



Exact solution of matricial Φ_2^3 quantum field theory

Harald Grosse^a, Akifumi Sako^{b,a}, Raimar Wulkenhaar^{c,*}

^a *Fakultät für Physik, Universität Wien, Boltzmanngasse 5, A-1090 Wien, Austria*

^b *Department of Mathematics, Faculty of Science Division II, Tokyo University of Science, 1-3 Kagurazaka, Shinjuku-ku, Tokyo 162-8601, Japan*

^c *Mathematisches Institut der Westfälischen Wilhelms-Universität, Einsteinstraße 62, D-48149 Münster, Germany*

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Abstract

We apply a recently developed method to exactly solve the Φ^3 matrix model with covariance of a two-dimensional theory, also known as regularised Kontsevich model. Its correlation functions collectively describe graphs on a multi-punctured 2-sphere. We show how Ward–Takahashi identities and Schwinger–Dyson equations lead in a special large- \mathcal{N} limit to integral equations that we solve exactly for all correlation functions.

The solved model arises from noncommutative field theory in a special limit of strong deformation parameter. The limit defines ordinary 2D Schwinger functions which, however, do not satisfy reflection positivity.

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1. Introduction

Matrix models [1] were intensely studied around 1990. Highlights include the non-perturbative solution of the Hermitian one-matrix model [2–4] and the understanding that it gives a rigorous meaning to quantum gravity in two dimensions. As proved by Kontsevich [5], there is an equivalent formulation by a model for Hermitian matrices Φ with action $\text{tr}(E\Phi^2 + \frac{i}{6}\Phi^3)$,

* Corresponding author.

E-mail addresses: harald.grosse@univie.ac.at (H. Grosse), sako@rs.tus.ac.jp (A. Sako), raimar@math.uni-muenster.de (R. Wulkenhaar).

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where E is a fixed external matrix. Equivalently, the external structure can be moved to the linear term. The resulting partition function

$$Z[J] = \int_{M_{\mathcal{N}}(\mathbb{C})} \mathcal{D}\Phi \exp\left(-\operatorname{tr}\left(-J\Phi + \frac{\mathcal{N}}{2}\beta\Phi^2 + \frac{\mathcal{N}}{3}\alpha\Phi^3\right)\right) \quad (1.1)$$

(all matrices self-adjoint) was solved by Makeenko and Semenov [6]. The strategy consists in a diagonalisation of Φ thanks to the Itzykson–Zuber–Harish-Chandra formula, leaving an integral over the eigenvalues x_i of the random matrix Φ . Since these x_i are dummy integration variables, the partition function is invariant under variations $x_i \mapsto x_i + \epsilon_n x_i^{n+1}$. These give rise to Virasoro constraints on $Z[j_1, \dots, j_N]$, which Makeenko–Semenov were able to solve.

A renewed interest in matrix models came from field theories on noncommutative spaces of Moyal–Weyl type. We mention the magnetic field model studied in [7], which is also exactly solvable but trivial as a field theory. The field theory of the Φ^3 model on Moyal space with harmonic term (see below) has been studied by one of us (HG) and H. Steinacker in [8,9]. The novel aspect was a renormalisation procedure for the Kontsevich model. Only partial information on correlation functions was obtained; this is the point where the present paper goes much further.

Two of us (HG+RW) worked on the Φ^4 -theory on four-dimensional Moyal–Weyl deformed space and cured the ultraviolet–infrared mixing by adding a harmonic oscillator potential to the action. This leads to a renormalisable model [10], which develops a zero of the β -function of the coupling constant [11] at a special value of the parameter space. At this special point the model becomes a dynamical matrix model. In [12] we (HG+RW) extended the idea of [11] to an alternative solution strategy for matrix models, avoiding the diagonalisation (which is useless for the Φ^4 interaction). We used instead the Ward–Takahashi identities which result from a variation $\Phi \mapsto U^* \Phi U$, with $U = \exp(i\epsilon B)$ unitary, to derive a different type of Schwinger–Dyson equations. We proved that one of them consists in a non-linear singular integral equation for the 2-point function alone (first obtained in [13]), which then determines all higher correlation functions. We subsequently reduced the problem to a fixed point equation for a single function on \mathbb{R}_+ and proved that a solution exists [14]. If one could prove that the solution is the Stieltjes transform of a positive measure, which is true for the computer [15], then one could convert the model into a 4-dimensional Euclidean quantum field theory with reflection-positive Schwinger 2-point function [16].

In this paper we apply the strategy of [12] to the Φ_2^3 matrix model.¹ Since a linear term would be generated by loop corrections, we add it from the beginning. We define first the model with cut-offs and give next Ward–Takahashi (WT) identities and Schwinger–Dyson (SD) equations. The 1-point function requires renormalisation, after which the cut-off can be sent to ∞ in the usual way [6]; for noncommutative field theory this corresponds to a limit of large matrices coupled with an infinitely strong deformation parameter – a limit which is called the “Swiss cheese limit”. This way one projects onto the genus zero sector, but keeps all possible boundary components. In this limit the infinite hierarchy of SD-equations decouples (as in the Φ^4 -model [12]). We find that a function $W(X)$ related to the 1-point function satisfies a non-linear integral equation which, up to the renormalisation problem, is *identical* to an equation solved by

¹ In our subsequent paper [17] we extend this work to four and six dimensions. Whereas the renormalisation of Φ_4^3 and Φ_6^3 is much more involved, the solution of the Schwinger–Dyson equations is easily adapted from the Φ_2^3 case. To avoid duplication of material we introduce in some formulae parameters Z, ν which at the end are set to $Z = 1$ and $\nu = 0$ for Φ_2^3 .

Makeenko–Semenoff [6] in the framework of the Kontsevich model. This coincidence is by no means surprising! We then proceed by resolving the entire hierarchy of linear equations for all genus-zero matrix correlation functions. Here combinatorial identities on Bell polynomials play a crucial rôle.

In the final section we relate the Φ^3 -matrix model to field theory on noncommutative Moyal space. We also perform in position space the limit of large deformation parameter. In this way a Euclidean quantum field theory on standard (undeformed) \mathbb{R}^2 is obtained for which we can explicitly describe all connected Schwinger functions. We deduce that already the Schwinger 2-point function *does not fulfil* reflection positivity for whatever (real or imaginary) non-zero coupling constant. This is in sharp contrast with the Φ^4 -model where numerical and partial analytic evidence was given that the Schwinger 2-point function is reflection positive.

Associating a quantum field theory with a matrix model is somewhat unusual in the traditional setup. We therefore begin in section 2 with a description of this relation, thereby giving a precise definition of correlation functions on the multi-punctured sphere, with N_β fields attached to the β th boundary component (= puncture).

2. Prelude: a QFT toy model

We consider planar graphs Γ on the 2-sphere with two sorts of vertices: any number of black (internal) vertices of valence 3, and $B \geq 1$ white vertices $\{v_\beta\}_{\beta=1}^B$ (external vertices, or punctures, or boundary components) of any valence $N_\beta \geq 1$. Every face is required to have at most one white vertex (separation of punctures). Faces with a white vertex are called external; they are labelled by positive real numbers $x_1^1, \dots, x_{N_1}^1, \dots, x_1^B, \dots, x_{N_B}^B$ (the upper index labels the unique white vertex of the face). Faces without white vertex are called internal; they are labelled by positive real numbers y_1, \dots, y_L . Such graphs are dual to triangulations of the B -punctured sphere.

We associate a weight $(-\tilde{\lambda})$ to each black vertex, weight 1 to each white vertex, and weight $\frac{1}{z_1+z_2+1}$ to an edge separating faces labelled by z_1 and z_2 . These can be internal or external, also $z_1 = z_2$ can occur. Multiply the weights of all edges and vertices of the graph and integrate over all internal face variables y_1, \dots, y_L from 0 to a cut-off Λ^2 , thus giving rise to a function $\tilde{G}_\Gamma^\Lambda(x_1^1, \dots, x_{N_1}^1 | \dots | x_1^B, \dots, x_{N_B}^B)$ of the external face variables.

Three examples are in order:

$$\Gamma_1 : \text{---} \circ \text{---} x_1^1 \text{---} \bullet \text{---} y_1 \text{---} \circ \text{---} \quad \tilde{G}_{\Gamma_1}^\Lambda(x_1^1) = \frac{(-\tilde{\lambda})}{2x_1^1 + 1} \int_0^{\Lambda^2} \frac{dy_1}{x_1^1 + y_1 + 1}, \quad (2.1)$$

$$\Gamma_2 : \text{---} \circ \text{---} x_1^1 \text{---} \bullet \text{---} y_1 \text{---} \bullet \text{---} x_2^1 \text{---} \circ \text{---} \quad \tilde{G}_{\Gamma_2}^\Lambda(x_1^1, x_2^1) = \frac{(-\tilde{\lambda})^2}{(x_1^1 + x_2^1 + 1)^2} \int_0^{\Lambda^2} \frac{dy_1}{(x_1^1 + y_1 + 1)(x_2^1 + y_1 + 1)}, \quad (2.2)$$

$$\Gamma_3 : \text{---} \circ \text{---} x_1^1 \text{---} \bullet \text{---} x_1^2 \text{---} \bullet \text{---} \circ \text{---} \quad \tilde{G}_{\Gamma_3}^\Lambda(x_1^1 | x_1^2) = \frac{(-\tilde{\lambda})^2}{(2x_1^1 + 1)(2x_1^2 + 1)(x_1^1 + x_1^2 + 1)^2}. \quad (2.3)$$

This setting defines a toy model of quantum field theory, sharing all typical features. It has the power-counting behaviour of the Φ^3_2 model, in particular has a single divergence: The limit $\lim_{\Lambda \rightarrow \infty} \tilde{G}_{\Gamma_1}^\Lambda(x_1^1)$ does not exist. The problem is cured by renormalisation. We assume the reader is familiar with the notion of one-particle irreducible (1PI) subgraphs. The renormalisation of the toy quantum field theory consists in recursively replacing all 1PI one-point subfunctions $f(z)$ by

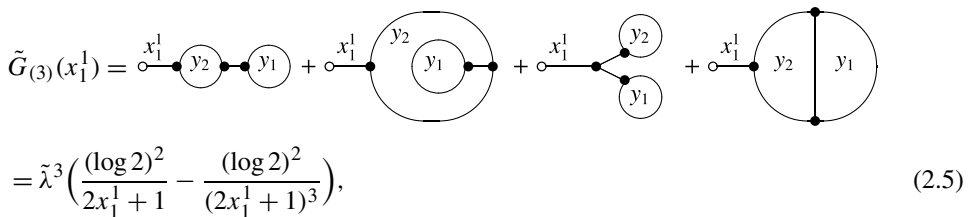
its Taylor subtraction $f(z) - f(0)$. This does more than necessary, but permits the global (i.e. non-perturbative) normalisation rule $\tilde{G}_\Gamma(0) = 0$ for any graph Γ with a single white vertex of valence 1. Omitting the superscript Λ on \tilde{G} means recursive renormalisation plus limit $\Lambda \rightarrow \infty$. We note

$$\tilde{G}_{\Gamma_1}(x_1^1) = \frac{(-\tilde{\lambda})}{2x_1^1 + 1} \int_0^\infty dy_1 \left(\frac{1}{x_1^1 + y_1 + 1} - \frac{1}{y_1 + 1} \right) = \tilde{\lambda} \frac{\log(x_1^1 + 1)}{2x_1^1 + 1}. \tag{2.4}$$

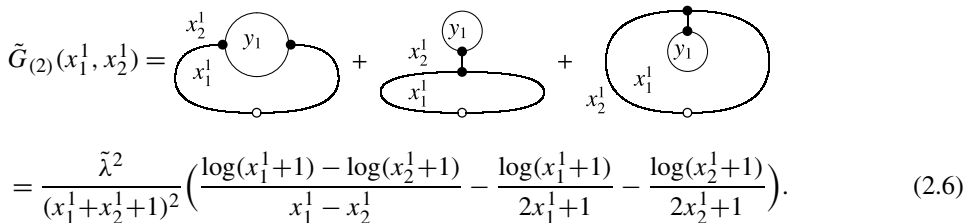
Consider the following challenge: Fix B white vertices of valences N_1, \dots, N_B , take an arbitrary number (there is a lower bound) of black vertices, and connect them in all possible ways to planar graphs. Assign the weights, perform the renormalisation, evaluate the face integrals (for $\Lambda \rightarrow \infty$) and sum everything up. What does this give?

