarrow \infty$ to compact operators Φ . Let E be an unbounded self-adjoint positive operator on \mathcal{H} with compact resolvent. We consider the following action functional

$$S[\Phi] := V \operatorname{tr} \left(ZE\Phi^2 + (\kappa + \nu E + \zeta E^2)\Phi + \frac{\lambda_{bare} Z^{\frac{3}{2}}}{3} \Phi^3 \right), \tag{2.1}$$

where products and trace are understood in the algebra of finite rank operators and only the projection PEP to the finite-dimensional space $(\ker \Phi)^\perp = P\mathcal{H}$ contributes to (2.1). The parameter V is a constant discussed later, λ_{bare} is the bare coupling constant (real or complex) and κ, ν, ζ, Z as well as λ_{bare} and the lowest eigenvalue μ_{bare}^2 of $2E$ are functions of (V, \mathcal{N}) and renormalised parameters (λ_r, μ^2) .

The operator E plays the rôle of a generalised Laplacian for which the Theorem of H. Weyl applies on the recovery of the dimension from the asymptotics of the spectrum:

Definition 1. The operator E in (2.1) encodes a *spectral dimension*

$$D = \inf \left\{ p \in \mathbb{R}_+ : \operatorname{tr}((1 + E)^{-\frac{p}{2}}) < \infty \right\}. \tag{2.2}$$

The dimension D does not need to be an integer. We shall see, however, that renormalisation is only sensitive to the even integer $2\lfloor \frac{D}{2} \rfloor$.

- In [14] we treated $2\lfloor \frac{D}{2} \rfloor = 2$ where only the renormalisation parameter κ is necessary: the other parameters are $\nu = \zeta = 0, Z = 1, \mu_{bare}^2 = \mu^2, \lambda_{bare} = \lambda$.
- The most involved case is $2\lfloor \frac{D}{2} \rfloor = 6$, six dimensions in short. We need all six renormalisation parameters $\kappa, \nu, \zeta, Z, \lambda_{bare}, \mu_{bare}^2$.

- $2[\frac{D}{2}] = 4$, four dimensions in short, is an intermediate case where we need $\kappa, \nu, \mu_{bare}^2$ whereas $\zeta = 0, Z = 1$ and $\lambda_{bare} = \lambda$.
- $2[\frac{D}{2}] > 6$ breaks the structures and cannot be renormalised.

Local details about the eigenvalues of E (e.g. their degeneracy) are not important. In the simplest case all eigenvalues are different; here we would notationally proceed as in [14]. Spaces with larger symmetry, such as spheres and tori, have degenerate eigenvalues. In this paper we notationally assume a $\frac{D}{2}$ -fold cartesian product of a two-dimensional space with simple eigenvalues $\{E_{|\underline{n}|}\}_{|\underline{n}|=0}^\infty$, growing at most linearly in $|\underline{n}|$, which on the D -dimensional space arise with multiplicity $\binom{|\underline{n}|+\frac{D}{2}-1}{\frac{D}{2}-1}$. This allows us to label the eigenspaces by tuples $\underline{n} := (n_1, \dots, n_{\frac{D}{2}})$, $n_i \in \mathbb{N}$, of natural numbers (which include 0). Indeed, defining $|\underline{n}| := n_1 + \dots + n_{\frac{D}{2}}$, there are precisely $\binom{|\underline{n}|+\frac{D}{2}-1}{\frac{D}{2}-1}$ different tuples \underline{n} with the same $|\underline{n}|$. But again, these choices¹ only affect the notation in the first part; any E of the same $2[\frac{D}{2}] \leq 6$ leads to a solvable model with the same renormalisation prescription in the large-matrix limit.

From now on we assume that $D \in \{2, 4, 6\}$, that the projector P commutes with E and that the eigenvalues of E are a discretisation of a monotonously increasing C^1 -function e ,

$$E = (E_{\underline{n}} \delta_{\underline{m}, \underline{n}}), \quad E_{\underline{n}} := \frac{\mu_{bare}^2}{2} + \mu^2 e\left(\frac{|\underline{n}|}{\mu^2 V^{\frac{D}{2}}}\right), \quad e(0) \equiv 0. \tag{2.3}$$

The parameter $\mu > 0$ will become the renormalised mass, whereas the bare mass μ_{bare} is a function of $(V, \mathcal{N}, \lambda_r, \mu)$ identified later. Note that e being increasing amounts to a particular choice of the projection P : Representing $\Phi = (\Phi_{\underline{m}\underline{n}})_{\underline{m}, \underline{n} \in \mathbb{N}^{D/2}}$ in this special eigenbasis, the rank condition becomes $\Phi_{\underline{m}\underline{n}} = 0$ if $|\underline{m}| > \mathcal{N}$ or $|\underline{n}| > \mathcal{N}$. In the very end we are interested in a special limit $\mathcal{N} \rightarrow \infty$ which is independent of the ordering.

In these conventions, and with $\mathbb{N}_{\mathcal{N}}^{D/2} := \{\underline{m} \in \mathbb{N}^{D/2} : |\underline{m}| \leq \mathcal{N}\}$, the action (2.1) takes the following form (simplifications arise for $D = 2$ and $D = 4$):

$$\begin{aligned} S[\Phi] = & V \left(\sum_{\underline{n}, \underline{m} \in \mathbb{N}_{\mathcal{N}}^{D/2}} Z \Phi_{\underline{m}\underline{n}} \Phi_{\underline{n}\underline{m}} \frac{H_{\underline{m}\underline{n}}}{2} + \sum_{\underline{n} \in \mathbb{N}_{\mathcal{N}}^{D/2}} (\kappa + \nu E_{\underline{n}} + \zeta E_{\underline{n}}^2) \Phi_{\underline{n}\underline{n}} \right. \\ & \left. + \frac{\lambda_{bare} Z^{\frac{3}{2}}}{3} \sum_{\underline{n}, \underline{m}, \underline{l} \in \mathbb{N}_{\mathcal{N}}^{D/2}} \Phi_{\underline{n}\underline{m}} \Phi_{\underline{m}\underline{l}} \Phi_{\underline{l}\underline{n}} \right), \\ H_{\underline{m}\underline{n}} := & E_{\underline{m}} + E_{\underline{n}}. \end{aligned} \tag{2.4}$$

The action formally defines a partition function with external field J (which is also a self-adjoint matrix):

$$\begin{aligned} \mathcal{Z}[J] := & \int \mathcal{D}\Phi \exp(-S[\Phi] + V \text{tr}(J\Phi)) \\ = & K \exp\left(-\frac{\lambda_{bare} Z^{\frac{3}{2}}}{3V^2} \sum_{\underline{m}, \underline{n}, \underline{k} \in \mathbb{N}_{\mathcal{N}}^{D/2}} \frac{\partial^3}{\partial J_{\underline{m}\underline{n}} \partial J_{\underline{n}\underline{k}} \partial J_{\underline{k}\underline{m}}}\right) \mathcal{Z}_{free}[J], \end{aligned} \tag{2.5}$$

¹ Induced by the natural matrix representation of noncommutative Moyal space.

$$\mathcal{Z}_{free}[J] := \exp\left(\frac{V}{2} \sum_{\underline{m}, \underline{n} \in \mathbb{N}_{\mathcal{N}}^{D/2}} \frac{(J_{\underline{m}\underline{n}} - (\kappa + \nu E_{\underline{m}} + \zeta E_{\underline{m}}^2)\delta_{\underline{m}, \underline{n}})(J_{\underline{n}\underline{m}} - (\kappa + \nu E_{\underline{n}} + \zeta E_{\underline{n}}^2)\delta_{\underline{n}, \underline{m}})}{ZH_{\underline{m}\underline{n}}}\right), \tag{2.6}$$

where $K = \int \mathcal{D}\Phi \exp\left(-\frac{VZ}{2} \sum_{\underline{m}, \underline{n} \in \mathbb{N}_{\mathcal{N}}^{D/2}} \Phi_{\underline{m}\underline{n}} H_{\underline{m}\underline{n}} \Phi_{\underline{m}\underline{n}}\right)$.

The integral in the first line of (2.5) does not make any sense for real λ . We rather view the partition function as a tool to derive identities between the (formal) expansion coefficients $G_{|\underline{p}_1^1 \dots \underline{p}_{N_1}^1 | \dots | \underline{p}_1^B \dots \underline{p}_{N_B}^B|}$ of the logarithm

$$\log \frac{\mathcal{Z}[J]}{\mathcal{Z}[0]} =: \sum_{B=1}^{\infty} \sum_{1 \leq N_1 \leq \dots \leq N_B} \sum_{\underline{p}_1^1, \dots, \underline{p}_{N_B}^B \in \mathbb{N}_{\mathcal{N}}^{D/2}} V^{2-B} \frac{G_{|\underline{p}_1^1 \dots \underline{p}_{N_1}^1 | \dots | \underline{p}_1^B \dots \underline{p}_{N_B}^B|}}{S_{(N_1, \dots, N_B)}} \prod_{\beta=1}^B \frac{\mathbb{J}_{\underline{p}_1^{\beta} \dots \underline{p}_{N_{\beta}}^{\beta}}}{N_{\beta}}, \tag{2.7}$$

where $\mathbb{J}_{\underline{p}_1 \dots \underline{p}_{N_{\beta}}} := \prod_{j=1}^{N_{\beta}} J_{\underline{p}_j \underline{p}_{j+1}}$, with $N_{\beta} + 1 \equiv 1$, and with symmetry factor $S_{(N_1, \dots, N_B)} = \prod_{i=1}^s v_i!$ if $(N_1, \dots, N_B) = (\underbrace{N'_1, \dots, N'_1}_{v_1}, \dots, \underbrace{N'_s, \dots, N'_s}_{v_s})$ for pairwise different N'_i .

These $(N_1 + \dots + N_B)$ -point functions $G_{|\underline{p}_1^1 \dots \underline{p}_{N_1}^1 | \dots | \underline{p}_1^B \dots \underline{p}_{N_B}^B|}$ are obtained by partial J -derivatives of $\log \mathcal{Z}$ at $J = 0$. These external derivatives combine with the internal derivatives in (2.5) to identities between the $(N_1 + \dots + N_B)$ -point functions called *Schwinger–Dyson equations*. Our point of view is to declare these Schwinger–Dyson equations (although their derivation was purely formal) as *exact identities between their rigorous solutions* $G_{|\underline{p}_1^1 \dots \underline{p}_{N_1}^1 | \dots | \underline{p}_1^B \dots \underline{p}_{N_B}^B|}$. These Schwinger–Dyson equations alone are not enough. As decisive tool we also need *Ward–Takahashi identities* which reflect the fact that the partition function is unchanged under renaming the dummy integration variable Φ . This was first pointed out and used in [16]. For actions $S[\Phi] = V \text{Tr}(ZE\Phi^2 + \nu E\Phi + \zeta E^2\Phi) + S_{inv}[\Phi]$, where S_{inv} is invariant under unitary transformation $\Phi \mapsto U^*\Phi U$, one has

$$0 = \int \mathcal{D}\Phi V \left(Z(E\Phi^2 - \Phi^2 E) + \nu(E\Phi - \Phi E) + \zeta(E^2\Phi - \Phi E^2) - (J\Phi + \Phi J) \right) \times \exp(-S[\Phi] + V \text{tr}(\Phi J)). \tag{2.8}$$

The $(\underline{a}\underline{b})$ -component reads for diagonal E

$$\begin{aligned} & \frac{Z(E_{\underline{a}} - E_{\underline{b}})}{V} \sum_{\underline{n} \in \mathbb{N}_{\mathcal{N}}^{D/2}} \frac{\partial^2 \mathcal{Z}[J]}{\partial J_{\underline{b}\underline{n}} \partial J_{\underline{n}\underline{a}}} + \nu(E_{\underline{a}} - E_{\underline{b}}) \frac{\partial \mathcal{Z}[J]}{\partial J_{\underline{b}\underline{a}}} + \zeta(E_{\underline{a}}^2 - E_{\underline{b}}^2) \frac{\partial \mathcal{Z}[J]}{\partial J_{\underline{b}\underline{a}}} \\ & = \sum_{\underline{n} \in \mathbb{N}_{\mathcal{N}}^{D/2}} \left(J_{\underline{a}\underline{n}} \frac{\partial \mathcal{Z}[J]}{\partial J_{\underline{b}\underline{n}}} - J_{\underline{n}\underline{b}} \frac{\partial \mathcal{Z}[J]}{\partial J_{\underline{n}\underline{a}}} \right). \end{aligned} \tag{2.9}$$

We need a formula for $\sum_{\underline{n} \in \mathbb{N}_{\mathcal{N}}^{D/2}} \frac{\partial^2 \mathcal{Z}[J]}{\partial J_{\underline{b}\underline{n}} \partial J_{\underline{n}\underline{a}}}$, naively obtained by multiplication with $\frac{V}{Z(E_{\underline{a}} - E_{\underline{b}})}$.

But this needs discussion. There is first a kernel $W_{\underline{a}} \delta_{\underline{a}, \underline{b}} \mathcal{Z}[J]$ in $\sum_{\underline{n} \in \mathbb{N}_{\mathcal{N}}^{D/2}} \frac{\partial^2 \mathcal{Z}[J]}{\partial J_{\underline{b}\underline{n}} \partial J_{\underline{n}\underline{a}}}$ which was identified in [15]. In the present case we can restrict ourselves to $\underline{a} \neq \underline{b}$ where this kernel is absent. Remains the degeneracy problem: There is $E_{\underline{a}} - E_{\underline{b}} = 0$ even for $\underline{a} \neq \underline{b}$. The solution, already sketched in [15], goes as follows: Because their definition via the action (2.1) and

the definition (2.3) of E only involves the spectrum of E , the $(N_1 + \dots + N_B)$ -point functions $G_{|p_1^1 \dots p_{N_1}^1 | \dots | p_1^B \dots p_{N_B}^B |}$ only depend on $|p_i^\beta|$ (and λ), but not on p_i^β individually. If we can afford to exclude the diagonal $\underline{a} = \underline{b}$, then we can also afford to exclude $|\underline{a}| = |\underline{b}|$ and use the following identity, valid for $|\underline{a}| \neq |\underline{b}|$,

$$\sum_{\underline{n} \in \mathbb{N}_{\mathcal{N}}^{D/2}} \frac{\partial^2 \mathcal{Z}[J]}{\partial J_{\underline{b}\underline{n}} \partial J_{\underline{n}\underline{a}}} = \sum_{\underline{n} \in \mathbb{N}_{\mathcal{N}}^{D/2}} \frac{V}{Z(E_{\underline{a}} - E_{\underline{b}})} \left(J_{\underline{a}\underline{n}} \frac{\partial}{\partial J_{\underline{b}\underline{n}}} - J_{\underline{n}\underline{b}} \frac{\partial}{\partial J_{\underline{n}\underline{a}}} \right) \mathcal{Z}[J] - \frac{V}{Z} (\nu + \zeta H_{\underline{a}\underline{b}}) \frac{\partial \mathcal{Z}[J]}{\partial J_{|\underline{b}\underline{a}|}}. \tag{2.10}$$

Recall [14] that for $D = 2$ we had $Z = 1$ and $\nu = \zeta = 0$, whereas $Z = 1$ and $\zeta = 0$ for $D = 4$. These are minor differences: Only the prefactors of terms already present in the equations receive a modification; no new structure is created. This is the essence of multiplicative renormalisation. We will see that the new parameters permit to regularise the integrals up to $D = 6$. By extrapolation we would need for $D \geq 8$ a term $\propto E^3 \Phi$ in the action. But such a term gives rise to $E_{\underline{a}}^2 + E_{\underline{a}} E_{\underline{b}} + E_{\underline{b}}^2$ in (2.10) which completely destroys the previous structures. Hence, renormalisability is lost for $D \geq 8$.