A main difficulty in quantum field theory is that there are too many graphs. Our situation is more favourable: The number of connected planar graphs with n black vertices can be estimated by the number n^{n-2} of ordered trees with n vertices. With the typical tools of quantum field theory, see e.g. [18], one can prove uniform bounds of the type $|\tilde{G}_\Gamma| \leq C_1 \cdot |\tilde{\lambda}|^n C_2^n$. Together with the $\frac{1}{n!}$ -prefactor from the expansion of the exponential one can expect to give a meaning to $\tilde{G}(x_1^1, \dots, x_{N_1}^1 | \dots | x_1^B, \dots, x_{N_B}^B) = \sum_\Gamma \tilde{G}_\Gamma(x_1^1, \dots, x_{N_1}^1 | \dots | x_1^B, \dots, x_{N_B}^B)$ for $|\tilde{\lambda}|$ small enough.²

In this paper we achieve more than a proof of convergence: We will provide *exact formulae*, analytic in $\tilde{\lambda}^2$, for any $\tilde{G}(x_1^1, \dots, x_{N_1}^1 | \dots | x_1^B, \dots, x_{N_B}^B)$. For convenience we refer to the simplest cases: $\tilde{G}(x_1^1)$ will be given in (4.18), $\tilde{G}(x_1^1, x_2^1)$ implicitly in (4.21) and $\tilde{G}(x_1^1 | x_1^2)$ implicitly in (5.9). One has to insert $X_i^\beta = (2x_i^\beta + 1)^2$ and the formulae for $W(X)$ and $c(\tilde{\lambda})$ given in Proposition 4.1. The order- n Taylor term reproduces the sum of all graphs with n black vertices and B white vertices of valences N_1, \dots, N_B . The reader is invited to convince herself/himself that these formulae (restricted to the relevant order in $\tilde{\lambda}$) and the graphical rules agree on the following examples:



$$\tilde{G}_{(3)}(x_1^1) = \text{diagram 1} + \text{diagram 2} + \text{diagram 3} + \text{diagram 4} = \tilde{\lambda}^3 \left(\frac{(\log 2)^2}{2x_1^1 + 1} - \frac{(\log 2)^2}{(2x_1^1 + 1)^3} \right), \tag{2.5}$$



$$\tilde{G}_{(2)}(x_1^1, x_2^1) = \text{diagram 1} + \text{diagram 2} + \text{diagram 3} = \frac{\tilde{\lambda}^2}{(x_1^1 + x_2^1 + 1)^2} \left(\frac{\log(x_1^1 + 1) - \log(x_2^1 + 1)}{x_1^1 - x_2^1} - \frac{\log(x_1^1 + 1)}{2x_1^1 + 1} - \frac{\log(x_2^1 + 1)}{2x_2^1 + 1} \right). \tag{2.6}$$

² We are grateful to the referee for her/his clarifying comments on this point.

In fact we solve a more general case with weight functions $\frac{1}{e(z_1)+e(z_2)+1}$ for the edges, where $e : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is a differentiable function of positive derivative. Equivalently, one can keep the old face variables y_i but assign a weight $\tilde{\rho}(y_i) = \frac{1}{e'(e^{-1}(y_i))}$ to the faces. The asymptotic behaviour of $\tilde{\rho}(y) \sim y^{\frac{D}{2}-1}$ for $y \rightarrow \infty$ encodes a dimensionality D , where actually only the even integer $2[\frac{D}{2}]$ matters. This paper treats $2[\frac{D}{2}] = 2$. For $2[\frac{D}{2}] = 0$ we have a finite model where no renormalisation is necessary. In [17] we extend this work to $2[\frac{D}{2}] = 4$ (which also has a finite number of divergences) and to the just renormalisable case $2[\frac{D}{2}] = 6$.

3. The setup

Consider the following action functional for Hermitian matrix-valued ‘fields’ $\Phi = \Phi^* \in M_{\mathcal{N}}(\mathbb{C})$:

$$S = V \operatorname{tr}(E\Phi^2 + \kappa\Phi + \frac{\lambda}{3}\Phi^3), \tag{3.1}$$

or explicitly (in symmetrised form)

$$S = V \left(\sum_{n,m=0}^{\mathcal{N}} \frac{1}{2} \Phi_{nm} \Phi_{mn} H_{nm} + \kappa \sum_{m=0}^{\mathcal{N}} \Phi_{mm} + \frac{\lambda}{3} \sum_{k,l,m=0}^{\mathcal{N}} \Phi_{kl} \Phi_{lm} \Phi_{mk} \right),$$

$$H_{mn} := E_m + E_n. \tag{3.2}$$

Here V is a constant discussed later, λ is the coupling constant (real or complex), and κ will be needed for renormalising the 1-point function. The self-adjoint positive matrix $E = (E_m \delta_{mn})$ plays a crucial rôle. We assume that the eigenvalues E_m are a discretisation of a monotonously increasing differentiable function e with $e(0) = 0$,

$$E_m = \mu^2 \left(\frac{1}{2} + e \left(\frac{m}{\mu^2 V} \right) \right), \tag{3.3}$$

thus identifying $2E_0 = \mu^2$ with a squared mass. The resulting covariance functions $\frac{1}{H_{mn}} = \frac{1}{\mu^2(e(\frac{m}{\mu^2 V})+e(\frac{n}{\mu^2 V})+1)}$ are nothing else than the (discretised) edge weights considered in section 2. In particular, the discussion on the dimensionality encoded in e (i.e. in the spectrum of E) applies.

Comparison with (1.1) suggests that V is proportional to the size \mathcal{N} of the matrices. This is precisely what we will do. The only reason to keep them distinct is the fact that, as recalled in section 6, the action (3.1) naturally arises in noncommutative field theory. There, V is related to the deformation parameter, so that the limit $\mathcal{N} \sim V \rightarrow \infty$ defines the strong-deformation regime.

The partition function with an external field J , which is also a self-adjoint matrix, is formally defined by

$$\begin{aligned} Z[J] &:= \int \mathcal{D}\Phi \exp(-S + V \operatorname{tr}(J\Phi)) \\ &= K \exp\left(-\frac{\lambda}{3V^2} \sum_{m,n,k=0}^{\mathcal{N}} \frac{\partial^3}{\partial J_{mn} \partial J_{nk} \partial J_{km}}\right) Z_{free}[J], \end{aligned} \tag{3.4}$$

$$Z_{free}[J] := \exp \left(\sum_{m,n=0}^{\mathcal{N}} \frac{V}{2} (J_{nm} - \kappa \delta_{nm}) H_{nm}^{-1} (J_{mn} - \kappa \delta_{nm}) \right), \tag{3.5}$$

where $K = \int \mathcal{D}\Phi \exp \left(-\frac{V}{2} \sum_{m,n=0}^{\mathcal{N}} \Phi_{mn} H_{mn} \Phi_{nm} \right)$.

A perturbative expansion of $\log Z[J]$ gives exactly the graphical setup described in section 2 – up to discretisation and temporary admission of non-planar graphs. The matrix indices correspond to face variables, edges between faces m, n have weight $\frac{1}{H_{mn}}$, and the Φ^3 vertices are the black ones with weight $(-\lambda)$. Identifying the white vertices is a little tricky. It turns out that the source matrices J partition into cycles $\mathbb{J}_{p_1 \dots p_{N_\beta}} := \prod_{j=1}^{N_\beta} J_{p_j p_{j+1}}$, with $N_\beta + 1 \equiv 1$. Such a cycle of length N_β is what we call a white vertex of valence N_β . Indeed, a ‘star’ of covariances $\prod_{j=1}^{N_\beta} \frac{1}{H_{p_j p_{j+1}}}$ attaches to the source matrices, which graphically means that the white vertex is the common corner of the N_β external faces labelled by p_1, \dots, p_{N_β} .

With this identification we can represent $\log Z$ as a sum over the number and the valences of the white vertices, i.e. the cycles of source matrices:

$$\log \frac{Z[J]}{Z[0]} =: \sum_{B=1}^{\infty} \sum_{1 \leq N_1 \leq \dots \leq N_B} \sum_{p_1^1, \dots, p_{N_B}^B=0}^{\mathcal{N}} V^{2-B} \frac{G_{|p_1^1 \dots p_{N_1}^1| \dots |p_1^B \dots p_{N_B}^B|}}{S_{(N_1, \dots, N_B)}} \prod_{\beta=1}^B \frac{\mathbb{J}_{p_1^\beta \dots p_{N_\beta}^\beta}}{N_\beta}, \tag{3.6}$$

where the symmetry factor $S_{(N_1, \dots, N_B)}$ is chosen as follows: If we regroup identical valence numbers N_β as $(N_1, \dots, N_B) = (\underbrace{N'_1, \dots, N'_1}_{\nu_1}, \dots, \underbrace{N'_s, \dots, N'_s}_{\nu_s})$, then $S_{(N_1, \dots, N_B)} = \prod_{i=1}^s \nu_i!$. The

expansion coefficients $G_{|p_1^1 \dots p_{N_1}^1| \dots |p_1^B \dots p_{N_B}^B|}$ are called $(N_1 + \dots + N_B)$ -point function. In principle they further expand into graphs Γ with all possible numbers of black vertices and their connections. Being interested in exact formulae, we keep the $(N_1 + \dots + N_B)$ -point functions intact and never expand into graphs. We will prove in this paper (similarly to [12]) that these functions have a well-defined large- (\mathcal{N}, V) limit precisely for the given a scaling factor V^{2-B} in $\log Z[J]$. For later purpose we note the first terms of the resulting expansion of the partition function itself:

$$\begin{aligned} \frac{Z[J]}{Z[0]} &= 1 + V \sum_m G_{|m|} J_{mm} \\ &+ \frac{V}{2} \sum_{m,n} G_{|mn|} J_{mn} J_{nm} + \sum_{m,n} \left(\frac{1}{2} G_{|m|n|} + \frac{V^2}{2} G_{|m|} G_{|n|} \right) J_{mm} J_{nn} \\ &+ \frac{V}{3} \sum_{m,n,k} G_{|mnk|} J_{mn} J_{nk} J_{km} + \sum_{m,n,k} \left(\frac{1}{2} G_{|mn|k|} + \frac{V^2}{2} G_{|mn|} G_{|k|} \right) J_{mn} J_{nm} J_{kk} \\ &+ \sum_{m,n,k} \left(\frac{1}{6V} G_{|m|n|k|} + \frac{V}{2} G_{|m|n|} G_{|k|} + \frac{V^3}{6} G_{|m|} G_{|n|} G_{|k|} \right) J_{mm} J_{nn} J_{kk} + \dots \end{aligned} \tag{3.7}$$

All sums run from 0 to a cut-off \mathcal{N} .

We repeat the remark pointed out in [12] that these correlation functions have common source factors on the diagonal, e.g. $(V^1 G_{|aa|} + G_{|a|a|}) J_{aa} J_{aa}$. The functions $G_{|a|a|}$ and $G_{|a|a|}$ are clearly distinguished by their topology (number and valence of white vertices) and most conveniently identified by continuation of $G_{|ab|}$ and $G_{|a|b|}$ to the diagonal.

Finally, we introduce our main tool: the Ward–Takahashi identities. As proved in [11,12], the invariance of the partition function (3.4) under inner automorphisms $\Phi \mapsto U^* \Phi U$ boils down to the WT-identities

$$\sum_m \frac{\partial}{\partial J_{am}} \frac{\partial}{\partial J_{mb}} Z[J] = W_a \delta_{ab} Z[J] + \sum_m \frac{V}{E_a - E_b} \left(J_{ma} \frac{\partial}{\partial J_{mb}} - J_{bm} \frac{\partial}{\partial J_{am}} \right) Z[J], \tag{3.8}$$

where the precise form of W_a (which we shall not need) is given in [12, Thm 2.3]. These identities are exactly the counterpart of the Virasoro constraints in the traditional approach to matrix models [6].

4. Schwinger–Dyson equations and their solution for $B = 1$

4.1. 1- and 2-point functions

We now derive a formula for the connected 1-point function $G_{|a|}$ by inserting (3.4), (3.5) into the corresponding term of (3.6):

$$\begin{aligned} G_{|a|} &= \left. \frac{\partial \log Z[J]}{V \partial J_{aa}} \right|_{J=0} \\ &= \frac{K}{Z[0]} \exp \left(-\frac{\lambda}{3V^2} \sum_{m,n,k} \frac{\partial^3}{\partial J_{mn} \partial J_{nk} \partial J_{km}} \right) \left((J_{aa} - \kappa) H_{aa}^{-1} Z_{free}[J] \right) \Big|_{J=0} \\ &= H_{aa}^{-1} \left(-\kappa - \frac{\lambda}{V^2 Z[0]} \sum_{m=0}^{\mathcal{N}} \frac{\partial}{\partial J_{am}} \frac{\partial}{\partial J_{ma}} Z[J] \right) \Big|_{J=0} \\ &= H_{aa}^{-1} \left(-\kappa - \lambda G_{|a|}^2 - \frac{\lambda}{V} \sum_{m=0}^{\mathcal{N}} G_{|am|} - \frac{\lambda}{V^2} G_{|a|a|} \right). \end{aligned} \tag{4.1}$$

The last line follows from a two-fold differentiation of (3.7). Of course the sum $\sum_{m=0}^{\mathcal{N}} G_{|am|}$ includes $m = a$!