3. Schwinger–Dyson equations and solution for $B = 1$

3.1. Equations

In the same way as in our previous paper [14], namely by inserting (2.5), (2.6) into the corresponding term of (2.7), we derive formulae for the connected N -point functions. We have to discuss separately the cases $N = 1$, $N = 2$ and $N \geq 3$. The equation for $G_{|\underline{a}|}$ is straightforward:

$$G_{|\underline{a}|} = \frac{1}{ZH_{\underline{a}\underline{a}}} \left\{ -\kappa - \nu E_{\underline{a}} - \zeta E_{\underline{a}}^2 - \lambda_{bare} Z^{\frac{3}{2}} \left(G_{|\underline{a}|}^2 + \frac{1}{V} \sum_{\underline{m} \in \mathbb{N}_{\mathcal{N}}^{D/2}} G_{|\underline{a}\underline{m}|} + \frac{G_{|\underline{a}|\underline{a}|}}{V^2} \right) \right\}. \tag{3.1}$$

The equation for $G_{|\underline{a}\underline{b}|}$ with $|\underline{a}| \neq |\underline{b}|$ reads:

$$\begin{aligned} G_{|\underline{a}\underline{b}|} &= \frac{1}{ZH_{\underline{a}\underline{b}}} - \frac{\lambda_{bare} Z^{\frac{1}{2}}}{V^2 H_{\underline{a}\underline{b}}} \frac{1}{\mathcal{Z}[0]} \sum_{\underline{m} \in \mathbb{N}_{\mathcal{N}}^{D/2}} \frac{\partial}{\partial J_{\underline{a}\underline{b}}} \frac{\partial}{\partial J_{\underline{b}\underline{m}}} \frac{\partial}{\partial J_{\underline{m}\underline{a}}} \mathcal{Z}[J] \Big|_{J=0} \\ &= \frac{1}{ZH_{\underline{a}\underline{b}}} \left(1 + \lambda_{bare} Z^{\frac{1}{2}} \frac{(G_{|\underline{a}|} - G_{|\underline{b}|})}{E_{\underline{a}} - E_{\underline{b}}} + \lambda_{bare} Z^{\frac{1}{2}} (\nu + \zeta H_{\underline{a}\underline{b}}) G_{|\underline{a}\underline{b}|} \right). \end{aligned} \tag{3.2}$$

The last line is obtained by inserting the Ward–Takahashi identity (2.10) and comparing the resulting J -derivatives with (2.7). For $(N > 2)$ and pairwise different $|\underline{a}_i|$ we have

$$\begin{aligned} G_{|\underline{a}_1 \underline{a}_2 \dots \underline{a}_N|} & \\ &= \frac{(-\lambda_{bare}) Z^{\frac{1}{2}}}{V^2 H_{\underline{a}_1 \underline{a}_2} \mathcal{Z}[0]} \sum_{\underline{n} \in \mathbb{N}_{\mathcal{N}}^{D/2}} \frac{\partial}{\partial J_{\underline{a}_2 \underline{a}_3}} \dots \frac{\partial}{\partial J_{\underline{a}_N \underline{a}_1}} \frac{\partial}{\partial J_{\underline{a}_1 \underline{n}}} \frac{\partial}{\partial J_{\underline{n} \underline{a}_2}} \mathcal{Z}[J] \Big|_{J=0} \end{aligned} \tag{3.3}$$

$$= \frac{(-\lambda_{bare})Z^{\frac{1}{2}}}{V H_{\underline{a}_1 \underline{a}_2} \mathcal{Z}[0]} \frac{\partial}{\partial J_{\underline{a}_2 \underline{a}_3}} \dots \frac{\partial}{\partial J_{\underline{a}_N \underline{a}_1}} \left(\sum_{\underline{n} \in \mathbb{N}_{\mathcal{N}}^{D/2}} \frac{(J_{\underline{a}_2 \underline{n}} \frac{\partial \mathcal{Z}[J]}{\partial J_{\underline{a}_1 \underline{n}}} - J_{\underline{n} \underline{a}_1} \frac{\partial \mathcal{Z}[J]}{\partial J_{\underline{n} \underline{a}_2}}) - \frac{\nu + \zeta H_{\underline{a}_1 \underline{a}_2}}{Z} \frac{\partial \mathcal{Z}[J]}{\partial J_{\underline{a}_1 \underline{a}_2}}}{Z(E_{\underline{a}_2} - E_{\underline{a}_1})} \right) \Big|_{J=0}.$$

The step to the last line uses the Ward–Takahashi identity (2.10) for pairwise different 1-norms of indices.

We start to discuss (3.3) because this unambiguously fixes a particular combination of the renormalisation parameters. We multiply (3.3) by $H_{\underline{a}_1 \underline{a}_2}$, perform the J -differentiations and collect all coefficients of $G_{|\underline{a}_1 \underline{a}_2 \dots \underline{a}_N|}$ on the lhs:

$$\begin{aligned} & (1 - Z^{-\frac{1}{2}} \lambda_{bare} \zeta) \left(E_{\underline{a}_1} + E_{\underline{a}_2} - \frac{Z^{-\frac{1}{2}} \lambda_{bare} \nu}{1 - Z^{-\frac{1}{2}} \lambda_{bare} \zeta} \right) G_{|\underline{a}_1 \underline{a}_2 \dots \underline{a}_N|} \\ &= \lambda_{bare} Z^{\frac{1}{2}} \frac{G_{|\underline{a}_1 \underline{a}_3 \dots \underline{a}_N|} - G_{|\underline{a}_2 \underline{a}_3 \dots \underline{a}_N|}}{Z(E_{\underline{a}_1} - E_{\underline{a}_2})}. \end{aligned} \tag{3.4}$$

In terms of the functions

$$F_{\underline{a}} = E_{\underline{a}} - \frac{Z^{-\frac{1}{2}} \lambda_{bare} \nu}{2(1 - Z^{-\frac{1}{2}} \lambda_{bare} \zeta)}, \quad \lambda_r := \frac{\lambda_{bare}}{Z^{\frac{1}{2}}(1 - Z^{-\frac{1}{2}} \lambda_{bare} \zeta)}, \tag{3.5}$$

which necessarily must be finite, we thus obtain

$$G_{|\underline{a}_1 \underline{a}_2 \dots \underline{a}_N|} = \lambda_r \frac{G_{|\underline{a}_1 \underline{a}_3 \dots \underline{a}_N|} - G_{|\underline{a}_2 \underline{a}_3 \dots \underline{a}_N|}}{(F_{\underline{a}_1}^2 - F_{\underline{a}_2}^2)}. \tag{3.6}$$

The same steps yield for (3.2):

$$G_{|\underline{a} \underline{b}|} = \frac{1}{Z(1 - Z^{-\frac{1}{2}} \lambda_{bare} \zeta)(F_{\underline{a}} + F_{\underline{b}})} + \lambda_r \frac{G_{|\underline{a}|} - G_{|\underline{b}|}}{F_{\underline{a}}^2 - F_{\underline{b}}^2}. \tag{3.7}$$

Now we describe the renormalisation. We clearly need well-defined functions $G_{|\underline{a}|}$, $F_{\underline{a}}$, λ_r in the limit $\mathcal{N} \rightarrow \infty$. This in turn forces a particular singular behaviour of κ , ν , ζ , Z , μ_{bare} , λ_{bare} . As familiar from perturbative renormalisation, there is still considerable freedom in choosing finite terms of these singular functions. We follow Zimmermann’s prescription [18] and fix the finite terms by normalisation of the first Taylor expansion coefficients of relevant and marginal correlation functions (with $i = 1, \dots, \frac{D}{2}$ below):

$$\begin{aligned} D \geq 2: & \quad G_{|0|} = 0, & (3.8) \\ D \geq 4: & \quad \frac{\partial}{\partial a_i} G_{|\underline{a}|} \Big|_{\underline{a}=0} = 0, \quad G_{|00|} = \frac{1}{\mu^2}, \\ D = 6: & \quad \frac{\partial^2}{\partial a_i^2} G_{|\underline{a}|} \Big|_{\underline{a}=0} = 0, \quad \frac{\partial}{\partial a_i} G_{|\underline{a} \underline{b}|} \Big|_{\underline{a}=\underline{b}=0} = \frac{\partial}{\partial b_i} G_{|\underline{a} \underline{b}|} \Big|_{\underline{a}=\underline{b}=0} = -\frac{1}{\mu^4 V^{\frac{2}{D}}} e'(0). \end{aligned}$$

Inserting the conditions on $G_{|\underline{a}|}$ into (3.7) reduces the conditions on $G_{|\underline{a} \underline{b}|}$ for $D \geq 4$ to

$$\frac{1}{Z(1 - Z^{-\frac{1}{2}} \lambda_{bare} \zeta)(2F_0)} = \frac{1}{\mu^2}, \quad -\frac{1}{Z(1 - Z^{-\frac{1}{2}} \lambda_{bare} \zeta)(2F_0)^2} \frac{e'(0)}{V^{\frac{2}{D}}} = -\frac{1}{\mu^4 V^{\frac{2}{D}}} e'(0)$$

with solution

$$Z(1 - Z^{-\frac{1}{2}}\lambda_{bare}\zeta) = 1, \quad F_0 = \frac{\mu_{bare}^2}{2} - \frac{Z^{-\frac{1}{2}}\lambda_{bare}\nu}{2(1 - Z^{-\frac{1}{2}}\lambda_{bare}\zeta)} = \frac{\mu^2}{2}. \tag{3.9}$$

We thus conclude in (3.5)

$$\lambda_r := Z^{\frac{1}{2}}\lambda_{bare}, \quad E_{\underline{a}} = F_{\underline{a}} + \frac{1}{2}\lambda_r\nu, \quad \frac{\lambda_r\zeta}{Z} = 1 - \frac{1}{Z} \tag{3.10}$$

and define

$$\frac{W_{|\underline{a}|}}{2\lambda_r} := G_{|\underline{a}|} + \frac{F_{\underline{a}}}{\lambda_r} \quad \Rightarrow \quad G_{|\underline{a}|} = \frac{1}{2} \frac{W_{|\underline{a}|} - W_{|\underline{b}|}}{F_{\underline{a}}^2 - F_{\underline{b}}^2}. \tag{3.11}$$

The above identities and definitions are now inserted into (3.1). To make sense out of $\sum_{\underline{m} \in \mathbb{N}_{\mathcal{N}}^{D/2}} G_{|\underline{a}\underline{m}|}$ it is necessary to permit again the case $|\underline{a}| = |\underline{b}|$ in (3.7). This is achieved by a continuity argument,² $G_{|\underline{a}\underline{a}|} = \frac{1}{2F_{\underline{a}}} (1 + \lambda_r \lim_{|\underline{b}| \rightarrow |\underline{a}|} \frac{(G_{|\underline{a}|} - G_{|\underline{b}|})}{F_{\underline{a}} - F_{\underline{b}}})$. The limit is well-defined in perturbation theory where $G_{|\underline{a}|}$ is, before performing the loop sum, a rational function of the $\{F_{\underline{n}}\}$ so that a factor $F_{\underline{a}} - F_{\underline{b}}$ can be taken out of $G_{|\underline{a}|} - G_{|\underline{b}|}$. The result is

$$W_{|\underline{a}|}^2 + 2\lambda_r\nu W_{|\underline{a}|} = \frac{4}{Z}F_{\underline{a}}^2 - \left(4\frac{\lambda_r\kappa}{Z} + \left(1 + \frac{1}{Z}\right)(\lambda_r\nu)^2\right) - \frac{2\lambda_r^2}{V} \sum_{\underline{n} \in \mathbb{N}_{\mathcal{N}}^{D/2}} \frac{W_{|\underline{a}|} - W_{|\underline{n}|}}{F_{\underline{a}}^2 - F_{\underline{n}}^2} - \frac{4\lambda_r^2}{V^2} G_{|\underline{a}|\underline{a}|}. \tag{3.12}$$

3.2. Large- (\mathcal{N}, V) limit and integral equations

The action $S[\Phi]$, hence $\log \mathcal{Z}$ and $G_{|\underline{n}|}, W_{|\underline{n}|}$ only depend (via $E_{\underline{n}}$ defined in (2.3)) on the 1-norm $|\underline{n}| = n_1 + \dots + n_{D/2}$ and not individually on the components n_i of \underline{n} . Therefore, the sum in (3.12) translates into

$$\sum_{\underline{m} \in \mathbb{N}_{\mathcal{N}}^{D/2}} f(|\underline{m}|) = \sum_{|\underline{m}|=0}^{\mathcal{N}} \binom{|\underline{m}| + \frac{D}{2} - 1}{\frac{D}{2} - 1} f(|\underline{m}|),$$

giving

$$W_{|\underline{a}|}^2 + 2\lambda_r\nu W_{|\underline{a}|} - \frac{4}{Z}F_{\underline{a}}^2 + \frac{2\lambda_r^2}{V} \sum_{|\underline{m}|=0}^{\mathcal{N}} \binom{|\underline{m}| + \frac{D}{2} - 1}{\frac{D}{2} - 1} \frac{W_{|\underline{a}|} - W_{|\underline{m}|}}{F_{\underline{a}}^2 - F_{\underline{m}}^2} + \frac{4\lambda_r^2}{V^2} G_{|\underline{a}|\underline{a}|} = \text{const.} \tag{3.13}$$

We take the limit $\mathcal{N}, V \rightarrow \infty$ subject to fixed ratio

$$\frac{\mathcal{N}}{V^{\frac{2}{D}}} = \mu^2 \Lambda^2, \tag{3.14}$$

in which the sum converges to a Riemann integral

² Alternatively, the limit $V \rightarrow \infty$ permits a cheaper solution: We can restrict the sum to $\sum_{\underline{m} \in \mathbb{N}_{\mathcal{N}}^{D/2}, |\underline{m}| \neq |\underline{a}|} G_{|\underline{a}\underline{m}|}$ because $\frac{1}{V} \sum_{|\underline{m}|=|\underline{a}|} G_{|\underline{a}\underline{m}|} \rightarrow 0$. For the sake of easier notation we write unrestricted sums. We leave it to the reader's taste to exclude the terms " $\frac{0}{0}$ " or to define them by continuity.

$$\lim_{V \xrightarrow{\frac{1}{2}} \infty} \sum_{m=0}^{\mathcal{N}} f(m/V \xrightarrow{\frac{1}{2}}) = \mu^2 \Lambda^2 \int_0^1 d\tau f(\mu^2 \Lambda^2 \tau) = \mu^2 \int_0^{\Lambda^2} dx f(\mu^2 x). \tag{3.15}$$

Expressing the 1-norms of discrete matrix elements as $|a\rangle =: V \xrightarrow{\frac{1}{2}} \mu^2 x$, with $x \in [0, \Lambda^2]$, and taking the normalisation $2F_0 = \mu^2$ from (3.9) and the relations (3.5)+(2.3) into account, we arrive at $F_a|_{|a\rangle =: V \xrightarrow{\frac{1}{2}} \mu^2 x} = \mu^2 (e(x) + \frac{1}{2})$.