The connected 2-point function $G_{|ab|}$ is computed for $a \neq b$ as follows:

$$\begin{aligned} G_{|ab|} &= \left. \frac{\partial^2 \log Z[J]}{V \partial J_{ab} \partial J_{ba}} \right|_{J=0} \\ &= \frac{K}{Z[0]} \exp \left(-\frac{\lambda}{3V^2} \sum_{m,n,k} \frac{\partial^3}{\partial J_{mn} \partial J_{nk} \partial J_{km}} \right) \frac{\partial}{\partial J_{ab}} \left(J_{ab} H_{ab}^{-1} Z_{free} \right) \Big|_{J=0} \\ &= H_{ab}^{-1} - \frac{\lambda}{V^2} \frac{H_{ab}^{-1}}{Z[0]} \sum_{m=0}^{\mathcal{N}} \frac{\partial}{\partial J_{ab}} \frac{\partial}{\partial J_{bm}} \frac{\partial}{\partial J_{ma}} Z[J] \Big|_{J=0} \\ &= H_{ab}^{-1} - \frac{\lambda}{V(E_b - E_a)} \frac{H_{ab}^{-1}}{Z[0]} \sum_{m=0}^{\mathcal{N}} \frac{\partial}{\partial J_{ab}} \left(J_{mb} \frac{\partial Z}{\partial J_{ma}} - J_{am} \frac{\partial Z}{\partial J_{bm}} \right) \Big|_{J=0} \\ &= H_{ab}^{-1} - \frac{\lambda}{V(E_b - E_a)} \frac{H_{ab}^{-1}}{Z[0]} \left(\frac{\partial Z}{\partial J_{aa}} - \frac{\partial Z}{\partial J_{bb}} \right) \Big|_{J=0} \end{aligned}$$

$$= H_{ab}^{-1} \left(1 + \lambda \frac{(G_{|a|} - G_{|b|})}{E_a - E_b} \right). \tag{4.2}$$

In the step from the 3rd to 4th line we have used the Ward–Takahashi identity (3.8). The equation extends by continuity to $a = b$, i.e. $G_{|aa|} = H_{aa}^{-1} + \lambda H_{aa}^{-1} \lim_{b \rightarrow a} \frac{(G_{|a|} - G_{|b|})}{E_a - E_b}$. The limit is well-defined in perturbation theory where $G_{|a|}$ is, before performing the loop sum, a rational function of the E_n so that a factor $E_a - E_b$ can be taken out of $G_{|a|} - G_{|b|}$. We shall later see that our large- (\mathcal{N}, V) limit automatically gives a meaning also to $\lim_{b \rightarrow a}$.

The naïve limit $\mathcal{N} \rightarrow \infty$ in (4.1) will diverge unless $\kappa = \kappa(\mathcal{N})$ is carefully adjusted. We chose a renormalisation condition

$$G_0 = 0 \quad \Leftrightarrow \quad -\kappa(\mathcal{N}) = \frac{\lambda}{V} \sum_{m=0}^{\mathcal{N}} G_{0m} + \frac{\lambda}{V^2} G_{|0|0|}, \tag{4.3}$$

where a well-defined limit $G_{|0|0|}$ is assumed. Substituting (4.2) and (4.3) into (4.1), the Schwinger–Dyson equations are obtained as

$$G_{|a|} = H_{aa}^{-1} \left\{ -\lambda G_{|a|}^2 - \frac{\lambda}{V} \sum_{m=0}^{\mathcal{N}} (H_{am}^{-1} - H_{0m}^{-1}) - \frac{\lambda}{V^2} (G_{|a|a|} - G_{|0|0|}) - \frac{\lambda^2}{V} \sum_{m=0}^{\mathcal{N}} \left(H_{am}^{-1} \frac{(G_{|a|} - G_{|m|})}{E_a - E_m} - H_{0m}^{-1} \frac{G_{|m|}}{E_m - E_0} \right) \right\}. \tag{4.4}$$

This equation suggests to introduce

$$\frac{W_{|a|}}{2\lambda} := G_{|a|} + \frac{H_{aa}}{2\lambda} = G_{|a|} + \frac{E_a}{\lambda}. \tag{4.5}$$

Taking $H_{am}(E_a - E_m) = E_a^2 - E_m^2$ into account, we arrive at

$$W_{|a|}^2 = 4E_a^2 - \frac{4\lambda^2}{V^2} (G_{|a|a|} - G_{|0|0|}) - \frac{2\lambda^2}{V} \sum_{m=0}^{\mathcal{N}} \left(\frac{W_{|a|} - W_{|m|}}{E_a^2 - E_m^2} - \frac{W_{|m|} - W_{|0|}}{E_m^2 - E_0^2} \right), \tag{4.6}$$

$$G_{|ab|} = \frac{1}{2} \frac{W_{|a|} - W_{|b|}}{E_a^2 - E_b^2}. \tag{4.7}$$

4.2. Large- (\mathcal{N}, V) limit and integral equations

Let us take the limit $\mathcal{N}, V \rightarrow \infty$ subject to fixed ratio $\frac{\mathcal{N}}{V} = \mu^2 \Lambda^2$, in which the sum converges to a Riemann integral

$$\lim \frac{1}{V} \sum_{m=0}^{\mathcal{N}} f(m/V) = \mu^2 \Lambda^2 \int_0^1 du f(\mu^2 \Lambda^2 u) = \mu^2 \int_0^{\Lambda^2} dx f(\mu^2 x). \tag{4.8}$$

Expressing discrete matrix elements as $a =: V\mu^2 x$, the eigenvalues of E take the form $E_a = \mu^2(e(x) + \frac{1}{2})$, see (3.3). We introduce the dimensionless³ coupling constant $\tilde{\lambda} := \frac{\lambda}{\mu^2}$ and define

³ From the partition function (3.4) and its expansion (3.6) one reads off the following mass dimensions:

$$[\Phi] = \mu^0, \quad [J] = \mu^2, \quad [\kappa] = \mu^2, \quad [\tilde{\lambda}] = \mu^2, \quad [G_{|p_1^1 \dots p_{N_1}^1 | \dots | p_1^B \dots p_{N_B}^B |}] = \mu^{2(2-B-N)}.$$

$$\mu^2 \tilde{W}(x) := \lim_{\mathcal{N}, V \rightarrow \infty} W_{|V\mu^2x|}, \quad \tilde{G}(x) := \lim_{\mathcal{N}, V \rightarrow \infty} G_{|V\mu^2x|}, \tag{4.9}$$

related by $\frac{\tilde{W}(x)}{2\tilde{\lambda}} = \tilde{G}(x) + \frac{e(x)+\frac{1}{2}}{\tilde{\lambda}}$. Now the limit of (4.6) becomes

$$(\tilde{W}(x))^2 = (2e(x) + 1)^2 - 8\tilde{\lambda}^2 \int_0^{\Lambda^2} dy \left(\frac{\tilde{W}(x) - \tilde{W}(y)}{(2e(x) + 1)^2 - (2e(y) + 1)^2} - \frac{\tilde{W}(y) - \tilde{W}(0)}{(2e(y) + 1)^2 - 1} \right). \tag{4.10}$$

We assume here $G_{|V\mu^2x|V\mu^2x|} = \mathcal{O}(V^0)$ so that this term does not contribute to the limit; this will be checked later. It can be seen graphically that this term generates higher genus contributions, which are scaled away in the large- \mathcal{N} limit. A final transformation

$$X := (2e(x) + 1)^2, \quad W(X) = \tilde{W}(x(X)), \quad G(X) = \tilde{G}(x(X)), \tag{4.11}$$

and similarly for other capital letters $Y(y)$, $T(y)$ and functions $G(X, Y) = \tilde{G}(x(X), y(Y))$ etc., simplifies (4.10) to

$$W^2(X) + \int_1^{\Xi} dY \rho(Y) \frac{W(X) - W(Y)}{X - Y} = X + \int_1^{\Xi} dY \rho(Y) \frac{W(1) - W(Y)}{1 - Y},$$

$$\rho(Y) := \frac{2\tilde{\lambda}^2}{\sqrt{Y} \cdot e'(e^{-1}(\frac{\sqrt{Y}-1}{2}))}, \quad \Xi := (1 + 2e(\Lambda^2))^2. \tag{4.12}$$

Equation (4.12) closely resembles a problem solved in the appendix of Makeenko–Semenoff [6]. We take their solution (obtained by solving a Riemann–Hilbert problem) as an ansatz⁴

$$W(X) := \frac{\sqrt{X+c}}{\sqrt{Z}} - v + \frac{1}{2} \int_1^{\Xi} dT \frac{\rho(T)}{(\sqrt{X+c} + \sqrt{T+c})\sqrt{T+c}} \tag{4.13}$$

with constants Z, v, c determined by normalisation and consistency conditions (thus becoming functions of λ, Ξ). Straightforward computation using $\frac{\sqrt{X+c}-\sqrt{Y+c}}{X-Y} = \frac{1}{\sqrt{X+c}+\sqrt{Y+c}}$ yields

$$\begin{aligned} & \int_1^{\Xi} dY \rho(Y) \frac{W(X) - W(Y)}{X - Y} \\ &= \frac{\sqrt{X+c}}{\sqrt{Z}} \int_1^{\Xi} \frac{dY \rho(Y)}{(\sqrt{X+c} + \sqrt{Y+c})\sqrt{X+c}} \\ & - \frac{1}{2} \int_1^{\Xi} \frac{dT \rho(T)}{\sqrt{T+c}(\sqrt{X+c} + \sqrt{T+c})} \int_1^{\Xi} \frac{dY \rho(Y)}{(\sqrt{X+c} + \sqrt{Y+c})(\sqrt{Y+c} + \sqrt{T+c})}. \end{aligned}$$

⁴ Our ansatz is more general than necessary in 2 dimensions. We need Z, v in 4 and 6 dimensions [17] and treat already here the general case in order to avoid duplication in [17].

In the last line we can symmetrise $\frac{1}{\sqrt{T+c}} \mapsto \frac{1}{2}(\frac{1}{\sqrt{T+c}} + \frac{1}{\sqrt{Y+c}})$ so that the double integral factors. Converting the second line by rational fraction expansion, we arrive at

$$\int_1^\Xi dY \rho(Y) \frac{W(X) - W(Y)}{X - Y} = \frac{1}{\sqrt{Z}} \int_1^\Xi \frac{dY \rho(Y)}{\sqrt{Y+c}} - \frac{1}{\sqrt{Z}} \int_1^\Xi \frac{dY \rho(Y) \sqrt{X+c}}{\sqrt{Y+c}(\sqrt{X+c} + \sqrt{Y+c})} - \frac{1}{4} \left(\int_1^\Xi \frac{dT \rho(T)}{\sqrt{T+c}(\sqrt{X+c} + \sqrt{T+c})} \right)^2 = -(W(X) + \nu)^2 + \frac{X+c}{Z} + \frac{1}{\sqrt{Z}} \int_1^\Xi \frac{dY \rho(Y)}{\sqrt{Y+c}}. \tag{4.14}$$

This equation takes the form of (4.12) if we choose $\nu = 0$, $Z = 1$ and adjust⁵ c by

$$W(1) = 1 = \sqrt{1+c} + \frac{1}{2} \int_1^\Xi dT \frac{\rho(T)}{(\sqrt{1+c} + \sqrt{T+c})\sqrt{T+c}}. \tag{4.15}$$

For $\rho(T) \sim T^{-\alpha}$ and $\alpha > 0$, realised in our case, the formula (4.13) and the resulting condition on c have a limit $\Xi \rightarrow \infty$.

Inserting $\rho(T)$ from (4.12) into (4.15) we have an explicit expression of $\tilde{\lambda}^2$ in terms of c , either with $c > -1$ real or $c \in \mathbb{C} \setminus]-\infty, -1]$. Obviously, $c = 0$ corresponds to $\tilde{\lambda} = 0$. The implicit function theorem then provides a unique diffeomorphism $\tilde{\lambda}^2 \mapsto c(\tilde{\lambda})$ on a neighbourhood of $0 \in \mathbb{R}$ or $0 \in \mathbb{C}$. Since we will be able to express all correlation functions in terms of elementary functions of $c(\tilde{\lambda}, e)$ and $\rho(\tilde{\lambda}, e)$, this proves analyticity of all correlation functions in these neighbourhoods.

4.3. Linearly spaced eigenvalues of E

The noncommutative field theory model of section 6 translates to linearly spaced eigenvalues with $e(x) = x$. This yields $X = (2x + 1)^2$ and $\rho(Y) = \frac{2\tilde{\lambda}^2}{\sqrt{Y}}$. The integral can be evaluated for $\Xi \rightarrow \infty$:

Proposition 4.1. *Equation (4.12) is for eigenvalue functions $e(x) = x$ and $Z = 1, \nu = 0$ solved by:*

$$W(X) = \sqrt{X+c} + \frac{2\tilde{\lambda}^2}{\sqrt{X}} \log \left(\frac{(\sqrt{X+c} + \sqrt{X})(\sqrt{X} + 1)}{\sqrt{X}\sqrt{1+c} + \sqrt{X+c}} \right), \tag{4.16}$$

$$1 = \sqrt{c+1} + 2\tilde{\lambda}^2 \log \left(1 + \frac{1}{\sqrt{c+1}} \right). \tag{4.17}$$

⁵ In [6], c is determined by $c + \int_1^\Xi \frac{dY \rho(Y)}{\sqrt{Y+c}} = 0$ from (4.14). We are particularly interested in linearly spaced eigenvalues $e(x) = x$ where $\rho(Y) \propto \frac{1}{\sqrt{Y}}$, see (4.12). Then $\int_1^\Xi \frac{dY \rho(Y)}{\sqrt{Y+c}}$ diverges for $\Xi \rightarrow \infty$. This makes it necessary to normalise $W(1) = 1$.

We thus get for the renormalised 1-point function

$$\begin{aligned} \tilde{G}(x) &= \frac{1}{2\tilde{\lambda}} \left(W((2x+1)^2) - (2x+1) \right) \\ &= \frac{\sqrt{(2x+1)^2 + c} - (2x+1)}{2\tilde{\lambda}} + \frac{\tilde{\lambda}}{2x+1} \log \left(\frac{(2x+2)(\sqrt{(2x+1)^2 + c} + 2x+1)}{(2x+1)\sqrt{1+c} + \sqrt{(2x+1)^2 + c}} \right), \end{aligned} \tag{4.18}$$

again with c being the inverse solution of (4.17).