The mass μ is the only dimensionful parameter. The previously introduced functions and parameters have in spectral dimension $D \in \{2, 4, 6\}$ the following mass dimensions:

$$\begin{aligned} [E_a] &= [F_a] = [W_{|a\rangle}] = \mu^2, & [V] &= \mu^{-D}, & [Z] &= \mu^0, & [\Phi] &= \mu^{\frac{D-2}{2}}, & [J] &= \mu^{\frac{D+2}{2}}, \\ [\lambda_{bare}] &= [\lambda_r] = \mu^{3-\frac{D}{2}}, & [\kappa] &= \mu^{\frac{D+2}{2}}, & [v] &= \mu^{\frac{D-2}{2}}, & [\zeta] &= \mu^{\frac{D-6}{2}}, \\ [G_{|p_1^1 \dots p_{N_1}^1 | \dots | p_1^B \dots p_{N_B}^B |}] &= \mu^{(2-B-\frac{N}{2})D-N}, \end{aligned} \tag{3.16}$$

where $N = N_1 + \dots + N_B$. In terms of the dimensionless functions

$$\begin{aligned} W_{|a\rangle}|_{|a\rangle =: V \xrightarrow{\frac{1}{2}} \mu^2 x} &=: \mu^2 \tilde{W}(x), & \tilde{\lambda} &:= \mu^{\frac{D}{2}-3} \lambda_r, & \tilde{v} &:= \mu^{1-\frac{D}{2}} v, \\ G_{|p_1^1 \dots p_{N_1}^1 | \dots | p_1^B \dots p_{N_B}^B |}|_{|p_i^\beta| = V \xrightarrow{\frac{1}{2}} \mu^2 x_i^\beta} &=: \mu^{(2-B-\frac{N}{2})D-N} \tilde{G}(x_1^1, \dots, x_{N_1}^1 | \dots | x_1^B, \dots, x_{N_B}^B), \end{aligned} \tag{3.17}$$

the large- (\mathcal{N}, V) limit of μ^{-4} times (3.13) takes the form

$$\tilde{W}^2(x) + 2\tilde{\lambda}\tilde{v}\tilde{W}(x) - \frac{(2e(x)+1)^2}{Z} + \frac{2\tilde{\lambda}^2}{(\frac{D}{2}-1)!} \int_0^{\Lambda^2} dt t^{\frac{D}{2}-1} \frac{\tilde{W}(x) - \tilde{W}(t)}{(e(x)+\frac{1}{2})^2 - (e(t)+\frac{1}{2})^2} = \text{const.} \tag{3.18}$$

We have used here the fact proved later that $G_{|a\rangle|a\rangle}$ has a finite large- (\mathcal{N}, V) limit so that $\frac{4\lambda^2}{V^2} G_{|a\rangle|a\rangle}$ from (3.13) does not contribute to the limit. A final transformation

$$X := (2e(x) + 1)^2, \quad W(X) = \tilde{W}(x(X)), \quad G(X) = \tilde{G}(x(X)), \tag{3.19}$$

and similarly for other capital letters $Y(y)$, $T(t)$ and functions $G(X, Y) = \tilde{G}(x(X), y(Y))$, etc., simplifies (3.18) to

$$\begin{aligned} (W(X))^2 + 2\tilde{\lambda}\tilde{v}W(X) + \int_1^{\Xi} dT \rho(T) \frac{W(X) - W(T)}{X - T} - X &= \text{const}, \\ \rho(T) &:= \frac{2\tilde{\lambda}^2 (e^{-1}(\frac{\sqrt{T}-1}{2}))^{\frac{D}{2}-1}}{(\frac{D}{2}-1)! \sqrt{T} \cdot e'(e^{-1}(\frac{\sqrt{T}-1}{2}))}, & \Xi &:= (1 + 2e(\Lambda^2))^2. \end{aligned} \tag{3.20}$$

Building on [17] we proved in [14, eq. (4.14)] that (3.20) is solved by

$$W(X) := \frac{\sqrt{X+c}}{\sqrt{Z}} - \tilde{\lambda}\tilde{v} + \frac{1}{2} \int_1^{\Xi} dT \frac{\rho(T)}{(\sqrt{X+c} + \sqrt{T+c})\sqrt{T+c}}, \tag{3.21}$$

for some function $c(\tilde{\lambda}, e, \tilde{\nu}, Z)$. The functions $c, \tilde{\nu}, Z$ are fixed by the normalisation conditions (3.8) which translate into $W(X) = \sqrt{X} + \mathcal{O}((\sqrt{X} - 1)^{\frac{D}{2}})$, i.e.

$$\underbrace{W(1) = 1}_{D \geq 2}, \quad \underbrace{W'(1) = \frac{d}{dX} \sqrt{X} \Big|_{X=1} = \frac{1}{2}}_{D \geq 4}, \quad \underbrace{W''(1) = \frac{d^2}{dX^2} \sqrt{X} \Big|_{X=1} = -\frac{1}{4}}_{D=6}. \quad (3.22)$$

The first condition fixes $\tilde{\nu}(c, Z)$ to

$$1 =: \frac{\sqrt{1+c}}{\sqrt{Z}} - \tilde{\lambda} \tilde{\nu} + \frac{1}{2} \int_1^{\Xi} dT \frac{\rho(T)}{(\sqrt{1+c} + \sqrt{T+c})\sqrt{T+c}}. \quad (3.23)$$

For $D = 2$ we had $Z = 1$ and $\nu = \tilde{\nu} = 0$ so that in the limit $\Xi \rightarrow \infty$ (safe in $D = 2$ where $\rho(T) \propto \frac{1}{\sqrt{T}}$) we recover the solution of [14, eq. (4.15)]. For $D \geq 4$ equations (3.21)+(3.23) read

$$W(X) = 1 + (\sqrt{X+c} - \sqrt{1+c}) \left(\frac{1}{\sqrt{Z}} - \frac{1}{2} \int_1^{\Xi} \frac{dT \rho(T)}{(\sqrt{X+c} + \sqrt{T+c})(\sqrt{1+c} + \sqrt{T+c})\sqrt{T+c}} \right). \quad (3.24)$$

The condition on $W'(1)$ in (3.22) then fixes $Z(c)$ to

$$\frac{1}{2} = \frac{1}{2\sqrt{1+c}} \left(\frac{1}{\sqrt{Z}} - \frac{1}{2} \int_1^{\Xi} dT \frac{\rho(T)}{(\sqrt{1+c} + \sqrt{T+c})^2 \sqrt{T+c}} \right). \quad (3.25)$$

For $D = 4$ where $Z \equiv 1$, this equation determines the main function $c(\tilde{\lambda}, e)$ (in the limit $\Xi \rightarrow \infty$ which is safe for $\rho(T) \propto \text{const}$) to

$$D = 4: \quad 1 - \sqrt{1+c} = \frac{1}{2} \int_1^{\infty} dT \frac{\rho(T)}{(\sqrt{1+c} + \sqrt{T+c})^2 \sqrt{T+c}}. \quad (3.26)$$

The formula differs from $D = 2$ by a power of 2 in the denominator $(\sqrt{1+c} + \sqrt{T+c})$. With this solution for c we have

$$D = 4: \quad W(X) = \sqrt{X+c} + 1 - \sqrt{1+c} - \frac{1}{2} \int_1^{\infty} \frac{dT \rho(T) (\sqrt{X+c} - \sqrt{1+c})}{(\sqrt{X+c} + \sqrt{T+c})(\sqrt{1+c} + \sqrt{T+c})\sqrt{T+c}}. \quad (3.27)$$

For $D = 6$ we conclude

$$\frac{1}{\sqrt{Z}} = \sqrt{1+c} + \frac{1}{2} \int_1^{\Xi} dT \frac{\rho(T)}{(\sqrt{1+c} + \sqrt{T+c})^2 \sqrt{T+c}}. \quad (3.28)$$

This allows us to compute the β -function of the running coupling constant³ $\lambda_b(\Xi) = \frac{1}{\sqrt{Z(\Xi)}} \lambda_r$:

³ For the sake of readability we assume here that c is independent of Ξ and given by (3.31) below. Requiring $W''(1) = -\frac{1}{4}$ exactly restricts the integral in (3.31) to $[1, \Xi]$ and leads to $c(\tilde{\lambda}, e, \Xi)$. We are free to admit a finite renormalisation of $W''(1)$ which approaches $-\frac{1}{4}$ only in the limit $\Xi \rightarrow \infty$.

$$\begin{aligned} \beta_\lambda &:= \Lambda^2 \frac{d\lambda_{bare}(\Xi(\Lambda))}{d\Lambda^2} = 4\sqrt{\Xi}e^{-1(\frac{\sqrt{\Xi}-1}{2})} \cdot e'(e^{-1(\frac{\sqrt{\Xi}-1}{2})}) \frac{d}{d\Xi}\lambda_b(\Xi) \\ &= \frac{2\lambda_r^3\Lambda^6}{(\sqrt{1+c} + \sqrt{(2e(\Lambda^2)+1)^2+c})^2\sqrt{(2e(\Lambda^2)+1)^2+c}} > 0. \end{aligned} \tag{3.29}$$

We learn that the β -function is – for real coupling constant – strictly positive,⁴ with $\lambda_b(\Xi) \xrightarrow{\Xi \rightarrow \infty} +\infty$, but without developing a Landau pole (a singularity of λ_b already at finite Ξ_0). We also have $Z(\Xi) \in [0, 1]$, as it should. We remark that [12] addresses the case that $\lambda_{bare}, \lambda_r \in i\mathbb{R}$, hence $\lambda_r^2 < 0$, resulting in negative β -function and asymptotic freedom. We would like to point out, however, that this also implies $\rho(T) < 0$ and consequently the senseless result $\frac{1}{\sqrt{Z}} \rightarrow -\infty$. We shall see in the final section that an important positivity property only holds for λ_r real.

Inserted into (3.24) we get

$$\begin{aligned} D = 6: \quad W(X) &= \sqrt{X+c}\sqrt{1+c} - c \\ &+ \frac{1}{2} \int_1^\infty \frac{dT \rho(T) (\sqrt{X+c} - \sqrt{1+c})^2}{(\sqrt{X+c} + \sqrt{T+c})(\sqrt{1+c} + \sqrt{T+c})^2\sqrt{T+c}}, \end{aligned} \tag{3.30}$$

where the limit $\Xi \rightarrow \infty$ is now safe. Eventually, the condition on $W''(1)$ in (3.22) determines $c(\tilde{\lambda}, e)$ to

$$-c = \int_1^\infty \frac{dT \rho(T)}{(\sqrt{1+c} + \sqrt{T+c})^3\sqrt{T+c}}. \tag{3.31}$$

Obviously, $\tilde{\lambda} = 0$ and hence $\rho = 0$ corresponds to $c = 0$ for any D . For given $e(x)$, thus $\rho(T)$, the implicit function theorem provides a unique diffeomorphism $\tilde{\lambda}^2 \mapsto c(\tilde{\lambda}, e)$ on a neighbourhood of $0 \in \mathbb{R}$ or $0 \in \mathbb{C}$. Since we will be able to express all correlation functions in terms of elementary functions of $c(\tilde{\lambda}, e)$ and $\rho(\tilde{\lambda}, e)$, this proves analyticity of all correlation functions in these neighbourhoods.

According to [14, Prop 4.2] (which is unchanged due to (3.6)), all N -point functions are explicitly computable from this solution $W(X)$:

Proposition 3.1. *The connected ($N \geq 1$)-point functions in the scaling limit $\tilde{G}(x_1, \dots, x_N) = \lim_{V, N \rightarrow \infty} \mu^{N+(\frac{N}{2}-1)D} G|_{\underline{p}_1, \dots, \underline{p}_N} |_{|\underline{p}_l|=V^{2/D}\mu^{2x_l}}$ and $G(X_1, \dots, X_N) := \tilde{G}(x_1(X_1), \dots, x_N(X_N))$ read*

$$G(X_1) := \frac{W(X_1) - \sqrt{X_1}}{2\tilde{\lambda}} \quad \text{for } N = 1, \tag{3.32}$$

$$G(X_1, \dots, X_N) = \sum_{k=1}^N \frac{W(X_k)}{2\tilde{\lambda}} \prod_{l=1, l \neq k}^N \frac{4\tilde{\lambda}}{X_k - X_l} \quad \text{for } N > 1, \tag{3.33}$$

where $W(X)$ is given in [14, eq. (4.13)+(4.15)] for $D = 2$, in (3.25)+(3.27) for $D = 4$ and in (3.30)+(3.31) for $D = 6$. We have $\tilde{\lambda} = \lambda_r$ for $D = 6$ and $\tilde{\lambda} = \frac{\lambda}{\mu^{3-D/2}}$ for $D \in \{2, 4\}$.

⁴ More precisely, β_λ has the same sign as λ which means that $|\lambda_{bare}(\Lambda^2)|$ increases with Λ^2 .

4. N -points function with $B \geq 2$ boundaries

The derivation of the formula for the $(N_1 + \dots + N_B)$ -point function with one $N_i > 1$ proceeds along the same lines as in our previous paper [14]. Key ingredient is the Ward–Takahashi identity (2.10) which compared with [14] contains additional parameters $\nu, \zeta \neq 0$ and $Z \neq 1$. However, in identical manner as in the derivation of (3.6), the resulting contributions reconstruct the renormalised functions $F_{\underline{a}}$ instead of E_a in [14]. We thus have for $N_1 > 1$

$$G_{|\underline{a}_1^1 \dots \underline{a}_{N_1}^1 | \dots | \underline{a}_1^B \dots \underline{a}_{N_B}^B |} = \lambda_r \frac{G_{|\underline{a}_1^1 \underline{a}_3^1 \dots \underline{a}_{N_1}^1 | \underline{a}_1^2 \dots \underline{a}_{N_2}^2 | \dots | \underline{a}_1^B \dots \underline{a}_{N_B}^B |} - G_{|\underline{a}_2^1 \underline{a}_3^1 \dots \underline{a}_{N_1}^1 | \underline{a}_1^2 \dots \underline{a}_{N_2}^2 | \dots | \underline{a}_1^B \dots \underline{a}_{N_B}^B |}}{F_{\underline{a}_1^1}^2 - F_{\underline{a}_2^1}^2}. \tag{4.1}$$

Its reduction to $(1 + \dots + 1)$ -point functions is, up to renormalisations $E \mapsto F$ and $\lambda \mapsto \lambda_r$, identically as in [14, Prop. 5.2]:

Proposition 4.1. *Let $B \geq 2$. The connected $(N_1 + \dots + N_B)$ -point function with one $N_i > 1$ is given for any $D \in \{2, 4, 6\}$ by*

$$G_{|\underline{a}_1^1 \dots \underline{a}_{N_1}^1 | \dots | \underline{a}_1^B \dots \underline{a}_{N_B}^B |} = \lambda_r^{N_1 + \dots + N_B - B} \sum_{k_1=1}^{N_1} \dots \sum_{k_B=1}^{N_B} G_{|\underline{a}_{k_1}^1 | \dots | \underline{a}_{k_B}^B |} \prod_{\beta=1}^B \prod_{l_{\beta}=1, l_{\beta} \neq k_{\beta}}^{N_{\beta}} \frac{1}{F_{\underline{a}_{k_{\beta}}^{\beta}}^2 - F_{\underline{a}_{l_{\beta}}^{\beta}}^2} \tag{4.2}$$

(where $\lambda_r \equiv \lambda$ for $D \in \{2, 4\}$), its large- (\mathcal{N}, V) limit by

$$G(X_1^1, \dots, X_{N_1}^1 | \dots | X_1^B, \dots, X_{N_B}^B) = \tilde{\lambda}^{N_1 + \dots + N_B - B} \sum_{k_1=1}^{N_1} \dots \sum_{k_B=1}^{N_B} G(X_{k_1}^1 | \dots | X_{k_B}^B) \prod_{\beta=1}^B \prod_{l_{\beta}=1, l_{\beta} \neq k_{\beta}}^{N_{\beta}} \frac{4}{X_{k_{\beta}}^{\beta} - X_{l_{\beta}}^{\beta}}. \tag{4.3}$$

4.1. $(1 + \dots + 1)$ -point function

The Schwinger–Dyson equation for the $(1 + \dots + 1)$ -point function is at an intermediate step and up to taking multiple matrix indices and bare parameters Z, λ_{bare} identical as in our previous paper [14, eq. (5.5)]:

$$G_{|\underline{a}^1 | \underline{a}^2 | \dots | \underline{a}^B |} = \frac{(-\lambda_{bare} Z^{\frac{3}{2}})}{Z H_{\underline{a}^1 \underline{a}^1}} \left\{ \frac{1}{V} \sum_{\underline{n} \in \mathbb{N}_{\mathcal{N}}^{D/2}} G_{|\underline{a}^1 \underline{n} | \underline{a}^2 | \dots | \underline{a}^B |} + \sum_{\beta=2}^B G_{|\underline{a}^1 \underline{a}^{\beta} \underline{a}^{\beta} | \underline{a}^2 | \dots | \underline{a}^B |} \right. \\ \left. + \frac{1}{V^2} G_{|\underline{a}^1 | \underline{a}^1 | \underline{a}^2 | \dots | \underline{a}^B |} + 2G_{|\underline{a}^1 |} G_{|\underline{a}^1 | \underline{a}^2 | \dots | \underline{a}^B |} \right. \\ \left. + \sum_{p=1}^{B-2} \sum_{2 \leq i_1 < \dots < i_p \leq B} G_{|\underline{a}^1 | \underline{a}^{i_1} | \dots | \underline{a}^{i_p} |} G_{|\underline{a}^1 | \underline{a}^{j_1} | \dots | \underline{a}^{j_{B-p-1}} |} \right\}, \tag{4.4}$$

where $\{j_1, \dots, j_{B-p-1}\} = \{2, \dots, B\} \setminus \{i_1, \dots, i_p\}$ and $\dots \dots$ denotes the omission of \underline{a}^{β} . We multiply by $\frac{H_{\underline{a}\underline{a}}}{\lambda_r} = \frac{2F_{\underline{a}}}{\lambda_r} = \frac{2F_{\underline{a}}}{\lambda_r} + \nu$, see (3.10), and bring $-2G_{|\underline{a}^1 |} G_{|\underline{a}^1 | \underline{a}^2 | \dots | \underline{a}^B |}$ to the lhs, thus

reconstructing the function $W_{|a^1|}$ defined in (3.11) but *here shifted* by $\nu\lambda_r$ compared with [14, eq. (5.4)]. Reducing the $(2+1+\dots+1)$ -point function by (4.1), where the remarks of footnote 2 apply, leads to

$$\begin{aligned} (W_{|a^1|} + \nu\lambda_r)G_{|a^1|a^2|\dots|a^B|} &+ \frac{\lambda_r^2}{V} \sum_{\underline{n} \in \mathbb{N}_{\mathcal{N}}^{D/2}} \frac{G_{|a^1|a^2|\dots|a^B|} - G_{|\underline{n}|a^2|\dots|a^B|}}{(F_{a^1}^2 - F_{\underline{n}}^2)} \\ &= -\lambda_r \sum_{\beta=2}^B G_{|a^1|a^\beta|a^2|\dots|a^B|} - \frac{\lambda_r}{V^2} G_{|a^1|a^1|a^2|\dots|a^B|} \\ &\quad - \lambda_r \sum_{p=1}^{B-2} \sum_{2 \leq i_1 < \dots < i_p \leq B} G_{|a^1|a^{i_1}|\dots|a^{i_p}|} G_{|a^1|a^{i_1}|\dots|a^{i_p-1}|}. \end{aligned} \tag{4.5}$$

Taking the scaling limit $\tilde{G}(x^1|\dots|x^B) := \mu^{(2-\frac{3}{2}B)D-B} \lim_{\mathcal{N}, V \rightarrow \infty} G_{|a^1|\dots|a^B|} \Big|_{|a^i|=V^{\frac{2}{D}}\mu^2 x^i}$ and transforming variables x^i to X^i , we essentially obtain [14, eq. (5.7)],

$$\begin{aligned} (W(X^1) + \tilde{\lambda}\tilde{\nu})G(X^1|X^{\triangleleft\{2,\dots,B\}}) &+ \frac{1}{2} \int_1^\Xi dT \rho(T) \frac{G(X^1|X^{\triangleleft\{2,\dots,B\}}) - G(T|X^{\triangleleft\{2,\dots,B\}})}{(X-T)} \\ &= -\tilde{\lambda} \sum_{\beta=2}^B G(X^1, X^\beta, X^\beta|X^{\triangleleft\{2,\dots,B\}}) - \tilde{\lambda} \sum_{\substack{J \subset \{2,\dots,B\} \\ 1 \leq |J| \leq B-2}} G(X^1|X^{\triangleleft J})G(X^1|X^{\triangleleft\{2,\dots,B\} \setminus J}), \end{aligned} \tag{4.6}$$

where the measure $\rho(T)$ was defined in (3.20) and $G(X|Y^{\triangleleft\{i_1,\dots,i_p\}}) := G(X|Y^{i_1}|\dots|Y^{i_p})$. This equation was solved in our previous paper [14]:

Theorem 4.2. *The scaling limit of the $(1+\dots+1)$ -point function is given by*

$$G(X|Y) = \frac{4\tilde{\lambda}^2}{\sqrt{X+c} \cdot \sqrt{Y+c} \cdot (\sqrt{X+c} + \sqrt{Y+c})^2}, \tag{4.7}$$

for $B = 2$ and for $B \geq 3$ by

$$G(X^1|\dots|X^B) = (-2\tilde{\lambda})^{3B-4} \frac{d^{B-3}}{dt^{B-3}} \left(\frac{1}{(R(t))^{B-2} \sqrt{X^1+c-2t}^3 \dots \sqrt{X^B+c-2t}^3} \right) \Big|_{t=0}, \tag{4.8}$$

$$R(t) := \lim_{\Xi \rightarrow \infty} \left(\frac{1}{\sqrt{Z}} - \int_1^\Xi \frac{dT \rho(T)}{\sqrt{T+c}} \frac{1}{(\sqrt{T+c} + \sqrt{T+c-2t})\sqrt{T+c-2t}} \right). \tag{4.9}$$

Here, $c(\tilde{\lambda}, e)$ is defined in [14, eq. (4.15)] for $D = 2$, in (3.27) for $D = 4$ and in (3.31) for $D = 6$. The wavefunctions renormalisation equals $Z = 1$ for $D \in \{2, 4\}$ and is given in (3.28) for $D = 6$.

Explicitly we have for $D = 6$

$$R(t) := \sqrt{1+c} - \int_1^\infty dT \rho(T) \frac{\{\sqrt{1+c}(2\sqrt{T+c} + \sqrt{1+c})(\sqrt{T+c-2t} + \sqrt{T+c}) + t(\sqrt{T+c-2t} + 2\sqrt{T+c})\}}{\sqrt{T+c}(\sqrt{1+c} + \sqrt{T+c})^2(\sqrt{T+c} + \sqrt{T+c-2t})^2\sqrt{T+c-2t}}. \tag{4.10}$$

5. Linearly spaced eigenvalues and Feynman graphs for $D = 6$

5.1. Expanding the exact result

The noncommutative field theory model discussed in sec. 6 (see also [14, sec. 6]) translates to linearly spaced eigenvalues with $e(x) = x$ and $e'(x) = 1$. This yields $X = (2x + 1)^2$ and $\rho(T) = \frac{\tilde{\lambda}^2(\sqrt{T}-1)^2}{4\sqrt{T}}$. The integrals (3.30)+(3.31) can be evaluated⁵ for $\Xi \rightarrow \infty$:

Proposition 5.1. *Let $D = 6$. Equation (3.20) with normalisation (3.22) is for eigenvalue functions $e(x) = x$ solved by:*

$$W(X) = \sqrt{X+c}\sqrt{1+c} - c + \frac{\tilde{\lambda}^2}{2} \left\{ \sqrt{1+c} - \sqrt{1+X} + \log \left(\frac{\sqrt{X+c} + \sqrt{1+c}}{2(1 + \sqrt{1+c})} \right) + \frac{(1+X)}{2\sqrt{X}} \log \left(\frac{(\sqrt{X} + \sqrt{X+c})(1 + \sqrt{X})}{\sqrt{X}\sqrt{1+c} + \sqrt{X+c}} \right) \right\}, \tag{5.1}$$

where $c(\tilde{\lambda})$ is the inverse solution of

$$\tilde{\lambda}^2 = \frac{(-4c)}{1 - 2\sqrt{1+c} + 2(1+c)\log(1 + \frac{1}{\sqrt{1+c}})}. \tag{5.2}$$

The first terms of the inverse solution are

$$c = -\frac{2 \log 2 - 1}{4} \tilde{\lambda}^2 + \frac{(2 \log 2 - 1)(4 \log 2 - 3)}{32} \tilde{\lambda}^4 - \frac{(2 \log 2 - 1)(35 - 94 \log 2 + 64(\log 2)^2)}{1024} \tilde{\lambda}^6 + \mathcal{O}(\tilde{\lambda}^8). \tag{5.3}$$

This gives the following perturbative expansion of the 1-point function:

$$\begin{aligned} \tilde{G}(x) &:= \frac{W((2x+1)^2) - (2x+1)}{2\tilde{\lambda}} \\ &= \frac{\tilde{\lambda}}{4(2x+1)} (2(1+x)^2 \log(1+x) - x(2+3x)) \\ &\quad + \frac{\tilde{\lambda}^3}{16(2x+1)^3} (x^3(2+3x)(2 \log 2 - 1)^2) + \mathcal{O}(\tilde{\lambda}^5). \end{aligned} \tag{5.4}$$

⁵ We actually compute $\frac{1}{\sqrt{z}}$ from (3.28) and the integral (3.24). A primitive of the integrand is obtained by computer algebra; the limit $\Xi \rightarrow \infty$ is done by hand. Also identities of the type $\frac{\sqrt{1+c} + \sqrt{X+c}}{c + \sqrt{X} + \sqrt{1+c}\sqrt{X+c}} = \frac{1 + \sqrt{X}}{\sqrt{X}\sqrt{1+c} + \sqrt{X+c}}$ are used. The series expansions (5.3) and (5.4) are found by computer algebra.