A numerical investigation shows that (4.17) has a solution⁶ for $-\tilde{\lambda}_c \leq \tilde{\lambda} \leq \tilde{\lambda}_c$ and $\tilde{\lambda}_c = 0.490686\dots$ attained at $c_c = -0.873759\dots$. By choosing $c > 0$ it is possible to simulate purely imaginary $\tilde{\lambda}$. A perturbative solution of (4.17) gives as first terms

$$c = -4\tilde{\lambda}^2 \log 2 - 4\tilde{\lambda}^4 (\log 2 - (\log 2)^2) - 2\tilde{\lambda}^6 (2 \log 2 - (\log 2)^2) + \mathcal{O}(\tilde{\lambda}^8). \tag{4.19}$$

This leads to the following series expansion of the renormalised 1-point function:

$$\begin{aligned} \tilde{G}(x) &= \frac{\tilde{\lambda}}{2x+1} \log(x+1) + \tilde{\lambda}^3 \left(\frac{(\log(2))^2}{2x+1} - \frac{(\log(2))^2}{(2x+1)^3} \right) \\ &+ \tilde{\lambda}^5 \left(\frac{(\log(2))^2}{2x+1} + \frac{2(\log(2))^3 - (\log(2))^2}{(2x+1)^3} - \frac{2(\log(2))^3}{(2x+1)^5} \right) + \mathcal{O}(\tilde{\lambda}^7). \end{aligned} \tag{4.20}$$

It matches perfectly the Feynman graph computation (2.5) of section 2.

The scaling limit $\tilde{G}(x, y) = \lim_{\mathcal{N}, V \rightarrow \infty} \mu^2 G_{|V\mu^2x, V\mu^2y|}$ of (4.7) for the 2-point function is

$$G(X, Y) = \tilde{G}(x(X), y(Y)) = 2 \frac{W(X) - W(Y)}{X - Y}. \tag{4.21}$$

We refrain from spelling out the insertion of (4.18). There is no problem going to the diagonal: $\tilde{G}(x, x) = 2W'(X)$.

4.4. N -point functions

According to (3.6) the connected ($N > 2$)-point functions are

$$G_{|a_1 a_2 \dots a_N|} = \frac{1}{V} \frac{\partial}{\partial J_{a_N a_1}} \frac{\partial}{\partial J_{a_1 a_2}} \dots \frac{\partial}{\partial J_{a_{N-1} a_N}} \log \frac{Z[J]}{Z[0]} \Big|_{J=0}. \tag{4.22}$$

For pairwise different indices we compute, similarly to (4.2),

$$\begin{aligned} G_{|a_1 \dots a_N|} &= \frac{K}{Z[0]} \frac{\partial^{N-1} \left\{ \exp \left(-\frac{\lambda}{3V^2} \sum_{m,n,k} \frac{\partial^3}{\partial J_{mn} \partial J_{nk} \partial J_{km}} \right) (J_{a_2 a_1} H_{a_1 a_2}^{-1} Z_{free}) \right\}}{\partial J_{a_2 a_3} \dots \partial J_{a_{N-1} a_N}} \Big|_{J=0} \\ &= -\frac{\lambda}{V^2} \frac{H_{a_1 a_2}^{-1}}{Z[0]} \sum_{m=0}^{\mathcal{N}} \frac{\partial}{\partial J_{a_2 a_3}} \dots \frac{\partial}{\partial J_{a_N a_1}} \frac{\partial}{\partial J_{a_1 m}} \frac{\partial}{\partial J_{m a_2}} Z[J] \Big|_{J=0} \\ &= -\frac{\lambda}{V} \frac{H_{a_1 a_2}^{-1}}{Z[0]} \sum_{m=0}^{\mathcal{N}} \frac{\partial}{\partial J_{a_2 a_3}} \dots \frac{\partial}{\partial J_{a_N a_1}} \frac{(J_{m a_1} \frac{\partial Z[J]}{\partial J_{m a_2}} - J_{a_2 m} \frac{\partial Z[J]}{\partial J_{a_1 m}})}{(E_{a_1} - E_{a_2})} \Big|_{J=0} \end{aligned}$$

⁶ In general, the critical value corresponds to $\rho_0 := 1 - \frac{1}{2} \int_1^\infty \frac{dZ\rho(Z)}{\sqrt{Z+c}} = 0$. This function ρ_0 plays a key rôle in higher correlation functions.

$$= \lambda H_{a_1 a_2}^{-1} \frac{G_{|a_1 a_3 \dots a_N|} - G_{|a_2 \dots a_N|}}{(E_{a_1} - E_{a_2})} = \lambda \frac{G_{|a_1 a_3 \dots a_N|} - G_{|a_2 \dots a_N|}}{(E_{a_1}^2 - E_{a_2}^2)}. \tag{4.23}$$

The 2nd line is the result of the $\frac{\partial}{\partial J_{a_1 a_2}}$ differentiation, and in the step from the 3rd to 4th line we have used the Ward–Takahashi identity (3.8) for pairwise different indices. Formula (4.23) together with (4.7) expresses N -point functions recursively by factors $\frac{1}{(E_{a_i}^2 - E_{a_j}^2)}$ and $W_{|a_k|}$. We can solve this recursion:

Proposition 4.2. *The connected ($N \geq 2$)-point function is given for pairwise different indices by*

$$G_{|a_1 a_2, \dots, a_N|} = \frac{\lambda^{N-2}}{2} \sum_{k=1}^N W_{|a_k|} \prod_{l=1, l \neq k}^N P_{a_k a_l}, \quad P_{ab} := \frac{1}{E_a^2 - E_b^2}. \tag{4.24}$$

Proof. The formula is proved by induction, starting with $N = 2$ which is formula (4.7) when inserting $P_{a_1 a_2} = -P_{a_2 a_1}$. Assume it holds for N . Then using (4.23) and $P_{a_1 a_2} = -P_{a_2 a_1}$ we have

$$\begin{aligned} G_{|a_1 \dots a_{N+1}|} &= \lambda P_{a_1 a_2} (G_{|a_1 a_3 \dots a_{N+1}|} - G_{|a_2 \dots a_{N+1}|}) \\ &= \frac{\lambda^{N-1}}{2} P_{a_1 a_2} \left(\sum_{k=1, k \neq 2}^{N+1} W_{|a_k|} \prod_{l=1, l \notin \{2, k\}}^{N+1} P_{a_k a_l} - \sum_{k=2}^{N+1} W_{|a_k|} \prod_{l=2, l \neq k}^{N+1} P_{a_k a_l} \right) \\ &= \frac{\lambda^{N-1}}{2} \left(W_{|a_1|} \prod_{l=2}^{N+1} P_{a_1 a_l} + W_{|a_2|} \prod_{l=1, l \neq 2}^{N+1} P_{a_2 a_l} \right. \\ &\quad \left. + \sum_{k=3}^{N+1} W_{|a_k|} P_{a_1 a_2} \left(P_{a_k a_1} \prod_{l=3, l \neq k}^{N+1} P_{a_k a_l} - P_{a_k a_2} \prod_{l=3, l \neq k}^{N+1} P_{a_k a_l} \right) \right). \end{aligned}$$

Now the definition on $P_{a_k a_l}$ implies

$$P_{a_1 a_2} (P_{a_k a_1} - P_{a_k a_2}) = P_{a_k a_1} P_{a_k a_2},$$

so that (4.24) follows for $N \mapsto N + 1$. \square

We can easily perform the scaling limit $\mathcal{N}, V \rightarrow \infty$ to functions $\tilde{G}(x_1, \dots, x_N) = \lim_{\mathcal{N}, V \rightarrow \infty} \mu^{2(N-1)} G_{|V \mu^2 x_1, \dots, V \mu^2 x_N|}$ and $G(X_1, \dots, X_N) := \tilde{G}(x_1(X_1), \dots, x_N(X_N))$. With $\lim(2E_{V \mu^2 x_k}) = \mu^2 \sqrt{X_k}$ and thus $\lim(\mu^4 P_{kl}) = \frac{4}{X_k - X_l}$ we have

$$G(X_1, \dots, X_N) = \sum_{k=1}^N \frac{W(X_k)}{2\tilde{\lambda}} \prod_{l=1, l \neq k}^N \frac{4\tilde{\lambda}}{X_k - X_l}. \tag{4.25}$$

5. N -point function with $B \geq 2$ boundaries

5.1. $(N_1 + \dots + N_B)$ -point function with one $N_i > 1$

To simplify notation let $\frac{\partial^N}{\partial J_{a_1 \dots a_N}} := \frac{\partial^N}{\partial J_{a_1 a_2} \dots \partial J_{a_{N-1} a_N} \partial J_{a_N a_1}}$. We prove:

Proposition 5.1. For $N_1 > 1$ one has

$$G_{|a_1^1 \dots a_{N_1}^1| \dots |a_1^B \dots a_{N_B}^B|} = \lambda \frac{G_{|a_1^1 a_3^1 \dots a_{N_1}^1| a_1^2 \dots a_{N_2}^2| \dots |a_1^B \dots a_{N_B}^B|} - G_{|a_2^1 a_3^1 \dots a_{N_1}^1| a_1^2 \dots a_{N_2}^2| \dots |a_1^B \dots a_{N_B}^B|}}{E_{a_1^1}^2 - E_{a_2^1}^2}. \tag{5.1}$$

Proof. For pairwise different a_i, b_j we have from (3.6)

$$\begin{aligned} G_{|a_1^1 \dots a_{N_1}^1| \dots |a_1^B \dots a_{N_B}^B|} &= V^{B-2} \frac{\partial^{N_1}}{\partial \mathbb{J}_{a_1^1 \dots a_{N_1}^1}} \dots \frac{\partial^{N_B}}{\partial \mathbb{J}_{a_1^B \dots a_{N_B}^B}} \log \frac{Z[J]}{Z[0]} \Big|_{J=0} \\ &= V^{B-1} \frac{\partial^{N_1-1}}{\partial J_{a_2^1 a_3^1} \dots \partial J_{a_{N_1}^1 a_1^1}} \frac{\partial^{N_2}}{\partial \mathbb{J}_{a_1^2 \dots a_{N_2}^2}} \dots \frac{\partial^{N_B}}{\partial \mathbb{J}_{a_1^B \dots a_{N_B}^B}} \left\{ \frac{K}{Z[J]} \right. \\ &\quad \left. \times \exp \left(-\frac{\lambda}{3V^2} \sum_{m,n,k} \frac{\partial^3}{\partial J_{mn} \partial J_{nk} \partial J_{km}} \right) \left(J_{a_2^1 a_1^1} H_{a_1^1 a_2^1}^{-1} Z_{free}[J] \right) \right\} \Big|_{J=0} \tag{*} \\ &= \frac{V^{B-3}(-\lambda)}{H_{a_1^1 a_2^1}} \frac{\partial^{N_1-1}}{\partial J_{a_2^1 a_3^1} \dots \partial J_{a_{N_1}^1 a_1^1}} \frac{\partial^{N_2}}{\partial \mathbb{J}_{a_1^2 \dots a_{N_2}^2}} \dots \frac{\partial^{N_B}}{\partial \mathbb{J}_{a_1^B \dots a_{N_B}^B}} \left\{ \frac{1}{Z[J]} \sum_{m=0}^{\mathcal{N}} \frac{\partial^2 Z}{\partial J_{a_1^m} \partial J_{ma_2^1}} \right\} \Big|_{J=0} \\ &= \frac{V^{B-2}(-\lambda)}{E_{a_1^1}^2 - E_{a_2^1}^2} \frac{\partial^{N_1-1}}{\partial J_{a_2^1 a_3^1} \dots \partial J_{a_{N_1}^1 a_1^1}} \frac{\partial^{N_2}}{\partial \mathbb{J}_{a_1^2 \dots a_{N_2}^2}} \dots \frac{\partial^{N_B}}{\partial \mathbb{J}_{a_1^B \dots a_{N_B}^B}} \left\{ \frac{1}{Z[J]} \right. \\ &\quad \left. \times \sum_{m=0}^{\mathcal{N}} \left(J_{ma_1^1} \frac{\partial Z}{\partial J_{ma_2^1}} - J_{a_2^1 m} \frac{\partial Z}{\partial J_{a_1^1 m}} \right) \right\} \Big|_{J=0} \tag{**} \\ &= \frac{V^{B-2}(-\lambda)}{E_{a_1^1}^2 - E_{a_2^1}^2} \left(\frac{\partial^{N_1-1}}{\partial \mathbb{J}_{a_2^1 \dots a_{N_1}^1}} - \frac{\partial^{N_1-1}}{\partial \mathbb{J}_{a_1^1 a_3^1 \dots a_{N_1}^1}} \right) \frac{\partial^{N_2}}{\partial \mathbb{J}_{a_1^2 \dots a_{N_2}^2}} \dots \frac{\partial^{N_B}}{\partial \mathbb{J}_{a_1^B \dots a_{N_B}^B}} \log Z[J] \Big|_{J=0}. \tag{***} \end{aligned}$$

Precisely for $N_1 = 2$ there is a surviving term of the $J_{a_2^1 a_1^1}$ differentiation, but the result cancels with $\frac{K}{Z[J]}$ so that further differentiations due to $B \geq 2$ give zero. Therefore, all surviving differentiations of $J_{a_2^1 a_1^1}$ in (*) come from $\exp(-\frac{\lambda}{3V^2} \sum \frac{\partial^3}{\partial J^3})$. In (**) the Ward–Takahashi identity (3.8) and $H_{ab}(E_a - E_b) = E_a^2 - E_b^2$ are used. Then $J_{ma_1^1}$ must be hit by $\frac{\partial}{\partial J_{a_{N_1}^1 a_1^1}}$ and $J_{a_2^1 m}$ by $\frac{\partial}{\partial J_{a_2^1 a_3^1}}$, thus giving (***). The final line gives with (3.6) the assertion (5.1). \square