5.2. Perturbative expansion of the partition function

On the other hand, expanding the original action as a formal power series in λ leads to a ribbon graph representation of $\log \mathcal{Z}[J]$. Ignoring the renormalisation constants, i.e. setting $\mu_{bare} = \mu$, $\lambda_{bare} = \lambda$, $Z = 1$, $\kappa = \nu = \zeta = 0$ gives in the large- (\mathcal{N}, V) limit ‘Feynman’ rules for planar ribbon graphs with B boundary components. We formulate these rules in an equivalent description [14, sec. 2] by planar graphs Γ on the 2-sphere with two sorts of vertices: any number of black (internal) vertices of valence 3, and $B \geq 1$ white vertices $\{v_\beta\}_{\beta=1}^B$ (external vertices, or punctures, or boundary components) of any valence $N_\beta \geq 1$. Every face is required to have at most one white vertex (separation of punctures). Faces with a white vertex are called external; they are labelled by positive real numbers $x_1^1, \dots, x_{N_1}^1, \dots, x_1^B, \dots, x_{N_B}^B$ (the upper index labels the unique white vertex of the face). Faces without white vertex are called internal; they are labelled by positive real numbers y_1, \dots, y_L . Such graphs are dual to triangulations of the B -punctured sphere. To such a graph Γ we assign an *unrenormalised* amplitude \tilde{G}_Γ^Λ as follows:

- Associate a weight $(-\tilde{\lambda})$ to each black vertex, weight 1 to each white vertex of Γ .
- Associate weight $\frac{1}{z_1+z_2+1}$ to an edge of Γ separating faces labelled by z_1 and z_2 . These can be internal or external, also $z_1 = z_2$ can occur.
- Multiply these weights of Γ and integrate over all internal face variables y_1, \dots, y_L with measure $\frac{1}{2} y_i^2 \chi_{[0, \Lambda^2]}(y_i) dy_i$, where $\chi_{[0, \Lambda^2]}$ is the characteristic function of $[0, \Lambda^2]$ and dy_i the Lebesgue measure on \mathbb{R}_+ .
- The result is a function $\tilde{G}_\Gamma^\Lambda(x_1^1, \dots, x_{N_1}^1 | \dots | x_1^B, \dots, x_{N_B}^B)$ of the external face variables.

We list the simplest graphs contributing to the 1-point function $\tilde{G}^\Lambda(x)$ and their unrenormalised integrals:

$\Gamma_1 = \circ \overset{x}{\text{---}} \bullet \text{---} \bigcirc_{y_1}$	$\tilde{G}_{\Gamma_1}^\Lambda(x) = \frac{(-\tilde{\lambda})}{2x+1} \int_0^{\Lambda^2} \frac{dy_1 y_1^2/2}{x+y_1+1},$	(5.5)
$\Gamma_2 = \circ \overset{x}{\text{---}} \bullet \text{---} \bigcirc_{y_1} \text{---} \bigcirc_{y_2}$	$\tilde{G}_{\Gamma_2}^\Lambda(x) = \frac{(-\tilde{\lambda})^3}{(2x+1)^2} \int_0^{\Lambda^2} \frac{dy_1 y_1^2/2}{(x+y_1+1)^2} \int_0^{\Lambda^2} \frac{dy_2 y_2^2/2}{x+y_2+1},$	
$\Gamma_3 = \circ \overset{x}{\text{---}} \bullet \text{---} \bigcirc_{y_1} \text{---} \bigcirc_{y_2}$	$\tilde{G}_{\Gamma_3}^\Lambda(x) = \frac{(-\tilde{\lambda})^3}{2x+1} \int_0^{\Lambda^2} \frac{dy_1 y_1^2/2}{(x+y_1+1)^2 (2y_1+1)} \int_0^{\Lambda^2} \frac{dy_2 y_2^2/2}{y_1+y_2+1},$	
$\Gamma_4 = \circ \overset{x}{\text{---}} \bullet \text{---} \bigcirc_{y_1} \text{---} \bigcirc_{y_2}$	$\tilde{G}_{\Gamma_4}^\Lambda(x) = \frac{(-\tilde{\lambda})^3}{(2x+1)^3} \int_0^{\Lambda^2} \frac{dy_1 y_1^2/2}{x+y_1+1} \int_0^{\Lambda^2} \frac{dy_2 y_2^2/2}{x+y_2+1},$	
$\Gamma_5 = \circ \overset{x}{\text{---}} \bullet \text{---} \bigcirc_{y_1} \text{---} \bigcirc_{y_2}$	$\tilde{G}_{\Gamma_5}^\Lambda(x) = \frac{(-\tilde{\lambda})^3}{(2x+1)} \int_0^{\Lambda^2} \int_0^{\Lambda^2} \frac{dy_1 dy_2 (y_1^2/2)(y_2^2/2)}{(x+y_1+1)^2 (y_1+y_2+1)(x+y_2+1)}.$	

Clearly, all these integrals diverge for $\Lambda \rightarrow \infty$. By employing the standard techniques of perturbative renormalisation theory it should be possible to prove that there exist formal power series

$\mu_{bare} = \mu + \mathcal{O}(\lambda)$, $\lambda_{bare} = \lambda + \mathcal{O}(\lambda)$, $Z = 1 + \mathcal{O}(\lambda)$, $\kappa = \mathcal{O}(\lambda)$, $\nu = \mathcal{O}(\lambda)$, $\zeta = \mathcal{O}(\lambda)$, divergent at $\Lambda \rightarrow \infty$, such that all matrix correlation functions are finite to all orders in perturbation theory. Moreover, the value of these correlation functions is expected to be uniquely determined by normalisation conditions e.g. on

$$\tilde{G}(0), \quad (\partial\tilde{G})(0), \quad (\partial^2\tilde{G})(0), \quad \tilde{G}(0,0), \quad (\partial\tilde{G})(0,0), \quad \tilde{G}(0,0,0). \quad (5.6)$$

5.3. Zimmermann's forest formula for ribbon graphs

A key step in proving these claims is the *forest formula* of Wolhart Zimmermann [18]. Streamlining previous (and very essential) work of Klaus Hepp [20] on the Bogoliubov–Parasiuk renormalisation description [19], Zimmermann proved that those formal power series $\mu_{bare}(\lambda)$, $\lambda_{bare}(\lambda)$, $Z(\lambda)$, $\kappa(\lambda)$, $\nu(\lambda)$, $\zeta(\lambda)$ which enforce the normalisation conditions (3.8) simply amount to a well-defined modification, for every ribbon graph Γ , of the *integrand* I_Γ of Γ (product of vertex weights and edge weights). The modified integrand $\mathcal{R}(I_\Gamma)$ is Lebesgue-integrable over the whole space and by Fubini's theorem can be unambiguously integrated in any order.

We describe this modification \mathcal{R} for the Φ_G^3 -matrix model under consideration. The situation is far easier than in the papers by Hepp and Zimmermann because in Euclidean space there is no need to discuss the $i\epsilon$ -limit to tempered distributions. Moreover, the globally assigned face variables give unique edge weights, in contrast to choices of momentum routings in ordinary Feynman graphs that make recursive substitution operations necessary.

Let $\mathcal{B}_\Gamma, \mathcal{F}_\Gamma$ be the set of external and internal faces of Γ , respectively. A *ribbon subgraph* $\gamma \subset \Gamma$ consists of a subset $\mathcal{F}_\gamma \subset \mathcal{F}_\Gamma$ together with all edges and vertices bordering \mathcal{F}_γ , and not more, such that the (thus defined) set of edges and vertices of γ is connected. The ribbon subgraph defines a unique subset $\mathcal{E}_\gamma \subset \mathcal{B}_\Gamma \cup \mathcal{F}_\Gamma \setminus \mathcal{F}_\gamma$ of neighbouring faces to γ , i.e. any element of \mathcal{E}_γ has a common edge with γ . We let B_γ be the number of connected components of \mathcal{E}_γ . This number is most conveniently identified when drawing Γ on the 2-sphere where \mathcal{E}_γ will partition into B_γ disjoint regions.

We let $f(\gamma)$ be the set of those face variables $\{x, y\}$ which label the faces in \mathcal{E}_γ . Then for a rational function $r_\gamma(f(\gamma), y_1, \dots, y_l)$, where y_1, \dots, y_l label the faces of γ , we let $(T_{f(\gamma)}^\omega r_\gamma)(f(\gamma), y_1, \dots, y_l)$ be the order- ω Taylor polynomial of r_γ with respect to the variables $f(\gamma)$. We let $T_{f(\gamma)}^\omega r_\gamma \equiv 0$ for $\omega < 0$.

A forest \mathcal{U}_Γ in Γ is a collection of ribbon subgraphs $\gamma_1, \dots, \gamma_l$ such that for any pair γ_i, γ_j

$$\text{either } \gamma_i \subset \gamma_j, \quad \text{or } \gamma_j \subset \gamma_i, \quad \text{or } \gamma_i \cap \gamma_j = \emptyset.$$

Here $\gamma_i \cap \gamma_j = \emptyset$ means complete disjointness in the sense that γ_i, γ_j do not have any common edges or vertices. Similarly, $\gamma_j \subset \gamma_i$ means that all faces, edges and vertices of γ_j also belong to γ_i . We admit $\mathcal{U}_\Gamma = \emptyset$, but note that Γ itself cannot belong to \mathcal{U}_Γ because Γ has purely external edges which are not part of a ribbon subgraph.

Consequently, a forest is endowed with a partial order which we symbolise by disjoint trees (hence the name). The tree structure identifies for any $\gamma_i \in \mathcal{U}_\Gamma$ a unique set $o(\gamma_i) = \gamma_{i_1} \cup \dots \cup \gamma_{i_k}$ of offsprings, i.e. mutually disjoint ribbon subgraphs $\gamma_{i_h} \subset \gamma_i$ such that for any other $\gamma_j \in \mathcal{U}_\Gamma \setminus (\gamma_i \cup o(\gamma_i))$ either $\gamma_j \subset o(\gamma_i)$, or $\gamma_i \subset \gamma_j$, or $\gamma_i \cap \gamma_j = \emptyset$. A ribbon subgraph γ with $o(\gamma) = \emptyset$ is called a leaf.

Let I_Γ be the integrand encoded in a ribbon graph, given as the product of weights of edges of Γ (the constant vertex weights play no rôle). A forest \mathcal{U}_Γ defines a partition of the integrand I_Γ into

$$I_\Gamma = I_{\Gamma \setminus \mathcal{U}_\Gamma} \prod_{\gamma \in \mathcal{U}_\Gamma} I_{\gamma \setminus \circ(\gamma)}, \quad \prod_{\gamma \in \emptyset} I_\emptyset = 1. \tag{5.7}$$

Then the following holds:

Theorem 5.2 (after Zimmermann). *The formal power series $\mu_{bare}(\lambda)$, $\lambda_{bare}(\lambda)$, $Z(\lambda)$, $\kappa(\lambda)$, $v(\lambda)$, $\zeta(\lambda)$ which enforce the normalisation conditions (3.8) amount for any ribbon graph Γ to replace the integrand I_Γ as follows:*

$$I_\Gamma \mapsto \mathcal{R}(I_\Gamma) := \sum_{\mathcal{U}_\Gamma} I_{\Gamma \setminus \mathcal{U}_\Gamma} \prod_{\gamma \in \mathcal{U}_\Gamma} (-T_{f(\gamma)}^{\omega(\gamma)} I_{\gamma \setminus \circ(\gamma)}), \tag{5.8}$$

where the sum is over all forests \mathcal{U}_Γ of Γ including the empty forest \emptyset . For the planar sector of the Φ_D^3 -matrix model, the degree is defined as $\omega(\gamma) := \frac{D}{2}(2 - B_\gamma) - N_\gamma$, where N_γ is the number of edges of Γ which connect (within Γ) to vertices of γ but are not in γ itself.

5.4. Forest formula applied to 1-point function

We exemplify Zimmermann’s rules for the ribbon graphs given in (5.5). The graph Γ_1 has two forests $\mathcal{U}_{\Gamma_1} = \emptyset$ and $\mathcal{U}_{\Gamma_1} = \{\gamma_1\}$, where γ_1 is the ribbon subgraph consisting of the face y_1 and its bordering line and vertex. Taking $N_{\gamma_1} = 1$ and $B_{\gamma_1} = 1$ into account, we have to replace

$$\frac{1}{x + y_1 + 1} \mapsto \underbrace{\frac{1}{x + y_1 + 1}}_{\text{for } \mathcal{U}_{\Gamma_1} = \emptyset} + \underbrace{(-T_x^2) \left(\frac{1}{x + y_1 + 1} \right)}_{\text{for } \mathcal{U}_{\Gamma_1} = \{\gamma_1\}}. \tag{5.9}$$

With $(-T_x^2) \left(\frac{1}{x + y_1 + 1} \right) = -\frac{1}{y_1 + 1} + \frac{x}{(1 + y_1)^2} - \frac{x^2}{(1 + y_1)^3}$ we compute the renormalised amplitude of the ribbon graph Γ_1 to

$$\begin{aligned} \tilde{G}_{\Gamma_1}(x) &= \frac{(-\tilde{\lambda})}{2x + 1} \int_0^\infty \frac{y_1^2 dy_1}{2} \left(\frac{1}{x + y_1 + 1} - \frac{1}{y_1 + 1} + \frac{x}{(y_1 + 1)^2} - \frac{x^2}{(y_1 + 1)^3} \right) \\ &= \frac{(-\tilde{\lambda})}{4(2x + 1)} (x(2 + 3x) - 2(1 + x)^2 \log(1 + x)). \end{aligned} \tag{5.10}$$

We confirm the perfect agreement with the λ -expansion (5.4) of the exact formula!

The next graph Γ_2 has four forests $\mathcal{U}_{\Gamma_2} = \emptyset$, $\mathcal{U}_{\Gamma_2} = \{\gamma_1\}$, $\mathcal{U}_{\Gamma_2} = \{\gamma_2\}$, $\mathcal{U}_{\Gamma_2} = \{\gamma_1, \gamma_2\}$. By γ_i we denote the ribbon subgraph with face labelled y_i together with its bordering edges and vertices. Note that the faces labelled y_1, y_2 together do not give rise to a ribbon subgraph because its edge+vertex set would be disconnected. We have $N_{\gamma_1} = 2$, $N_{\gamma_2} = 1$ and $B_{\gamma_1} = B_{\gamma_2} = 1$. We have $f(\gamma_1) = f(\gamma_2) = \{x\}$. Hence, the Taylor subtraction operator for γ_1 is $(-T_x^1) \left(\frac{1}{(x + y_1 + 1)^2} \right) = -\frac{1}{(y_1 + 1)^2} + \frac{2x}{(y_1 + 1)^3}$, and $(-T_x^2) \left(\frac{1}{(x + y_2 + 1)} \right)$ is analogous to Γ_1 . Then Zimmermann’s forest formula factors as follows:

$$\begin{aligned} \tilde{G}_{\Gamma_2}(x) &:= \frac{(-\tilde{\lambda})^3}{(2x + 1)^2} \int_0^\infty \frac{y_1^2 dy_1}{2} \left(\frac{1}{(x + y_1 + 1)^2} - \frac{1}{(y_1 + 1)^2} + \frac{2x}{(y_1 + 1)^3} \right) \\ &\quad \times \int_0^\infty \frac{y_2^2 dy_2}{2} \left(\frac{1}{x + y_2 + 1} - \frac{1}{y_2 + 1} + \frac{x}{(y_2 + 1)^2} - \frac{x^2}{(y_2 + 1)^3} \right) \end{aligned}$$

$$= \frac{\tilde{\lambda}^3}{4(2x+1)^2} \left((1+x) \log(1+x) - x \right) \left(2(1+x)^2 \log(1+x) - x(2+3x) \right). \quad (5.11)$$

In Γ_3 we let the ribbon subgraph γ_2 as before and define γ_{12} as consisting of *both* faces labelled by y_1 and y_2 together with all bordering edges and vertices. There is also the ribbon subgraph γ_1 consisting of y_1 but not γ_2 . Then \mathcal{E}_{γ_2} has $B_{\gamma_2} = 2$ components, namely the disconnected faces labelled by y_2 and by x . In principle we could include γ_1 in the forest formula, but $\omega(\gamma_1) = -1$ forces the Taylor operator T to vanish. Therefore, we can work with the reduced set of forests $U_\Gamma = \emptyset$, $U_\Gamma = \{\gamma_2\}$, $U_\Gamma = \{\gamma_{12}\}$ and $U_\Gamma = \{\gamma_{12}, \gamma_2\}$. The external face variables are $f(\gamma_2) = \{y_1\}$ and $f(\gamma_{12}) = \{x\}$. The resulting analytic contribution is⁶

$$\begin{aligned} \tilde{G}_{\Gamma_3}(x) &= \frac{(-\tilde{\lambda})^3}{2x+1} \int_0^\infty \frac{y_1^2 dy_1}{2} \left(\frac{1}{(x+y_1+1)^2} - \frac{1}{(y_1+1)^2} + \frac{2x}{(y_1+1)^3} - \frac{3x^2}{(y_1+1)^4} \right) \\ &\quad \times \frac{1}{2y_1+1} \int_0^\infty \frac{y_2^2 dy_2}{2} \left(\frac{1}{y_1+y_2+1} - \frac{1}{y_2+1} + \frac{y_1}{(y_2+1)^2} - \frac{(y_1)^2}{(y_2+1)^3} \right) \\ &= \frac{\tilde{\lambda}^3}{4(2x+1)^3} \left\{ x^3(2+3x) \left(\frac{(1-2\log 2)^2}{4} - \frac{\pi^2}{2} \right) + (1+x)^2(2+7x+7x^2) \log(1+x) \right. \\ &\quad \left. + x(1+x)(1+3x+3x^2) \left((\log(1+x))^2 - 2\log(1+x) \log x + 2\text{Li}_2\left(\frac{1}{1+x}\right) \right) + \frac{x^4}{4} \right\} \\ &\quad - \frac{\tilde{\lambda}^3}{2(2x+1)} \left(\frac{\pi^2}{6} + 1 \right) x. \end{aligned} \quad (5.12)$$

We see that the whole number-theoretic features of quantum field theory are reproduced!

The graph Γ_4 poses no difficulty:

$$\begin{aligned} \tilde{G}_{\Gamma_4}(x) &= \frac{(-\tilde{\lambda})^3}{(2x+1)^3} \int_0^\infty \frac{y_1^2 dy_1}{2} \left(\frac{1}{x+y_1+1} - \frac{1}{(y_1+1)} + \frac{x}{(y_1+1)^2} - \frac{x^2}{(y_1+1)^3} \right) \\ &\quad \times \int_0^\infty \frac{y_2^2 dy_2}{2} \left(\frac{1}{x+y_2+1} - \frac{1}{y_2+1} + \frac{x}{(y_2+1)^2} - \frac{x^2}{(y_2+1)^3} \right) \\ &= \frac{-\tilde{\lambda}^3}{16(2x+1)^3} \left(2(1+x)^2 \log(1+x) - x(2+3x) \right)^2. \end{aligned} \quad (5.13)$$

The graph Γ_5 shows a new quality: the overlapping divergence made of the ribbon subgraphs γ_1 and γ_2 which share a common edge. Overlapping divergences were a problem in the first days of quantum field theory. Now with the forest formula at disposal there is nothing to worry. The definition simply forbids γ_1 and γ_2 in the same forest. The following forests remain:

$$\begin{aligned} \mathcal{U}_{\Gamma_5} &= \emptyset, \quad \mathcal{U}_{\Gamma_5} = \{\gamma_1\}, \quad \mathcal{U}_{\Gamma_5} = \{\gamma_2\}, \quad \mathcal{U}_{\Gamma_5} = \{\gamma_{12}\}, \quad \mathcal{U}_{\Gamma_5} = \{\gamma_{12}, \gamma_1\}, \\ \mathcal{U}_{\Gamma_5} &= \{\gamma_{12}, \gamma_2\}. \end{aligned}$$

⁶ We compute the primitive by computer algebra and take the limit $\Lambda^2 \rightarrow \infty$ by hand, where polylogarithmic identities are employed.

We have $N_{\gamma_{12}} = 1$, $N_{\gamma_1} = 3$, $N_{\gamma_2} = 2$ and $B_{\gamma_{12}} = B_{\gamma_1} = B_{\gamma_2} = 1$, giving $\omega(\gamma_{12}) = 2$, $\omega(\gamma_1) = 0$, $\omega(\gamma_2) = 1$. The face variables are $f(\gamma_{12}) = \{x\}$, $f(\gamma_1) = \{x, y_2\}$, $f(\gamma_2) = \{x, y_1\}$. It follows:

$$\begin{aligned}
 \tilde{G}_{\Gamma_5}(x) &= \frac{(-\tilde{\lambda})^3}{(2x+1)} \int_0^\infty \frac{y_1^2 dy_1}{2} \int_0^\infty \frac{y_2^2 dy_2}{2} \left\{ \underbrace{\frac{1}{(x+y_1+1)^2} \frac{1}{(y_1+y_2+1)} \frac{1}{(x+y_2+1)}}_{I_{\gamma_{12}}} \right. \\
 &\quad + \underbrace{\frac{1}{x+y_2+1}}_{I_{\gamma_{12} \setminus \gamma_1}} \underbrace{\left(-\frac{1}{(y_1+1)^3}\right)}_{(-T_{x,y_2}^0)(I_{\gamma_1})} + \underbrace{\frac{1}{(x+y_1+1)^2}}_{I_{\gamma_{12} \setminus \gamma_2}} \underbrace{\left(-\frac{1}{(y_2+1)^2} + \frac{y_1+x}{(y_2+1)^3}\right)}_{(-T_{x,y_1}^1)(I_{\gamma_2})} \\
 &\quad + \underbrace{\frac{1}{(y_1+y_2+1)} \left(-\frac{1}{(y_1+1)^2(y_2+1)} + \frac{2x}{(y_1+1)^3(y_2+1)} + \frac{x}{(y_1+1)^2(y_2+1)^2} - \frac{1}{(y_1+1)^4(y_2+1)} - \frac{1}{(y_1+1)^2(y_2+1)^3} - \frac{1}{(y_1+1)^3(y_2+1)^2}\right)}_{(-T_x^2)(I_{\gamma_{12}})} \\
 &\quad + \underbrace{\left(-\frac{1}{(y_1+1)^3}\right) \left(-\frac{1}{y_2+1} + \frac{x}{(y_2+1)^2} - \frac{x^2}{(y_2+1)^3}\right)}_{(-T_x^2)(I_{\gamma_{12} \setminus \gamma_1} (-T_{x,y_2}^0)(I_{\gamma_1}))} \\
 &\quad \left. + \underbrace{\left(\left(-\frac{1}{(y_1+1)^2} + \frac{2x}{(y_1+1)^3} - \frac{3x^2}{(y_1+1)^4}\right) \left(-\frac{1}{(y_2+1)^2} + \frac{y_1}{(y_2+1)^3}\right) + \left(-\frac{1}{(y_1+1)^2} + \frac{2x}{(y_1+1)^3}\right) \left(\frac{x}{(y_2+1)^3}\right)\right)}_{(-T_x^2)(I_{\gamma_{12} \setminus \gamma_2} (-T_{x,y_1}^1)(I_{\gamma_2}))} \right\} \\
 &= \frac{-\tilde{\lambda}^3}{4(2x+1)^3} \left\{ (x+1)(2x+1)(3x+2) \log(1+x) + (x+1)^3(3x+1)(\log(1+x))^2 \right. \\
 &\quad + x(1+x)(1+3x+3x^2) \left((\log(1+x))^2 - 2 \log(1+x) \log x + 2 \text{Li}_2\left(\frac{1}{1+x}\right) \right) \\
 &\quad \left. - x^3(2+3x) \frac{\pi^2}{2} \right\} + \frac{\tilde{\lambda}^3}{2(2x+1)} \left(\frac{\pi^2}{6} + 1 - \frac{x}{2} \right) x. \tag{5.14}
 \end{aligned}$$

The sum is indeed the λ^3 -part of (5.4):

$$\tilde{G}_{\Gamma_2}(x) + \tilde{G}_{\Gamma_3}(x) + \tilde{G}_{\Gamma_4}(x) + \tilde{G}_{\Gamma_5}(x) = \frac{\tilde{\lambda}^3}{16(2x+1)^3} x^3 (2+3x)(1-2\log 2)^2. \tag{5.15}$$

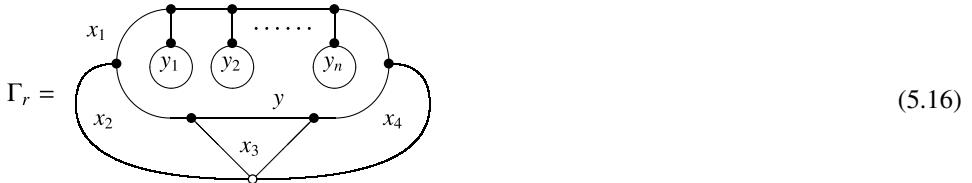
This coincidence looks remarkable, but it isn't. It is the unavoidable consequence that two correct theorems about the same object must agree.

5.5. Renormalons

Renormalised amplitudes typically involve a logarithmic dependence on the external parameters. Our model is no exception, see (5.10). In just renormalisable QFT models one finds that n -fold insertion of these subgraphs into a superficially convergent graph produces an unprotected amplification of the logarithms which let the amplitude of the renormalised graph grow

with $\mathcal{O}(n!)$. Since there are also $\mathcal{O}(n!)$ graphs contributing to order n , there is no hope of a Borel-summable perturbation series. This phenomenon is called the *renormalon problem*. It is an artefact of the strictly local Taylor subtraction at 0. Constructive renormalisation theory [30] avoids the problem by a reorganisation into an effective series of infinitely many (but related) coupling constants. On the downside we loose the nice forest formula so that it is hard to compute the amplitudes in practice.

In this subsection we convince ourselves that the Φ_6^3 -matrix model under consideration has, in its perturbative expansion, graphs showing the renormalon problem. Consider the finite ($N = 4, B = 1$)-graph with a single internal face labelled y . We replace the edge between two black vertices by a chain of n vertices and $n + 1$ edges and attach to every additional black vertex and towards the inner face the simplest 1-point function Γ_1 :



$$\Gamma_r = \tag{5.16}$$

Taking the renormalised amplitude (5.10) into account, the amplitude of the total ribbon graph becomes in the simplified case that all external face variables are equal,

$$\tilde{G}_{\Gamma_r}(x, x, x, x) = \frac{(-\tilde{\lambda})^{4+n} \cdot \tilde{\lambda}^n}{(2x+1)^4} \int_0^\infty \frac{y^2 dy}{2} \frac{1}{(x+y+1)^{n+4}} \left(\frac{2(1+y)^2 \log(1+y) - y(2+3y)}{4(2y+1)} \right)^n. \tag{5.17}$$

For large y the integral behaves as

$$\tilde{G}_{\Gamma_r}(x, x, x, x) \sim \frac{(-1)^n \tilde{\lambda}^{4+2n}}{2 \cdot 4^n (2x+1)^4} \int_R^\infty \frac{dy}{y^2} (\log y)^n = \frac{(-1)^n \tilde{\lambda}^{4+2n}}{2 \cdot 4^n (2x+1)^4} \underbrace{\int_{\log R}^\infty dt e^{-t} t^n}_{\sim n!}.$$

This is the renormalon problem for a single graph.

On the other hand we know that the exact formula is analytic in $\tilde{\lambda}$. There is a subtle cancellation between different graphs, similar to the $\log(1+x)$ contributions of $\tilde{G}_{\Gamma_{2\dots 5}}$ which all cancelled in the sum. In a certain sense, this cancellation is another instance of the integrability of the model.

6. From $\phi_D^{\star 3}$ model on Moyal space to Schwinger functions on \mathbb{R}^D

This section parallels the treatment of $\phi_4^{\star 4}$ in [15,21] and $\phi_2^{\star 3}$ in [14] where more details are given. The (unrenormalised) $\phi_D^{\star 3}$ -model on Moyal-deformed Euclidean space with harmonic propagation is defined by the action

$$S[\phi] := \int_{\mathbb{R}^D} \frac{d\xi}{(8\pi)^{D/2}} \left(\frac{1}{2} \phi \star (-\Delta + \|4\Theta^{-1} \cdot \xi\|^2 + \mu^2) \phi + \frac{\lambda}{3} \phi \star \phi \star \phi \right) (\xi), \tag{6.1}$$

where \star denotes the associative, noncommutative Moyal product parametrised by a skew-symmetric matrix Θ . The Moyal space possesses a convenient matrix basis $f_{\underline{mn}}(x)$ labelled by

pairs of $\frac{D}{2}$ -tuples $\underline{m} = (m_1, \dots, m_{D/2})$ for which we set $|\underline{m}| := m_1 + \dots + m_{D/2}$. The matrix basis satisfies $(f_{\underline{k}l} \star f_{\underline{m}n})(\xi) = \delta_{\underline{m}l} f_{\underline{k}n}(\xi)$ and $\int_{\mathbb{R}^D} d\xi f_{\underline{m}n}(\xi) = \sqrt{|\det(2\pi\Theta)|} \delta_{\underline{m}n}$. A convenient regularisation consists in restricting the fields ϕ to those with finite expansion $\phi(\xi) = \sum_{\underline{m}, \underline{n} \in \mathbb{N}_{\mathcal{N}}^{D/2}} \Phi_{\underline{m}\underline{n}} f_{\underline{m}\underline{n}}(\xi)$, where we recall $\mathbb{N}_{\mathcal{N}}^{D/2} := \{\underline{m} \in \mathbb{N}^{D/2} : |\underline{m}| \leq \mathcal{N}\}$. Then (6.1) takes precisely the form of our starting point (2.4) with undone renormalisation ($\kappa = \nu = \zeta = 0$, $Z = 1$, $\lambda_{bare} = \lambda$, $\mu_{bare} = \mu$) and identification

$$V = \frac{1}{2^D} \sqrt{|\det \Theta|}, \quad E_{\underline{n}} = \frac{|\underline{n}|}{V^{2/D}} + \frac{\mu^2}{2} = \mu^2 \left(\frac{1}{2} + \frac{|\underline{n}|}{\mu^2 V^{2/D}} \right). \tag{6.2}$$

Comparing with (2.3), the Moyal space leads to a linear eigenvalue function $e(x) = x$.