By symmetry in the boundary components we can recursively use (5.1) to express any $(N_1 + \dots + N_B)$ -point function with one $N_i > 1$ in terms of $G_{|a^1|a^2| \dots |a^B|}$. Since further boundaries play a spectator rôle in (5.1), we can easily adapt the arguments of Proposition 4.2 to resolve this recursion:

Proposition 5.2. Let $B \geq 2$. The connected $(N_1 + \dots + N_B)$ -point function with one $N_i > 1$ is given in terms of $P_{ab} := \frac{1}{E_a^2 - E_b^2}$ by

$$\begin{aligned}
 &G_{|a^1 \dots a^1_{N_1} | \dots | a^B \dots a^B_{N_B} |} \\
 &= \lambda^{N_1 + \dots + N_B - B} \sum_{k_1=1}^{N_1} \dots \sum_{k_B=1}^{N_B} G_{|a^1_{k_1} | \dots | a^B_{k_B} |} \left(\prod_{l_1=1, l_1 \neq k_1}^{N_1} P_{a^1_{k_1} a^1_{l_1}} \right) \dots \left(\prod_{l_B=1, l_B \neq k_B}^{N_B} P_{a^B_{k_B} a^B_{l_B}} \right),
 \end{aligned} \tag{5.2}$$

its large- (\mathcal{N}, V) limit by

$$\begin{aligned}
 &G(X^1_1, \dots, X^1_{N_1} | \dots | X^B_1, \dots, X^B_{N_B}) \\
 &= \tilde{\lambda}^{N_1 + \dots + N_B - B} \sum_{k_1=1}^{N_1} \dots \sum_{k_B=1}^{N_B} G(X^1_{k_1} | \dots | X^B_{k_B}) \prod_{\beta=1}^B \prod_{l_\beta=1, l_\beta \neq k_\beta}^{N_\beta} \frac{4}{X^\beta_{k_\beta} - X^\beta_{l_\beta}}.
 \end{aligned} \tag{5.3}$$

5.2. SD-equation for $(1 + \dots + 1)$ -point function

Proposition 5.3. *Let $B \geq 2$. Then the $(1 + \dots + 1)$ -point function satisfies*

$$\begin{aligned}
 &W_{|a^1 | G_{|a^1 | a^2 | \dots | a^B |} + \frac{\lambda^2}{V} \sum_{m=0}^{\mathcal{N}} \frac{G_{|a^1 | a^2 | \dots | a^B |} - G_{|m | a^2 | \dots | a^B |}}{(E_{a^1}^2 - E_m^2)} \\
 &= -\lambda \sum_{\beta=2}^B G_{|a^1 a^\beta a^\beta | a^2 | \dots | a^B |} - \frac{\lambda}{V^2} G_{|a^1 | a^1 | a^2 | \dots | a^B |} \\
 &- \lambda \sum_{p=1}^{B-2} \sum_{2 \leq i_1 < \dots < i_p \leq B} G_{|a^1 | a^{i_1} | \dots | a^{i_p} |} G_{|a^1 | a^{j_1} | \dots | a^{j_{B-p-1}} |},
 \end{aligned} \tag{5.4}$$

where $2 \leq j_1 < \dots < j_{B-p-1} \leq B$ and $\{i_1, \dots, i_p, j_1, \dots, j_{B-p-1}\} = \{2, \dots, B\}$, and $\overset{\beta}{\dots}$ denotes the omission of a^β .

Proof. We write down for pairwise different indices a^β the formula for the $(1 + \dots + 1)$ -point function in (3.6) with $B \geq 2$ boundary components and perform the $J_{a^1 a^1}$ -differentiation:

$$\begin{aligned}
 &G_{|a^1 | a^2 | \dots | a^B |} = V^{B-2} \frac{\partial^B}{\partial J_{a^1 a^1} \dots \partial J_{a^B a^B}} \log \frac{Z[J]}{Z[0]} \Big|_{J=0} \\
 &= \frac{V^{B-1} \partial^{B-1}}{\partial J_{a^2 a^2} \dots \partial J_{a^B a^B}} \left\{ \frac{K}{Z[J]} \exp \left(-\frac{\lambda}{3V^2} \sum_{m,n,k} \frac{\partial^3}{\partial J_{mn} \partial J_{nk} \partial J_{km}} \right) \right. \\
 &\times \left. \left((J_{a^1 a^1} - \kappa) H_{a^1 a^1}^{-1} Z_{free}[J] \right) \right\} \Big|_{J=0} \\
 &= V^{B-1} \frac{\partial^{B-1}}{\partial J_{a^2 a^2} \dots \partial J_{a^B a^B}} \left\{ \frac{1}{Z[J]} \frac{(-\lambda)}{V^2 H_{a^1 a^1}} \sum_{m=0}^{\mathcal{N}} \frac{\partial}{\partial J_{a^1 m}} \frac{\partial}{\partial J_{m a^1}} Z \right\} \Big|_{J=0} \\
 &= V^{B-3} \frac{(-\lambda)}{H_{a^1 a^1}} \sum_{m=0}^{\mathcal{N}} \frac{\partial^{B-1}}{\partial J_{a^2 a^2} \dots \partial J_{a^B a^B}} \left\{ \frac{\partial^2 \log(Z)}{\partial J_{a^1 m} \partial J_{m a^1}} + \frac{\partial \log(Z)}{\partial J_{a^1 m}} \frac{\partial \log(Z)}{\partial J_{m a^1}} \right\} \Big|_{J=0} \\
 &= \frac{(-\lambda)}{H_{a^1 a^1}} \left\{ \frac{1}{V} \sum_{m=0}^{\mathcal{N}} G_{|a^1 | m | a^2 | \dots | a^B |} + \sum_{\beta=2}^B G_{|a^1 a^\beta a^\beta | a^2 | \dots | a^B |} + \frac{1}{V^2} G_{|a^1 | a^1 | a^2 | \dots | a^B |} \right\}
 \end{aligned}$$

$$+ 2G_{|a^1|}G_{|a^1|a^2|\dots|a^B|} + \sum_{p=1}^{B-2} \sum_{2 \leq i_1 < \dots < i_p \leq B} G_{|a^1|a^{i_1}|\dots|a^{i_p}|}G_{|a^1|a^{j_1}|\dots|a^{j_{B-p-1}|}} \}, \tag{5.5}$$

with notations introduced in the proposition. We multiply by $\frac{H_{aa}}{\lambda}$ and bring $-2G_{|a^1|}G_{|a^1|a^2|\dots|a^B|}$ to the lhs, thus reconstructing the function $W_{|a^1|}$ defined in (4.5):

$$\begin{aligned} & \frac{1}{\lambda}W_{|a^1|}G_{|a^1|a^2|\dots|a^B|} + \frac{1}{V} \sum_{m=0}^{\mathcal{N}} G_{|a^1m|a^2|\dots|a^B|} \\ &= - \sum_{\beta=2}^B G_{|a^1a^\beta a^\beta|a^2|\dots|a^B|} - \frac{1}{V^2}G_{|a^1|a^1|a^2|\dots|a^B|} \\ & - \sum_{p=1}^{B-2} \sum_{2 \leq i_1 < \dots < i_p \leq B} G_{|a^1|a^{i_1}|\dots|a^{i_p}|}G_{|a^1|a^{j_1}|\dots|a^{j_{B-p-1}|}}. \end{aligned} \tag{5.6}$$

Reducing the $(2+1+\dots+1)$ -point function by (5.1) leads to the assertion (5.4). \square

In the scaling limit $G(x^1|\dots|x^B) := \mu^{2(2-B)} \lim_{\mathcal{N}, V \rightarrow \infty} G_{|V\mu^2x^1|\dots|V\mu^2x^B|}$, the term $\frac{1}{V^2}G_{|a^1|a^1|a^2|\dots|a^B|}$ in (5.4) goes away, and we obtain a recursive system of affine equations for the function with B boundary components. To write these equations in more condensed form, let us abbreviate for a set $I = \{i_1, \dots, i_p\}$ of indices $G(X|Y^{\triangleleft I}) := G(X|Y^{i_1}|\dots|Y^{i_p})$. With these notations, and including ν (here $= 0$) from (4.13) for later use in [17], we can express the limit of (5.4) in terms of $X^i := (2e(x^i) + 1)^2$ as follows:

$$\begin{aligned} & (W(X^1)+\nu)G(X^1|X^{\triangleleft\{2,\dots,B\}}) + \frac{1}{2} \int_1^{\Xi} dT \rho(T) \frac{G(X^1|X^{\triangleleft\{2,\dots,B\}}) - G(T|X^{\triangleleft\{2,\dots,B\}})}{(X - T)} \\ &= -\tilde{\lambda} \sum_{\beta=2}^B G(X^1, X^\beta, X^\beta|X^{\triangleleft\{2,\dots,B\}}) - \tilde{\lambda} \sum_{\substack{J \subset \{2,\dots,B\} \\ 1 \leq |J| \leq B-2}} G(X^1|X^{\triangleleft J})G(X^1|X^{\triangleleft\{2,\dots,B\} \setminus J}). \end{aligned} \tag{5.7}$$

The measure $\rho(T)$ was defined in (4.12). In presence of $\nu \neq 0$ we need a finite cut-off Ξ ; the limit $\Xi \rightarrow \infty$ is only possible for the solutions. The inhomogeneity only involves known functions with $< B$ boundary components.

5.3. Solution for the (1+1)-point function

We specify the problem (5.7) to the $1 + 1$ -point function

$$(W(X) + \nu)G(X|Y) = -\tilde{\lambda}G(X, Y, Y) - \frac{1}{2} \int_1^{\Xi} dT \rho(T) \frac{G(X|Y) - G(T|Y)}{X - T}. \tag{5.8}$$

A perturbative solution of (5.8) to $\mathcal{O}(\tilde{\lambda}^4)$ suggests:

Proposition 5.4. *The (1+1)-point function is given by*

$$G(X|Y) = \frac{4\tilde{\lambda}^2}{\sqrt{X+c} \cdot \sqrt{Y+c} \cdot (\sqrt{X+c} + \sqrt{Y+c})^2}, \tag{5.9}$$

where $c(e, \tilde{\lambda})$ was defined in (4.15).

Proof. We insert the *ansatz* (5.9) into the following integral:

$$\begin{aligned} & -\frac{1}{2} \int_1^{\Xi} dT \rho(T) \frac{G(X|Y) - G(T|Y)}{X - T} \\ &= -\frac{2\tilde{\lambda}^2}{\sqrt{Y+c}} \int_1^{\Xi} dT \rho(T) \frac{\frac{1}{\sqrt{X+c} \cdot (\sqrt{X+c} + \sqrt{Y+c})^2} - \frac{1}{\sqrt{T+c} \cdot (\sqrt{T+c} + \sqrt{Y+c})^2}}{X - T} \\ &= \frac{2\tilde{\lambda}^2}{\sqrt{X+c} \cdot \sqrt{Y+c} \cdot (\sqrt{X+c} + \sqrt{Y+c})^2} \int_1^{\Xi} \frac{dT \rho(T)}{\sqrt{T+c} \cdot (\sqrt{X+c} + \sqrt{T+c})} \\ &+ \frac{2\tilde{\lambda}^2}{\sqrt{Y+c} \cdot (\sqrt{X+c} + \sqrt{Y+c})^2} \int_1^{\Xi} \frac{dT \rho(T)}{\sqrt{T+c}} \frac{(\sqrt{X+c} + \sqrt{T+c} + 2\sqrt{Y+c})}{(\sqrt{X+c} + \sqrt{T+c})(\sqrt{T+c} + \sqrt{Y+c})^2} \\ &= (W(X) + \nu)G(X|Y) - \frac{4\tilde{\lambda}^2}{\sqrt{Z}\sqrt{Y+c} \cdot (\sqrt{X+c} + \sqrt{Y+c})^2} \\ &- 4\tilde{\lambda}^2 \frac{\partial}{\partial Y} \int_1^{\Xi} \frac{dT \rho(T)}{\sqrt{T+c}} \frac{1}{(\sqrt{X+c} + \sqrt{T+c})(\sqrt{X+c} + \sqrt{Y+c})(\sqrt{Y+c} + \sqrt{T+c})}. \tag{5.10} \end{aligned}$$

We have inserted the formula for W from (4.13). On the other hand, from (4.25),

$$\begin{aligned} G(X, Y, Y) &= 8\tilde{\lambda} \lim_{Y_1 \rightarrow Y} \frac{\frac{W(X) - W(Y)}{(X - Y)} - \frac{W(Y_1) - W(Y)}{(Y_1 - Y)}}{X - Y_1} = 8\tilde{\lambda} \frac{\partial}{\partial Y} \frac{W(X) - W(Y)}{X - Y} \\ &= 8\tilde{\lambda} \frac{\partial}{\partial Y} \left\{ \frac{1}{\sqrt{Z}(\sqrt{X+c} + \sqrt{Y+c})} \right. \\ &\left. - \frac{1}{2} \int_1^{\Xi} \frac{dT \rho(T)}{\sqrt{T+c}} \frac{1}{(\sqrt{X+c} + \sqrt{T+c})(\sqrt{X+c} + \sqrt{Y+c})(\sqrt{Y+c} + \sqrt{T+c})} \right\}. \tag{5.11} \end{aligned}$$

Adding $(-\tilde{\lambda})G(X, Y, Y)$ to (5.10) yields $(W(X) + \nu)G(X|Y)$, as required by (5.8). \square

Note that (5.9) is essentially the same as [8, eq. (93)].

5.4. Solution for the (1+1+1)-point function

We specify the problem (5.7) to the (1+1+1)-point function

$$(W(X) + \nu)G(X|Y^2|Y^3) + \frac{1}{2} \int_1^{\Xi} dT \rho(T) \frac{G(X|Y^2|Y^3) - G(T|Y^2|Y^3)}{X - T}$$

$$= -\tilde{\lambda}G(X, Y^2, Y^2|Y^3) - \tilde{\lambda}G(X, Y^3, Y^3|Y^2) - 2\tilde{\lambda}G(X|Y^2)G(X|Y^3). \tag{5.12}$$

We have with (5.3)

$$\begin{aligned} G(X, Y^2, Y^2|Y^3) &= 16\tilde{\lambda}^2 \frac{\partial}{\partial Y^2} \frac{G(X|Y^3) - G(Y^2|Y^3)}{X - Y^2} \\ &= -128\tilde{\lambda}^4 \frac{\partial^2}{\partial Y^2 \partial Y^3} \left\{ \frac{\frac{1}{\sqrt{X+c}(\sqrt{X+c}+\sqrt{Y^3+c})} - \frac{1}{\sqrt{Y^2+c}(\sqrt{Y^2+c}+\sqrt{Y^3+c})}}{X - Y^2} \right\} \\ &= \frac{\partial^2}{\partial Y^2 \partial Y^3} \left\{ \frac{128\tilde{\lambda}^4(\sqrt{X+c} + \sqrt{Y^2+c} + \sqrt{Y^3+c})}{\sqrt{X+c}\sqrt{Y^2+c}(\sqrt{X+c}+\sqrt{Y^3+c})(\sqrt{X+c}+\sqrt{Y^2+c})(\sqrt{Y^2+c}+\sqrt{Y^3+c})} \right\} \end{aligned}$$

and consequently

$$\begin{aligned} &G(X, Y^2, Y^2|Y^3) + G(X, Y^3, Y^3|Y^2) + 2G(X|Y^2)G(X|Y^3) \\ &= \frac{\partial^2}{\partial Y^2 \partial Y^3} \left\{ \frac{128\tilde{\lambda}^4(\sqrt{X+c} + \sqrt{Y^2+c} + \sqrt{Y^3+c})}{\sqrt{X+c}\sqrt{Y^2+c}\sqrt{Y^3+c}(\sqrt{X+c} + \sqrt{Y^3+c})(\sqrt{X+c} + \sqrt{Y^2+c})} \right. \\ &\quad \left. + \frac{128\tilde{\lambda}^4}{\sqrt{X+c}^2(\sqrt{X+c} + \sqrt{Y^3+c})(\sqrt{X+c} + \sqrt{Y^2+c})} \right\} \\ &= \frac{\partial^2}{\partial Y^2 \partial Y^3} \left\{ \frac{128\tilde{\lambda}^4}{\sqrt{X+c}^2\sqrt{Y^2+c}\sqrt{Y^3+c}} \right\} = \frac{32\tilde{\lambda}^4}{\sqrt{X+c}^2\sqrt{Y^2+c}^3\sqrt{Y^3+c}^3}. \tag{5.13} \end{aligned}$$

Because of the factorisation the only reasonable ansatz is

$$G(X|Y^2|Y^3) = \frac{(-32)\gamma\tilde{\lambda}^5}{\sqrt{X+c}^3\sqrt{Y^2+c}^3\sqrt{Y^3+c}^3}. \tag{5.14}$$

This gives as prefactor of $\frac{-32\tilde{\lambda}^5}{\sqrt{Y^2+c}^3\sqrt{Y^3+c}^3}$ in (5.12) (with exchanged lhs and rhs and use of (4.13)):

$$\begin{aligned} \frac{1}{X+c} &= \frac{\gamma}{\sqrt{Z}(X+c)} + \frac{\gamma}{2} \int_1^\Xi \frac{dT\rho(T)}{\sqrt{(T+c)}\sqrt{(X+c)}^3(\sqrt{X+c} + \sqrt{T+c})} \\ &\quad + \frac{\gamma}{2} \int_1^\Xi dT\rho(T) \frac{\frac{1}{\sqrt{(X+c)}^3} - \frac{1}{\sqrt{(T+c)}^3}}{X-T} \\ &= \frac{\gamma}{\sqrt{Z}(X+c)} - \frac{\gamma}{2(X+c)} \int_1^\Xi \frac{dT\rho(T)}{\sqrt{(T+c)}^3} \\ \Rightarrow \quad \gamma &= \frac{1}{\rho_0}, \quad \rho_0 := \frac{1}{\sqrt{Z}} - \int_1^\Xi \frac{dT\rho(T)}{2\sqrt{(T+c)}^3}. \tag{5.15} \end{aligned}$$

For linearly spaced eigenvalues $e(x) = x$ and $Z = 1$, i.e. $\rho(T) = \frac{2\tilde{\lambda}^2}{\sqrt{T}}$, this amounts to $\rho_0 = 1 - \frac{2\tilde{\lambda}^2}{\sqrt{1+c}(\sqrt{1+c}+1)}$.

5.5. Solution for the $(1 + \dots + 1)$ -point function for $B \geq 4$

This is the most elaborate section of the paper. Over the next 6 pages we prepare the proof of [Theorem 5.11](#). Eq. (5.14) suggests that all $(1 + \dots + 1)$ -point functions with $B \geq 3$ factorise. We make the ansatz

$$G(X^1 | \dots | X^B) = \frac{(-2\tilde{\lambda})^{3B-4}}{\rho_0} \sum_{M=0}^{B-3} \gamma_B^M \frac{d^M}{dt^M} \sqrt{X+c-2t}_{\{1, \dots, B\}}^{-3} \Big|_{t=0}, \tag{5.16}$$

where $\sqrt{X+c-2t}_I^{-3} := \prod_{\beta \in I} \frac{1}{\sqrt{X^\beta+c-2t}^3}$.

Our aim is to compute the coefficients γ_B^M starting with $\gamma_3^M = \delta_{M,0}$.

Lemma 5.5. Assume (5.16). Then

$$\begin{aligned} & (W(X^1) + \nu)G(X^1 | X_{\triangleleft \{2, \dots, B\}}) + \frac{1}{2} \int_1^{\Xi} dT \rho(T) \frac{G(X^1 | X_{\triangleleft \{2, \dots, B\}}) - G(T | X_{\triangleleft \{2, \dots, B\}})}{(X-T)} \\ &= \frac{(-2\tilde{\lambda})^{3B-4}}{\rho_0} \sum_{M=0}^{B-3} \gamma_B^M \sum_{j=0}^M \binom{M}{j} \sum_{l=0}^j \frac{(2j+1)!! \rho_{j-l}}{\sqrt{X^1+c}^{2l+2}} \frac{d^{M-j}}{dt^{M-j}} \sqrt{X+c-2t}_{\{2, \dots, B\}}^{-3} \Big|_{t=0}, \end{aligned} \tag{5.17}$$

where $\rho_l := \frac{\delta_{l,0}}{\sqrt{Z}} - \frac{1}{2} \int_1^{\Xi} \frac{dT \rho(T)}{\sqrt{T+c}^{3+2l}}$. (5.18)

Proof. We distribute the t -derivatives by Leibniz rule. The prefactor of $\frac{(-2\tilde{\lambda})^{3B-4}}{\rho_0} \gamma_B^M (2j+1)!! \binom{M}{j} \frac{d^{M-j}}{dt^{M-j}} \sqrt{X+c-2t}_{\{2, \dots, B\}}^{-3} \Big|_{t=0}$ under the sum over j, M is

$$\begin{aligned} & (W(X^1) + \nu) \frac{1}{\sqrt{X^1+c}^{3+2j}} + \frac{1}{2} \int_1^{\Xi} dT \rho(T) \frac{\frac{1}{\sqrt{X^1+c}^{3+2j}} - \frac{1}{\sqrt{T+c}^{3+2j}}}{X^1-T} \\ &= \frac{1}{\sqrt{Z} \sqrt{X^1+c}^{2j+2}} - \frac{1}{2} \int_1^{\Xi} dT \rho(T) \frac{\sum_{l=1}^{2j+2} \sqrt{X^1+c}^l \sqrt{T+c}^{2j+2-l}}{\sqrt{X^1+c}^{3+2j} \sqrt{T+c}^{3+2j} (\sqrt{X^1+c} + \sqrt{T+c})} \\ &= \frac{1}{\sqrt{Z} \sqrt{X^1+c}^{2j+2}} - \frac{1}{2} \sum_{l=0}^j \frac{1}{\sqrt{X^1+c}^{2(j-l)+2}} \int_1^{\Xi} \frac{dT \rho(T)}{\sqrt{T+c}^{3+2l}} \\ &= \sum_{l=0}^j \frac{\rho_l}{\sqrt{X^1+c}^{2(j-l)+2}}, \end{aligned} \tag{5.19}$$

with ρ_l defined in (5.18). The step from the first to second line relies on

$$\frac{\frac{1}{\sqrt{X^1+c}^{3+2j}} - \frac{1}{\sqrt{X^2+c}^{3+2j}}}{X^1 - X^2} = - \sum_{l=0}^{2j+2} \frac{\sqrt{X^1+c}^l \sqrt{X^2+c}^{2j+2-l}}{\sqrt{X^1+c}^{3+2j} \sqrt{X^2+c}^{3+2j} (\sqrt{X^1+c} + \sqrt{X^2+c})} \tag{5.20}$$

with compensation of $l = 0$ with the integral in $W(X^1)$ according to (4.13). After a reflection $l \mapsto j - l$ we arrive at (5.17). \square

Lemma 5.6. Assume (5.16). Then the first term on the rhs of (5.7) and the $|J| = 1$ and $|J| = B - 2$ contributions to the last term combine to

$$\begin{aligned} & -\tilde{\lambda} \sum_{\beta=2}^B \left(G(X^1, X^\beta, X^\beta | X_{\triangleleft}^{\{2, \dots, B\}}) + 2G(X^1 | X^\beta) G(X^1 | X_{\triangleleft}^{\{2, \dots, B\}}) \right) \\ &= \frac{(-2\tilde{\lambda})^{3B-4}}{\rho_0} \sum_{M=0}^{B-4} \sum_{j=0}^M \sum_{l=0}^{j+1} \frac{\gamma_{B-1}^M}{\sqrt{X^1+c}^{4+2j-2l}} \binom{M}{j} \frac{(2j+1)!!(2l+1)}{(2l+1)!!} \\ & \times \sum_{\beta=2}^B \left(\frac{d^l}{dt^l} \frac{1}{\sqrt{X^\beta+c-2t}^3} \right) \left(\frac{d^{M-j}}{dt^{M-j}} \sqrt{X+c-2t}^{-3} \Big|_{\{2, \dots, B\}} \right) \Big|_{t=0}. \end{aligned} \tag{5.21}$$

Proof. It suffices to take $\beta = 2$ and then to permute. From (5.3) we have

$$G(X^1, X^2, X^2 | X_{\triangleleft}^{\{3, \dots, B\}}) = 16\tilde{\lambda}^2 \frac{\partial}{\partial X^2} \frac{G(X^1 | X_{\triangleleft}^{\{3, \dots, B\}}) - G(X^2 | X_{\triangleleft}^{\{3, \dots, B\}})}{X^1 - X^2}. \tag{5.22}$$

We insert (5.16) for $B \mapsto B - 1$. With Leibniz rule and (5.20) one has

$$\begin{aligned} & -\tilde{\lambda} G(X^1, X^2, X^2 | X_{\triangleleft}^{\{3, \dots, B\}}) \\ &= \sum_{M=0}^{B-4} \sum_{j=0}^M \binom{M}{j} \sum_{l=0}^{2j+2} \frac{\partial}{\partial X^2} \left\{ \frac{\gamma_{B-1}^M (2j+1)!! \sqrt{X^1+c}^l \sqrt{X^2+c}^{2j+3-(l+1)}}{\sqrt{X^1+c}^{3+2j} \sqrt{X^2+c}^{3+2j} (\sqrt{X^1+c} + \sqrt{X^2+c})} \right\} \\ & \times 16\tilde{\lambda}^3 \cdot \frac{(-2\tilde{\lambda})^{3B-7}}{\rho_0} \frac{d^{M-j}}{dt^{M-j}} \sqrt{X+c-2t}^{-3} \Big|_{\{3, \dots, B\}} \Big|_{t=0}. \end{aligned} \tag{5.23}$$

The other term reads with (5.9) as well as (5.16) for $B \mapsto B - 1$

$$\begin{aligned} & -2\tilde{\lambda} G(X^1 | X^2) G(X^1 | X_{\triangleleft}^{\{3, \dots, B\}}) \\ &= \sum_{M=0}^{B-4} \sum_{j=0}^M \binom{M}{j} \frac{(2j+1)!! \gamma_{B-1}^M}{\rho_0 \sqrt{X^1+c}^{3+2j}} \frac{\partial}{\partial X^2} \left\{ \frac{(-2\tilde{\lambda}) \cdot (-8\tilde{\lambda}^2) \cdot (-2\tilde{\lambda})^{3B-7}}{\sqrt{X^1+c} (\sqrt{X^1+c} + \sqrt{X^2+c})} \right\} \\ & \times \frac{d^{M-j}}{dt^{M-j}} \sqrt{X+c-2t}^{-3} \Big|_{\{3, \dots, B\}} \Big|_{t=0}. \end{aligned} \tag{5.24}$$