According to [21,14] the large- (\mathcal{N}, V) limit of the matrix model induces together with the convention $\frac{\delta J_{mn}}{\delta J(\xi)} := \mu^D f_{\underline{m}\underline{n}}(\xi)$ the following relation for connected Schwinger functions on extreme Moyal space (where $s_\beta := \sum_{i=1}^{\beta-1} N_i$):

$$\begin{aligned} & S_c(\mu\xi_1, \dots, \mu\xi_N) \\ &= \sum_{\substack{N_1 + \dots + N_B = N \\ N_\beta \text{ even}}} \sum_{\sigma \in \mathcal{S}_N} \prod_{\beta=1}^B \left(\frac{2^{\frac{DN_\beta}{2}}}{N_\beta} \int_{\mathbb{R}^D} \frac{dp^\beta}{(2\pi\mu^2)^{\frac{D}{2}}} e^{i(p^\beta, \xi_{\sigma(s_\beta+1)} - \xi_{\sigma(s_\beta+2)} + \dots - \xi_{\sigma(s_\beta+N_\beta)})} \right) \\ &\times \frac{1}{(8\pi)^{D/2} S_{(N_1, \dots, N_B)}} \tilde{G} \left(\underbrace{\frac{\|p^1\|^2}{2\mu^2}, \dots, \frac{\|p^1\|^2}{2\mu^2}}_{N_1} \mid \dots \mid \underbrace{\frac{\|p^B\|^2}{2\mu^2}, \dots, \frac{\|p^B\|^2}{2\mu^2}}_{N_B} \right). \end{aligned} \tag{6.3}$$

This shows that out of the Osterwalder–Schrader axioms [22,23], *Euclidean invariance* and *symmetry* are automatically fulfilled, whereas *clustering* does not hold. The remaining section addresses *reflection positivity* for the Schwinger 2-point function $S_c(\mu\xi_1, \mu\xi_2)$.

6.1. Reflection positivity of Schwinger 2-point function for $D = 6$

It was proved in [21] that the Schwinger 2-point function $S_c(\mu\xi_1, \mu\xi_2)$ given by (6.3) is reflection positive iff $x \mapsto \tilde{G}(x, x) = G((2x + 1)^2, (2x + 1)^2)$ is a Stieltjes function, i.e. there exists a positive measure ϱ on \mathbb{R}_+ with $G((2x + 1)^2, (2x + 1)^2) = \int_0^\infty \frac{dt \varrho(t)}{t + 2x}$. From (3.33) and a combination of (3.21) and (3.25) we have in $D = 6$ dimensions

$$\begin{aligned} G(X, X) &= 2W'(X) \tag{6.4} \\ &= \frac{\sqrt{1+c}}{\sqrt{X+c}} + \frac{1}{2\sqrt{X+c}} \int_1^\infty \frac{dT \rho(T)}{\sqrt{T+c}} \left(\frac{1}{(\sqrt{1+c} + \sqrt{T+c})^2} - \frac{1}{(\sqrt{X+c} + \sqrt{T+c})^2} \right), \end{aligned}$$

where $X = (2x + 1)^2$ and $c \in]-1, 0]$ for $\tilde{\lambda} \in \mathbb{R}$. Already at this point we can state that reflection positivity is impossible for $\tilde{\lambda} \in i\mathbb{R} \Leftrightarrow c > 0$. Namely, $0 = \sqrt{X+c} = \sqrt{2x + 1 + i\sqrt{c}} \times \sqrt{2x + 1 - i\sqrt{c}}$ has solutions – hence (6.4) a pole or end point of a branch cut – outside the real axis. This contradicts holomorphicity of Stieltjes functions on $\mathbb{C} \setminus]-\infty, 0]$.

So let $\tilde{\lambda} \in \mathbb{R}$. For linearly spaced eigenvalues with $\rho(T) = \frac{\tilde{\lambda}^2(\sqrt{T}-1)^2}{4\sqrt{T}}$ we either evaluate (6.4) or better differentiate (5.1) to

$$G(X, X) = \frac{\sqrt{1+c}}{\sqrt{X+c}} + \frac{\tilde{\lambda}^2(\sqrt{X}+1)(\sqrt{X}-1)}{4(\sqrt{X})^2} \left\{ \frac{1}{\sqrt{X+c}} \left(\frac{1}{\sqrt{1+c}+\sqrt{X+c}} - 1 \right) \right. \tag{6.5}$$

$$\left. + \frac{1}{\sqrt{X}} (\log(\sqrt{X} + \sqrt{X+c}) - \log(\sqrt{X}\sqrt{1+c} + \sqrt{X+c}) + \log(1 + \sqrt{X})) \right\}.$$

The following is the deepest result of this paper:

Theorem 6.1. *The diagonal 2-point function of the renormalised 6-dimensional Kontsevich model Φ_6^3 is, for linearly spaced eigenvalues of E , real $\tilde{\lambda}$ and in large- (N, V) limit, a Stieltjes function, i.e. the Stieltjes transform $\tilde{G}(x, x) = \int_0^\infty \frac{\varrho(t)dt}{t+2x}$ of a positive measure ϱ . This Stieltjes measure $\varrho(t)$ has support $[1 - \sqrt{-c}, 1 + \sqrt{-c}] \cup [2, \infty[$ consisting of an isolated region near $t = 1$ and the unbounded interval $t \geq 2$. The precise relation is*

$$\tilde{G}\left(\frac{p^2}{2\mu^2}, \frac{p^2}{2\mu^2}\right) = \frac{\tilde{\lambda}^2}{4\pi(\sigma^2-1)} \int_0^\pi d\phi \frac{\left\{ 2\frac{\log(1+\sigma)}{\sigma} - 1 + \sigma(\sigma-1)\tan^2\phi - \tan\phi(1+\sigma^2\tan^2\phi)(\arctan_{[0,\pi]}(\sigma\tan\phi) - \phi) \right\}}{1 - \frac{\sqrt{\sigma^2-1}}{\sigma}\cos\phi + \frac{p^2}{\mu^2}} \tag{6.6}$$

$$+ \frac{\tilde{\lambda}^2}{4} \int_2^\infty dt \frac{t(t-2)/(t-1)^3}{t + \frac{p^2}{\mu^2}},$$

where $\sigma := \frac{1}{\sqrt{1+c}} \in [1, -2W_{-1}(-\frac{1}{2\sqrt{e}}) - 1]$ is the inverse solution of $\tilde{\lambda}^2 = \frac{4(\sigma^2-1)}{\sigma^2-2\sigma+2\log(1+\sigma)} \in [1, \frac{8W_{-1}(-\frac{1}{2\sqrt{e}})}{1+2W_{-1}(-\frac{1}{2\sqrt{e}})}]$. Here, $W_{-1}(z)$ for $z \in [-\frac{1}{e}, 0]$ is the lower real branch of the Lambert- W function.

It should be possible to verify the claim directly. The function $\frac{\tan^n \phi}{a+b \cos \phi}$ has a known primitive so that integration by parts does the job. Below we explain how we obtained the result. Fig. 1 shows a plot of the Stieltjes measure $\varrho(t)$ for various values of σ .

6.2. Identification of the Stieltjes measure for $D = 6$

As long as $x > 0$ the formula (6.5) can be taken literally. But for discussing the Stieltjes property we have to extend it to complex x . Here we already made a choice for the logarithms: When arranging them as in (6.5) we understand that $\log z$ has a branch cut along the negative real axis and we choose the standard branch $\text{Im}(\log z) \in]0, \pi[$ for $\text{Im}(z) > 0$ and $\text{Im}(\log z) \in]-\pi, 0[$ for $\text{Im}(z) < 0$. Next, the only reasonable interpretation that applies to $c \in]-1, 0]$ is $\sqrt{X+c} := \sqrt{(2x+1) + \sqrt{-c}\sqrt{(2x+1) - \sqrt{-c}}}$ and $\sqrt{X} := 2x+1$. This shows that $x \mapsto 2W'((2x+1))$ is holomorphic on $\mathbb{C} \setminus]-\infty, 0[$ with branch cut along parts of the negative real axis. Such holomorphicity is one of the characterising properties of Stieltjes functions. If we knew that $\tilde{G}(x, x)$ is Stieltjes, then the measure is recovered from the inversion formula

$$\tilde{G}(x, x) = \int_0^\infty \frac{\varrho(t)dt}{t+2x} \quad \Rightarrow \quad \varrho(t) = \frac{1}{\pi} \text{Im}(\tilde{G}(-\frac{t}{2} - i\epsilon, -\frac{t}{2} - i\epsilon)). \tag{6.7}$$

To determine the nature of the branch cut we set $2x = -(t + i\epsilon)$, $t > 0$, and note

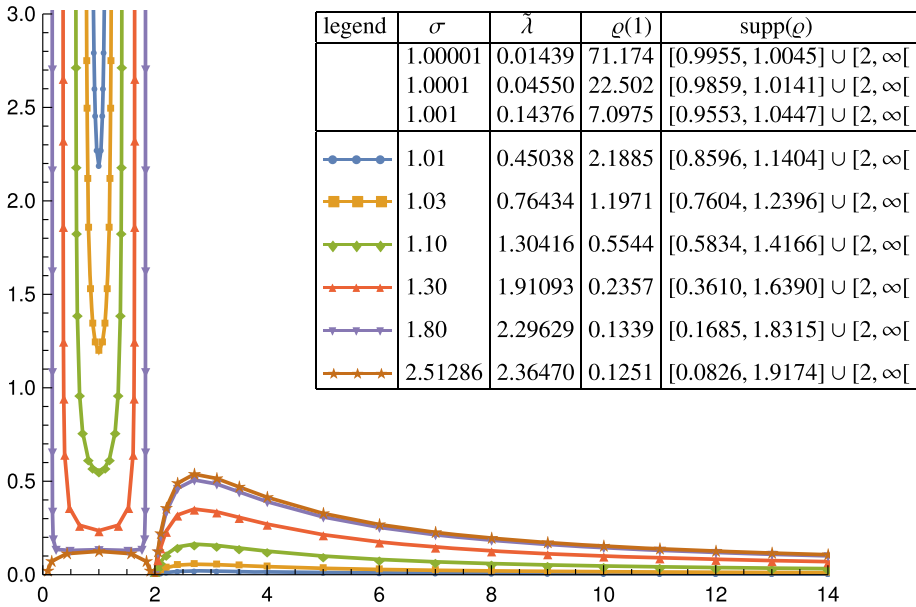


Fig. 1. Stieltjes measure $\varrho(t)$ of the diagonal 2-point function for $D = 6$ and selected coupling constants.

$$\sqrt{X+c} \Big|_{\sqrt{X}=1-t-i\epsilon} = \begin{cases} \sqrt{(1-t)^2+c-i\epsilon} & \text{for } 0 \leq t < 1 - \sqrt{-c}, \\ (-i)\sqrt{-c-(1-t)^2} & \text{for } 1 - \sqrt{-c} < t < 1 + \sqrt{-c}, \\ -\sqrt{(t-1)^2+c-i\epsilon} & \text{for } t > 1 + \sqrt{-c}. \end{cases} \quad (6.8)$$

We have to distinguish the three cases in (6.5):

- $[0 < t < 1 - \sqrt{-c}]$: We have $\sqrt{X}, \sqrt{X+c} > 0$, hence $\text{Im}(2W'(X)) \Big|_{\substack{\sqrt{X}=1-t-i\epsilon \\ 0 \leq t \leq 1+\sqrt{-c}}} = 0$.
- $[1 - \sqrt{-c} < t < 1 + \sqrt{-c}]$: Along the branch cut of $\sqrt{X+c}$, where this square root has strictly negative imaginary part, the logarithms in (6.5) are well-defined, and we have

$$\begin{aligned} & \text{Im}(2W'(X)) \Big|_{\substack{\sqrt{X}=1-t-i\epsilon \\ 1-\sqrt{-c} < t < 1+\sqrt{-c}}} \\ &= \frac{\sqrt{1+c}}{\sqrt{-c-(1-t)^2}} - \frac{\tilde{\lambda}^2(\sqrt{1+c}-t(2-t))}{4(1-t)^2\sqrt{-c-(1-t)^2}} \\ & \quad - \frac{\tilde{\lambda}^2 t(2-t)}{4(1-t)^3} \left\{ \arctan \Big|_{[0,\pi]} \left(\frac{\sqrt{-c-(1-t)^2}}{(1-t)\sqrt{1+c}} \right) - \arctan \Big|_{[0,\pi]} \left(\frac{\sqrt{-c-(1-t)^2}}{1-t} \right) \right\}. \end{aligned} \quad (6.9)$$

Positivity will be discussed below.

- $[t > 1 + \sqrt{-c}]$: The negative roots $\sqrt{X}, \sqrt{X+c} \in \mathbb{R}_- - i\epsilon$ are selected so that $\log(\sqrt{X} + \sqrt{X+c}) - \log(\sqrt{X}\sqrt{1+c} + \sqrt{X+c})$ is real. But $\log(1 + \sqrt{X}) = \log(2-t-i\epsilon)$ develops an imaginary part for $t > 2$ (recall that log is the standard branch):

$$\text{Im}(2W'(X)) \Big|_{\substack{\sqrt{X}=1-t-i\epsilon \\ t \geq 1+\sqrt{-c}}} = \chi_{[2,\infty[} \frac{\tilde{\lambda}^2 \pi t(t-2)}{4(t-1)^3}, \quad (6.10)$$

where $\chi_{[2, \infty]}$ is the characteristic function of $[2, \infty]$. The function (6.10) is manifestly non-negative and, remarkably, depends on $\tilde{\lambda}$ only via the global prefactor $\tilde{\lambda}^2$, but not on $c(\tilde{\lambda})$.