Bringing (5.23)+(5.24) to common $X^1 - X^2$ denominator $\frac{1}{\sqrt{X^1+c}^{4+2j} \sqrt{X^2+c}^{3+2j}}$ (before X^2 -differentiation) produces a total numerator

$$\sum_{l=0}^{2j+3} \sqrt{X^1+c}^l \sqrt{X^2+c}^{3+2j-l} = \sum_{l=0}^{j+1} (\sqrt{X^1+c} + \sqrt{X^2+c}) \sqrt{X^1+c}^{2l} \sqrt{X^2+c}^{2j+2-2l}.$$

After cancellation and differentiation with respect to X^2 we have

$$\begin{aligned}
 & -\tilde{\lambda}(G(X^1, X^2, X^2|X_{\triangleleft\{3,\dots,B\}}) + 2G(X^1|X^2)G(X^1|X_{\triangleleft\{3,\dots,B\}})) \\
 &= \frac{(-2\tilde{\lambda})^{3B-4}}{\rho_0} \sum_{M=0}^{B-4} \sum_{j=0}^M \binom{M}{j} \gamma_{B-1}^M \sum_{l=0}^{j+1} \frac{(2j+1)!(2l+1)}{\sqrt{X^1+c}^{4+2j-2l} \sqrt{X^2+c}^{2l+3}} \\
 & \times \frac{d^{M-j}}{dt^{M-j}} \sqrt{X+c-2t}_{\{3,\dots,B\}}^{-3} \Big|_{t=0}.
 \end{aligned} \tag{5.25}$$

We write $\frac{1}{\sqrt{X^2+c}^{2l+3}} = \frac{1}{(2l+1)!} \frac{d^l}{dt^l} \frac{1}{\sqrt{X^2+c-2t}^3}$, repeat these steps for all $X^{\beta \geq 2}$ and sum over β , thus establishing the formula. \square

The remaining terms with $2 \leq |J| \leq B - 3$ in the last term of (5.7) are straightforward:

$$\begin{aligned}
 & -\tilde{\lambda} \sum_{J \subset \{2,\dots,B\}, |J|=p} G(X^1|X_{\triangleleft^J})G(X^1|X_{\triangleleft\{2,\dots,B\} \setminus J}) \\
 &= \frac{1}{2} \cdot \frac{(-2\tilde{\lambda})^{3B-4}}{\rho_0^2} \sum_{M'=0}^{p+1-3} \sum_{M''=0}^{B-p-3} \sum_{j'=0}^{M'} \sum_{j''=0}^{M''} \binom{M'}{j'} \binom{M''}{j''} \frac{(2j'+1)!(2j''+1)!! \gamma_{p+1}^{M'} \gamma_{B-p}^{M''}}{\sqrt{X^1+c}^{6+2j'+2j''}} \\
 & \times \sum_{J \subset \{2,\dots,B\}, |J|=p} \left(\frac{d^{M'-j'}}{dt^{M'-j'}} \sqrt{X+c-2t_J}^{-3} \right) \left(\frac{d^{M''-j''}}{dt^{M''-j''}} \sqrt{X+c-2t_{\{2,\dots,B\} \setminus J}}^{-3} \right) \Big|_{t=0}.
 \end{aligned} \tag{5.26}$$

Symbolically we are left with the problem $[(5.17) = (5.21) + \sum_{p=2}^{B-3} (5.26)]$ to be solved for γ_B^M , provided the ansatz is consistent. By shifting indices we select the common coefficient of $\frac{(-2\tilde{\lambda})^{3B-4}}{\rho_0 \sqrt{X_1+c}^{6+2l}} \prod_{\beta=2}^B \left(\frac{1}{m_\beta!} \frac{d^{m_\beta}}{dt^{m_\beta}} \frac{1}{\sqrt{X^\beta+c-2t}} \right) \Big|_{t=0}$ in this equation:

Lemma 5.7. Assume (5.16). Then (5.7) amounts to the following system of equations for integers $l \geq -2$ and $(B-1)$ -tuples $\mathcal{M} = (m_2, \dots, m_B)$ with $M := m_2 + \dots + m_B$:

$$\begin{aligned}
 & \sum_{j=0}^{B-5-M-l} (M+2+l+j)! \gamma_B^{M+2+l+j} \frac{(2j+2l+5)!! \rho_j}{(l+2+j)!} \\
 &= (M+l+1)! \gamma_{B-1}^{M+l+1} \sum_{\beta=2}^B \frac{(2l+2m_\beta+3)!! (2m_\beta+1) m_\beta!}{(l+m_\beta+1)! (2m_\beta+1)!!} \\
 &+ \frac{1}{2\rho_0} \sum_{l'+l''=l} \frac{(2l'+1)!! (2l''+1)!!}{l'! l''!} \sum_{\mathcal{M}' \cup \mathcal{M}'' = \mathcal{M}} (M'+l')! \gamma_{\#\mathcal{M}'+1}^{M'+l'} (M''+l'')! \gamma_{\#\mathcal{M}''+1}^{M''+l''}.
 \end{aligned} \tag{5.27}$$

The sum in the last line (which contributes only for $l \geq 0$) is over all partitions of \mathcal{M} into two subtuples \mathcal{M}' , \mathcal{M}'' of $\#\mathcal{M}'$ and $\#\mathcal{M}''$ elements which sum up to M' and M'' , respectively. The initial condition is $\gamma_3^M = \delta_{M,0}$.

For the solution we have to introduce:

Definition 1. The Bell polynomials⁷ $B_{n,k}$ are defined by $B_{0,k}(\{ \}) = \delta_{k,0}$ and $B_{n,k}(\{x_j\}_{j=1}^{n-k+1}) = \sum \frac{n!}{j_1!j_2!\dots j_{n-k+1}!} \left(\frac{x_1}{1!}\right)^{j_1} \left(\frac{x_2}{2!}\right)^{j_2} \dots \left(\frac{x_{n-k+1}}{(n-k+1)!}\right)^{j_{n-k+1}}$, for $n \geq 1$, where the sum is over non-negative integers j_1, \dots, j_{n-k+1} with $j_1 + j_2 + \dots + j_{n-k+1} = k$ and $1j_1 + 2j_2 + \dots + (n-k-1)j_{n-k+1} = n$.

Lemma 5.8. The Bell polynomials satisfy the identity

$$\sum_{j=1}^{n-k} (\alpha j + \beta) \binom{n}{j} x_j B_{n-j,k}(x_1, \dots, x_{n-j-k+1}) = (\alpha n + \beta(k+1)) B_{n,k+1}(x_1, \dots, x_{n-k}). \tag{5.28}$$

Proof. This follows from [19, Lemma 8],

$$\binom{n}{m} B_{m,l}(\{x\}) B_{n-m,k-l}(\{x\}) = \sum_{v \in \pi(n,k)} \frac{n!}{v_1!v_2!\dots} W_{m,l}(v) \left(\frac{x_1}{1!}\right)^{v_1} \left(\frac{x_2}{2!}\right)^{v_2} \dots,$$

where the $\pi(n, k)$ is the set of $v_1, v_2, \dots \geq 0$ with $1v_1 + 2v_2 + \dots = n$ and $v_1 + v_2 + \dots = k$. We only need $l = 1$ where the general definition of $W_{m,l}(v)$ given in [19, eq. (2)] reduces to $W_{m,1}(v) = v_m$. Moreover, $B_{m,1}(\{x\}) = x_m$. Therefore,

$$\begin{aligned} \sum_{m=1}^{n-k+1} (\alpha m + \beta) \binom{n}{m} x_m B_{n-m,k-1}(\{x\}) &= \sum_{m=1}^{n-k+1} (\alpha m + \beta) \binom{n}{m} B_{m,1}(\{x\}) B_{n-m,k-1}(\{x\}) \\ &= \sum_{v \in \pi(n,k)} \frac{n!}{v_1!v_2!\dots} \left(\sum_{m=1}^{n-k+1} (\alpha m v_m + \beta v_m)\right) \left(\frac{x_1}{1!}\right)^{v_1} \left(\frac{x_2}{2!}\right)^{v_2} \dots = (\alpha n + \beta k) B_{n,k}(\{x\}). \end{aligned}$$

A shift in k yields the result. \square

Proposition 5.9. The solution of (5.27) for $l = -2$ and $l = -1$,

$$\sum_{j=0}^{B-3-M} \binom{M+j}{j} (2j+1)!! \rho_j \gamma_B^{M+j} = \gamma_{B-1}^{M-1}, \tag{5.29}$$

$$\sum_{j=0}^{B-4-M} \binom{M+1+j}{j+1} (2j+3)!! \rho_j \gamma_B^{M+1+j} = (2M+B-1) \gamma_{B-1}^M, \tag{5.30}$$

where $M \in \{0, \dots, B-3\}$ and under initial condition $\gamma_3^M = \delta_{M,0}$, is

$$\gamma_B^M = \frac{1}{\rho_0^{B-3}} \sum_{K=0}^{B-3-M} \frac{(B-3+K)!}{(B-3-M)!K!} B_{B-3-M,K} \left(\left\{ -\frac{(2r+1)!!\rho_r}{(r+1)\rho_0} \right\}_{r=1}^{B-2-M-K} \right). \tag{5.31}$$

Proof. We start with (5.29). The formula correctly captures the case $M = B - 3$ where only $j = 0$ contributes in (5.29), giving the solution $\gamma_B^{B-3} = \frac{1}{\rho_0^{B-3}}$. We proceed by twofold induction

⁷ For an overview about Bell polynomials, see https://en.wikipedia.org/wiki/Bell_polynomials or https://www.encyclopediaofmath.org/index.php/Bell_polynomial. Many identities are proved in [19] and references therein.

in increasing B and increasing $s := B - 3 - M$. We rearrange (5.29) as an equation for γ_B^M . All other terms either have less B (namely γ_{B-1}^{M-1}) or less s (namely γ_B^{M+j} , $j \geq 1$) so that the induction hypothesis applies. We have with $x_r := -\frac{(2r+1)!!\rho_r}{(r+1)\rho_0}$ and $s := B - 3 - M$ in (5.29):

$$\gamma_B^M = \frac{1}{\rho_0^{B-3}} \sum_{K=0}^s \frac{(B-4+K)!}{s!(M-1)!} B_{s,K}(\{x_r\}_{r=1}^{s-K+1}) \tag{*}$$

$$+ \frac{1}{\rho_0^{B-3}} \sum_{j=1}^s \binom{M+j}{j} (j+1)x_j \sum_{K=0}^{s-j} \frac{(B-3+K)!}{(s-j)!(M+j)!} B_{s-j,K}(\{x_r\}_{r=1}^{s-j-K+1}). \tag{**}$$

We exchange the summation order $\sum_{j=1}^s \sum_{K=0}^{s-j} = \sum_{K=0}^{s-1} \sum_{j=1}^{s-K}$ and use (5.28) to express the last line as

$$(**) = \frac{1}{\rho_0^{B-3}} \sum_{K=0}^{s-1} \frac{(B-3+K)!}{s!M!} (s+K+1) B_{s,K+1}(\{x_r\}_{r=1}^{s-K}).$$

We shift the index $K+1 \mapsto K$ and redistribute the resulting $(s+K) = (B-3+K) - M$: Its part $(-M)$ cancels the rhs of the first line (*), and $(B-3+K)$ increases the factorial to the claimed formula (5.31).