Case 2 needs careful discussion. For the recognition of $2W'(X)$ as a Stieltjes function we need $\text{Im}(2W'(X)) \geq 0$ for $\sqrt{X} = 1 - t - i\epsilon$. We find it convenient to introduce in (6.9) and (5.2) the substitution

$$\sqrt{-c} = \cos \psi, \quad \sqrt{1+c} = \sin \psi, \quad 1-t = \cos \psi \cos \phi. \tag{6.11}$$

In these variables we have, after extracting a common prefactor $\tilde{\lambda}^2$,

$$\underbrace{\text{Im}(2W'(X))}_{\sqrt{X} = \cos \psi \cos \phi - i\epsilon} = \frac{\tilde{\lambda}^2}{4 \sin \phi \cos^3 \psi} \left\{ \sin^2 \psi \left(2 \sin \psi \log \left(1 + \frac{1}{\sin \psi} \right) - 1 \right) + (1 - \sin \psi) \tan^2 \phi - \tan \phi (\sin^2 \psi + \tan^2 \phi) \left(\arctan_{[0, \pi]} \left(\frac{\tan \phi}{\sin \psi} \right) - \phi \right) \right\}. \tag{6.12}$$

For $\phi \rightarrow 0$ we see that positivity requires $\log(1 + \frac{1}{\sin \psi}) \geq \frac{1}{2 \sin \psi}$ or

$$\psi \geq \arcsin \left(\frac{1}{-2W_{-1}(-\frac{1}{2\sqrt{e}}) - 1} \right) = 0.409284\dots, \tag{6.13}$$

where $W_{-1}(z)$ is the lower branch of the Lambert-W function. This gives a bound on the coupling constant

$$\sqrt{1+c} = \sin \psi \geq \frac{1}{-2W_{-1}(-\frac{1}{2\sqrt{e}}) - 1} = 0.397953\dots, \quad |\lambda| \leq 2.3647\dots \tag{6.14}$$

Coincidentally, this critical value agrees with the critical value where $\frac{d\tilde{\lambda}^2(c)}{dc} = 0$. In other words, we have positivity precisely on the interval where $c \mapsto \tilde{\lambda}^2(c)$ is bijective.

The other critical value to discuss is $\phi \rightarrow \frac{\pi}{2}$, or $t \rightarrow 1$, where $\tan \phi$ becomes singular. The series expansion yields

$$\begin{aligned} \text{Im}(2W'(X)) \Big|_{\substack{\sqrt{X}=1-t-i\epsilon \\ t \rightarrow 1}} & \tag{6.15} \\ &= \frac{\tilde{\lambda}^2}{12 \cos^3 \psi} \left(1 - 6 \sin^2 \psi + 2 \sin^3 \psi + 6 \sin^3 \psi \log \left(1 + \frac{1}{\sin \psi} \right) \right) + \mathcal{O}((1-t)^2), \end{aligned}$$

which is manifestly positive. Inserting (6.12) and (6.10) into the inversion formula (6.7) and taking the Jacobian of $dt = \cos \psi \sin \phi d\phi$ into account we arrive at Theorem 6.1.

6.3. Reflection positivity of 2-point function for $D = 4$

For linearly spaced eigenvalues in $D = 4$ dimensions, hence summation measure $\rho(T) = \tilde{\lambda}^2 \frac{\sqrt{T-1}}{\sqrt{T}}$, we evaluate (3.26) to

$$\tilde{\lambda}^2 = \frac{1 - \sqrt{1+c}}{1 - \sqrt{1+c} \log(1 + \frac{1}{\sqrt{1+c}})}. \tag{6.16}$$

Next, (3.21) with $Z = 1$ is evaluated to

$$\begin{aligned}
 2W'(X) &= \frac{1}{\sqrt{X+c}} \left(1 - \frac{\tilde{\lambda}^2}{2} \int_1^\infty \frac{dT (\sqrt{T}-1)}{\sqrt{T}\sqrt{T+c}(\sqrt{X+c} + \sqrt{T+c})^2} \right) \\
 &= \frac{1}{\sqrt{X+c}} - \frac{\tilde{\lambda}^2}{\sqrt{X^2}} + \frac{\tilde{\lambda}^2(\sqrt{1+c}-1)}{\sqrt{X^2}\sqrt{X+c}} \\
 &\quad + \frac{\tilde{\lambda}^2}{\sqrt{X^3}} \left(\log(\sqrt{X}+1) + \log(\sqrt{X} + \sqrt{X+c}) - \log(\sqrt{X}\sqrt{1+c} + \sqrt{X+c}) \right).
 \end{aligned}
 \tag{6.17}$$

The same discussion as for $D = 6$ gives:

Theorem 6.2. *The diagonal 2-point function of the renormalised 4-dimensional Kontsevich model Φ_4^3 is, for linearly spaced eigenvalues of E , real $\tilde{\lambda}$ and in large- (\mathcal{N}, V) limit, a Stieltjes function. The Stieltjes measure $\varrho(t)$ has support $[1 - \sqrt{-c}, 1 + \sqrt{-c}] \cup [2, \infty[$ consisting of an isolated region near $t = 1$ and the unbounded interval $t \geq 2$. The precise relation is*

$$\begin{aligned}
 \tilde{G}\left(\frac{p^2}{2\mu^2}, \frac{p^2}{2\mu^2}\right) &= \frac{\tilde{\lambda}^2\sigma^2}{\pi(\sigma^2-1)} \int_0^\pi d\phi \frac{\left\{ \frac{(1+\sigma)}{\sigma} \left(1 - \frac{\log(1+\sigma)}{\sigma}\right) - (1 + \tan^2\phi) \left(1 - \frac{1}{\sigma} - \tan\phi (\arctan_{[0,\pi]}(\sigma \tan\phi) - \phi)\right) \right\}}{1 - \frac{\sqrt{\sigma^2-1}}{\sigma} \cos\phi + \frac{p^2}{\mu^2}} \\
 &\quad + \tilde{\lambda}^2 \int_2^\infty dt \frac{1/(t-1)^3}{t + \frac{p^2}{\mu^2}},
 \end{aligned}
 \tag{6.18}$$

where $\sigma := \frac{1}{\sqrt{1+c}} \in [1, -\frac{2}{W_0(-\frac{2}{c})} - 1]$ is the inverse solution of $\tilde{\lambda}^2 = \frac{\sigma-1}{\sigma-\log(1+\sigma)} \in [0, \frac{2}{2+W_0(-\frac{2}{c})}]$.

Here, $W_0(z)$ for $z \geq -\frac{1}{e}$ is the upper real branch of the Lambert-W function.

Positivity of the measure at $\phi \in \{0, \pi\}$ leads to the same condition $2(1 + \sigma) - (1 + \sigma) \log(1 + \sigma) \geq 2$ (solved in terms of Lambert-W) that restricts the bijectivity region of $c \mapsto \tilde{\lambda}^2(c)$. Fig. 2 shows a plot of the Stieltjes measure for various values of σ .

6.4. Two-point function for $D = 2$

We had already pointed out in [14] that the Φ_2^3 -model is *not* reflection positive. In this subsection we show what goes wrong compared with $D = 4$ and $D = 6$. Starting point is (3.21) which evaluates for the measure $\rho(T) = \frac{2\tilde{\lambda}^2}{\sqrt{T}}$ to

$$\begin{aligned}
 2W'(X) &= \frac{1}{\sqrt{X+c}} \left(1 - \tilde{\lambda}^2 \int_1^\infty \frac{dT}{\sqrt{T}\sqrt{T+c}(\sqrt{X+c} + \sqrt{T+c})^2} \right) \\
 &= \frac{1}{\sqrt{X+c}} + \frac{2\tilde{\lambda}^2}{\sqrt{X^2}(\sqrt{X+c} + \sqrt{1+c})\sqrt{1+c}} + \frac{2\tilde{\lambda}^2}{\sqrt{X^2}\sqrt{X+c}} \left(1 - \frac{1}{\sqrt{1+c}} \right) \\
 &\quad - \frac{2\tilde{\lambda}^2}{\sqrt{X^3}} \left(\log(\sqrt{X}+1) + \log(\sqrt{X} + \sqrt{X+c}) - \log(\sqrt{X}\sqrt{1+c} + \sqrt{X+c}) \right).
 \end{aligned}
 \tag{6.19}$$

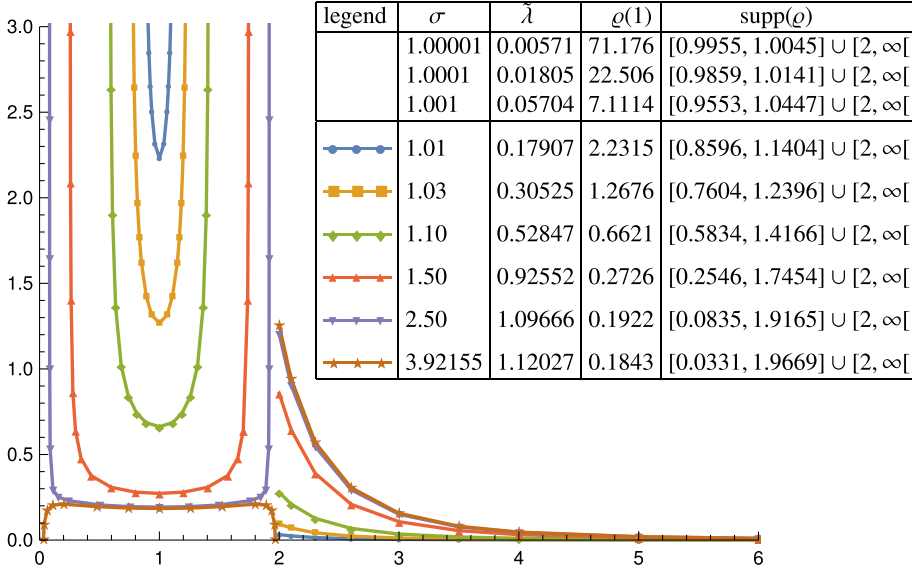


Fig. 2. Stieltjes measure of the diagonal 2-point function for $D = 4$ and selected coupling constants.

It is the opposite sign of the last line in (6.19) compared with (6.17) which lets the scattering measure supported at $t > 2$ arise with the wrong sign! In addition there is an atomic measure from the second term on the rhs of (6.19): Near $t = 2$ we have

$$\begin{aligned} & \text{Im}\left(\frac{2\tilde{\lambda}^2/\sqrt{1+c}}{\sqrt{X}^2(\sqrt{X+c}+\sqrt{1+c})}\right) \xrightarrow{\sqrt{X}=-\frac{t}{2}-i\epsilon} \text{Im}\left(\frac{2\tilde{\lambda}^2/\sqrt{1+c}}{(t-1)^2(-\sqrt{(t-1)^2+c}+\sqrt{1+c}-i\epsilon)}\right) \\ &= \frac{2\pi\tilde{\lambda}^2}{(t-1)^2\sqrt{1+c}}\delta(\sqrt{(t-1)^2+c}-\sqrt{1+c}) = 2\pi\tilde{\lambda}^2\delta(t-2). \end{aligned}$$

Adding also the measure on $[1 - \sqrt{-c}, 1 + \sqrt{-c}]$ we obtain the representation:

Proposition 6.3. *The diagonal 2-point function of the renormalised 2-dimensional Kontsevich model Φ_2^3 has, for linearly spaced eigenvalues of E , real $\tilde{\lambda}$ and in large- (N, V) limit, an integral representation*

$$\begin{aligned} \tilde{G}\left(\frac{p^2}{2\mu^2}, \frac{p^2}{2\mu^2}\right) &= \frac{2\tilde{\lambda}^2\sigma^2}{\pi(\sigma^2-1)} \int_0^\pi d\phi \frac{\left\{ (1+\sigma)\frac{\log(1+\sigma)}{\sigma} - \frac{\sigma(1+\tan^2\phi)^2}{1+\sigma^2\tan^2\phi} \right. \\ &\quad \left. + (1+\tan^2\phi)(1-\tan\phi(\arctan_{[0,\pi]}(\sigma\tan\phi)-\phi)) \right\}}{1-\frac{\sqrt{\sigma^2-1}}{\sigma}\cos\phi+\frac{p^2}{\mu^2}} \\ &\quad - 2\tilde{\lambda}^2 \int_2^\infty dt \frac{1/(t-1)^3}{t+\frac{p^2}{\mu^2}} + \frac{\tilde{\lambda}^2}{\frac{p^2}{\mu^2}+1}. \end{aligned} \tag{6.20}$$

Therefore, $p^2 \mapsto \tilde{G}\left(\frac{p^2}{2\mu^2}, \frac{p^2}{2\mu^2}\right)$ is not a Stieltjes function!

7. Summary and outlook

We extended our previous work [14] (on $D = 2$) to give an exact solution of the Φ_D^3 large- \mathcal{N} matrix model in $D = 4$ and $D = 6$ dimensions. Induction proofs and the difficult combinatorics were unchanged compared with $D = 2$, but the renormalisation, performed in the manner of Wolfhart Zimmermann, was much more involved. The main lesson is that our method is powerful enough to handle just renormalisable models with running coupling constant where a perturbative approach is plagued by overlapping divergences and renormalon problem. None of these perturbative artefacts arises: the exact renormalised correlation functions are analytic in the renormalised Φ^3 -coupling constant. Although the bare (and real) Φ_6^3 -coupling constant diverges (positive β -function), the exact solution does not develop a Landau pole.

The deepest result established in this paper is the completely unexpected proof that the Schwinger 2-point function arising from the large-deformation limit of the Φ_D^3 -model on non-commutative Moyal space is reflection positive in $D = 4$ and $D = 6$ dimensions, but not in $D = 2$. This result relied heavily on the explicit knowledge of the Φ_D^3 -matrix correlation functions which allowed us to perform the analytic continuation to the complex plane. Consequently, this sector of the theory defines unambiguously a Wightman 2-point function of a true relativistic quantum field theory [24] in $D \in \{4, 6\}$ dimensions.

We explicitly computed the Stieltjes measure of the Euclidean quantum field theory, which is the same as the Källén–Lehmann mass spectrum [25,26] of the Wightman theory. The mass shell around $|p|^2 = \mu^2$ is not sharp but broadened to non-zero width which depends on the coupling constant. In addition there is a scattering spectrum starting at $|p|^2 = 2\mu^2$ and ranging up to ∞ . The beginning at $|p|^2 = 2\mu^2$ and not at $|p|^2 = 4\mu^2$ is strange. It essentially means that the theory, although in dimension $D \in \{4, 6\}$, behaves like a one-dimensional theory where only the energy, and no momentum, is additive. This sounds somewhat disappointing, but mathematical physics in one dimension [31] is a rich subject! Of course it remains to be seen whether the interpretation as a scattering spectrum is correct. According to Aks [32], scattering in dimension $D \in \{4, 6\}$ must be accompanied by particle production, which however is absent in integrable models. The clarification of this question, and of reflection positivity of $(N > 2)$ -point functions, are left for future investigation.

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