We check consistency with (5.30). For $M = B - 4$ the lhs restricts to $j = 0$, and both sides evaluate to $\frac{3(M+1)}{\rho_0^{B-4}}$. For $M \geq B - 5$ we express the lhs in terms of $x_r := -\frac{(2r+1)!!\rho_r}{(r+1)\rho_0}$ and insert (5.31). Then the $j \geq 1$ part of the lhs becomes after exchanging the $K-j$ summation

$$\begin{aligned} (5.30)_{j \geq 1}^{\text{lhs}} &= -\frac{1}{\rho_0^{B-3}} \sum_{K=0}^{B-5-M} \sum_{j=1}^{B-4-M-K} \frac{(B-3+K)!}{(B-4-M)!M!} \binom{B-4-M}{j} \\ &\quad \times (2j+3)x_j B_{B-4-M-j,K}(\{x_r\}) \\ &= -\frac{1}{\rho_0^{B-3}} \sum_{K=0}^{B-5-M} \frac{(B-3+K)!}{(B-4-M)!M!} \underbrace{(2(B-M-4) + 3(K+1))}_{=3(B-2+K)-(B+2M-1)} B_{B-4-M,K+1}(\{x_r\}), \end{aligned}$$

where (5.28) has been used for $\alpha = 2$, $\beta = 3$. Its part $3(B-2+K)$, after a shift $K+1 \mapsto K$, evaluates to $-3(M+1)\gamma_B^{M+1}$. The remainder gives $\frac{(2M+B-1)}{\rho_0} \gamma_{B-1}^M$, so that (5.30) is true. \square

Remains (5.27) for $l \geq 0$. Because of permutation symmetry we can assume $\mathcal{M} = (\underbrace{0, \dots, 0}_{n_0}, \dots, \underbrace{p, \dots, p}_{n_p})$ with $n_0 + \dots + n_p = B - 1 =: N$ and $0n_0 + 1n_1 + \dots + pn_p = M$. Then the sum over subtuples $\mathcal{M}' = (\underbrace{0, \dots, 0}_{n'_0}, \dots, \underbrace{p, \dots, p}_{n'_p})$ amounts to the sum over $0 \leq n'_i \leq n_i$ with multiplicity $\binom{n_0}{n'_0} \dots \binom{n_p}{n'_p}$. Therefore (5.31) solves (5.27) for $l \geq 0$ iff the following is true:

Conjecture 5.10. For any $l, n_0, \dots, n_p \in \mathbb{N}$, the Bell polynomials satisfy the identity (with $n_0 + \dots + n_p = N$ and $0n_0 + 1n_1 + \dots + pn_p = M$)

$$\frac{(2l+5)!!}{(l+2)!} \sum_{K \geq 0} (N-2+K)! \frac{B_{N-M-l-4,K}(\{x_r\})}{(N-M-l-4)!} \tag{5.32}$$

$$\begin{aligned}
 & - \sum_{K \geq 0} (N-3+K)! \frac{B_{N-M-l-4,K}(\{x_r\})}{(N-M-l-4)!} \sum_{i=0}^p n_i \frac{(2l+2i+3)!!(2i+1)!}{(2i+1)!!(l+i+1)!} \\
 & = \sum_{j \geq 1} \sum_{K \geq 0} (N-2+K)! \frac{(2j+2l+5)!!(j+1)!}{(2j+1)!!(j+l+2)!} \cdot \frac{x_j}{j!} \cdot \frac{B_{N-M-l-j-4,K}(\{x_r\})}{(N-M-l-j-4)!} \\
 & + \frac{1}{2} \sum_{l'=0}^l \sum_{n'_0=0}^{n_0} \dots \sum_{n'_p=0}^{n_p} \frac{(2l'+1)!!(2l''+1)!!}{l'!l''!} \binom{n_0}{n'_0} \dots \binom{n_p}{n'_p} \\
 & \times \sum_{K',K'' \geq 0} (N'-2+K')! \frac{B_{N'-M'-l'-2,K'}(\{x_r\})}{(N'-M'-l'-2)!} (N''-2+K'')! \frac{B_{N''-M''-l''-2,K''}(\{x_r\})}{(N''-M''-l''-2)!},
 \end{aligned}$$

where $N' := n'_0 + \dots + n'_p$ and $M' := 0n'_0 + 1n'_1 + \dots + pn'_p$ as well as $l'' := l - l'$, $N'' := N - N'$ and $M'' := M - M'$. The sums over j, K, K', K'' are restricted to the range of non-trivial Bell polynomials and inverse Gamma functions.

We have checked (5.32) with a computer algebra program for many different l, p, n_i . Of course a direct proof will be necessary.⁸

The generating function of Bell polynomials is

$$\exp\left(u \sum_{j=1}^{\infty} \frac{x_j t^j}{j!}\right) = \sum_{n,k \geq 0} u^k \frac{t^n}{n!} B_{n,k}(\{x_r\}_{r=1}^{n-k+1}). \tag{5.33}$$

Multiplying by $e^{-u} u^{B-3}$, integrating over $u \in \mathbb{R}_+$ and differentiating with respect to t gives an alternative realisation of (5.31), where we also insert the definition (5.18) of ρ_r . With the series $\sum_{j=1}^{\infty} \frac{(2j+1)!!}{(j+1)!} y^j = \frac{1}{y} \sum_{k=2}^{\infty} \binom{-\frac{1}{2}}{k} (-2y)^k = \frac{1}{y} \left(\frac{1}{\sqrt{1-2y}} - 1 - y\right) = \frac{2}{(1+\sqrt{1-2y})\sqrt{1-2y}} - 1$, below with $y = \frac{t}{T+c}$, we arrive at

$$\begin{aligned}
 & \rho_0^{B-3} M!(B-3-M)! \gamma_B^M \\
 & = \int_0^{\infty} du e^{-u} u^{B-3} \frac{d^{B-3-M}}{dt^{B-3-M}} \exp\left(\frac{u}{\rho_0} \sum_{r=1}^{\infty} \frac{t^r (2r+1)!!}{(r+1)!} (-\rho_r)\right) \Big|_{t=0}
 \end{aligned}$$

⁸ Other identities found during this work include for any $m, p, n_2, \dots, n_p \in \mathbb{N}$:

$$\begin{aligned}
 & \sum_{n'_i+n''_i=n_i} \sum_{k'+k''=m} \frac{(2k'+1)!!(2k''+1)!!(k'+\sum_{j=2}^p jn'_j)!(k''+\sum_{j=2}^p jn''_j)!}{k'!k''!(2+k'+\sum_{j=2}^p (j-1)n'_j)!(2+k''+\sum_{j=2}^p (j-1)n''_j)!} \prod_{j=2}^p \binom{n_j}{n'_j} \\
 & = \frac{2 \cdot (m+1+\sum_{j=2}^p jn_j)!}{(m+4+\sum_{j=2}^p (j-1)n_j)!} \left\{ \frac{(2m+3)!!}{m!} + \sum_{j=2}^p n_j \left(\frac{(2m+3)!!}{(m+2)!} \binom{(m+3)j+m+2}{j} - \frac{j!(2j+2m+3)!!}{(j+m+1)!(2j-1)!!} \right) \right\}.
 \end{aligned}$$

$$\begin{aligned}
 &= \int_0^\infty du e^{-u} u^{B-3} \\
 &\times \frac{d^{B-3-M}}{dt^{B-3-M}} \exp\left(\frac{u}{2\rho_0} \int_1^\Xi \frac{dT\rho(T)}{\sqrt{T+c}^3} \left(\frac{2(T+c)}{(\sqrt{T+c} + \sqrt{T+c-2t})\sqrt{T+c-2t}} - 1\right)\right)\Big|_{t=0} \\
 &= \rho_0^{B-2} \frac{d^{B-3-M}}{dt^{B-3-M}} \frac{(B-3)!}{\left(\frac{1}{\sqrt{Z}} - \int_1^\Xi \frac{dT\rho(T)}{\sqrt{T+c}} \frac{1}{(\sqrt{T+c} + \sqrt{T+c-2t})\sqrt{T+c-2t}}\right)^{B-2}}\Big|_{t=0}. \tag{5.34}
 \end{aligned}$$

Combined with the ansatz (5.16) and with $Z = 1$ in 2 dimensions we have proved (provided that Conjecture 5.10 is true):

Theorem 5.11. *The $(1 + \dots + 1)$ -point function with $B \geq 3$ boundary components of the Φ_2^3 matrixial QFT-model has the solution*

$$\begin{aligned}
 &G(X^1 | \dots | X^B) \\
 &= (-2\tilde{\lambda})^{3B-4} \frac{d^{B-3}}{dt^{B-3}} \left(\frac{\frac{1}{\sqrt{X^1+c-2t}^3} \dots \frac{1}{\sqrt{X^B+c-2t}^3}}{\left(1 - \int_1^\infty \frac{dT\rho(T)}{\sqrt{T+c}} \frac{1}{(\sqrt{T+c} + \sqrt{T+c-2t})\sqrt{T+c-2t}}\right)^{B-2}} \right)\Big|_{t=0}. \tag{5.35}
 \end{aligned}$$

Together with (5.3) we have thus completely solved the combined large- (\mathcal{N}, V) limit of the Kontsevich model.

6. From Φ_2^3 model on Moyal space to Schwinger functions on \mathbb{R}^2

This section parallels the treatment of the ϕ_4^* case in [12]. We refer to that paper for more details. The ϕ_2^* -model on Moyal-deformed 2D Euclidean space with harmonic propagation is defined by the action

$$S[\phi] := \int_{\mathbb{R}^2} \frac{d\xi}{8\pi} \left(\kappa \phi + \frac{1}{2} \phi \star (-\Delta + \|4\Theta^{-1} \cdot \xi\|^2 + \mu^2) \phi + \frac{\lambda}{3} \phi \star \phi \star \phi \right) (\xi). \tag{6.1}$$

The tadpole contribution proportional to $\kappa \in \mathbb{R}$ is required for renormalisation. By \star we denote the 2D-Moyal product parametrised by $\theta \in \mathbb{R}$,

$$(f \star g)(\xi) := \int_{\mathbb{R}^2 \times \mathbb{R}^2} \frac{d\eta dk}{(2\pi)^2} f\left(\xi + \frac{1}{2}\Theta \cdot k\right) g(\xi + \eta) e^{i(k,\eta)}, \quad \Theta := \begin{pmatrix} 0 & \theta \\ -\theta & 0 \end{pmatrix}. \tag{6.2}$$

The Moyal space possesses a convenient matrix basis

$$f_{mn}(\xi) = 2(-1)^m \sqrt{\frac{m!}{n!}} \left(\sqrt{\frac{2}{\theta}} \xi\right)^{n-m} L_m^{n-m} \left(\frac{2\|\xi\|^2}{\theta}\right) e^{-\frac{\|\xi\|^2}{\theta}}, \quad m, n \in \mathbb{N}, \tag{6.3}$$

where the $L_m^\alpha(t)$ are associated Laguerre polynomials of degree m in t and $(\xi_1, \xi_2)^k := (\xi_1 + i\xi_2)^k$. The matrix basis satisfies $(f_{kl} \star f_{mn})(\xi) = \delta_{ml} f_{kn}(\xi)$ and $\int_{\mathbb{R}^2} d\xi f_{mn}(\xi) = (2\pi\theta)\delta_{mn}$. A convenient regularisation consists in restricting the fields ϕ to those with finite expansion $\phi(\xi) = \sum_{m,n=0}^{\mathcal{N}} \Phi_{mn} f_{mn}(\xi)$. Using formulae for Laguerre polynomials, the action (6.1) takes precisely the form (3.1) of a matrix model for $\phi = \phi^* \in M_{\mathcal{N}}(\mathbb{C})$, with the following identification:

$$V = \frac{\theta}{4}, \quad E_m = \frac{m}{V} + \frac{\mu^2}{2} = \mu^2 \left(\frac{1}{2} + \frac{m}{\mu^2 V} \right). \tag{6.4}$$

This explains our interest in linearly spaced eigenvalues $e(x) = x$.

Following [16] we *define* connected Schwinger functions in position space as

$$S_c(\mu\xi_1, \dots, \mu\xi_N) := \lim_{V\mu^2 \rightarrow \infty} \lim_{\Lambda \rightarrow \infty} \sum_{N_1 + \dots + N_B = N} \sum_{q_1^1, \dots, q_{N_B}^B = 0}^{\mathcal{N}} \frac{G_{|q_1^1 \dots q_{N_1}^1| \dots |q_1^B \dots q_{N_B}^B|}}{8\pi \mu^{2(2-B-N)} S_{(N_1, \dots, N_B)}} \\ \times \sum_{\sigma \in \mathcal{S}_N} \prod_{\beta=1}^B \frac{f_{q_1^\beta q_2^\beta}(\xi_{\sigma(s_\beta+1)}) \dots f_{q_{N_\beta}^\beta q_1^\beta}(\xi_{\sigma(s_\beta+N_\beta)})}{V \mu^2 N_\beta}, \tag{6.5}$$

where $s_\beta := N_1 + \dots + N_{\beta-1}$ and $\mathcal{N} = \Lambda^2 V \mu^2$. The G_{\dots} are the expansion coefficients of $\log Z[J]$ in (3.6), where we already absorbed their mass dimension given in footnote 3. These Schwinger functions are fully symmetric in $\mu\xi_1, \dots, \mu\xi_N$.

The various factors of V need explanation. We recall that the prefactor of G_{\dots} in (3.6) was V^{2-B} . The factor V^{-B} is distributed over the B cycles. In a first step we have thus defined the free energy density as $(\mu^2 V)^{-2} \log \frac{Z[J]}{Z[0]}$, in agreement with the usual procedure in matrix models (see e.g. the $\frac{1}{N^2}$ prefactor in [6, eq. (4.2)]). Then formally we set

$$S_c(\mu\xi_1, \dots, \mu\xi_N) = \frac{1}{8\pi} \frac{\delta^N ((\mu^2 V)^{-2} \log \frac{Z[J]}{Z[0]})}{\delta J(\xi_1) \dots \delta J(\xi_N)} \Big|_{J=0},$$

with a *special* definition of $\frac{\delta J_{mn}}{\delta J(\xi)}$. Since by properties of the matrix basis (6.3) one has $J_{mn} = \int_{\mathbb{R}^2} \frac{d\eta}{8\pi V} f_{nm}(\eta) J(\eta)$, the usual convention $\frac{\delta J(\eta)}{\delta J(\xi)} = \delta(\xi - \eta)$ gives $\frac{\delta J_{mn}}{\delta J(\xi)} = \frac{1}{8\pi V} f_{nm}(\xi)$. As part of the renormalisation process, we change these conventions into

$$\frac{\delta J_{mn}}{\delta J(\xi)} := \mu^2 f_{nm}(\xi), \tag{6.6}$$

or equivalently $S_c(\mu\xi_1, \dots, \mu\xi_N) = \frac{1}{8\pi} \frac{(8\pi V \mu^2)^N \delta^N ((\mu^2 V)^{-2} \log \frac{Z[J]}{Z[0]})}{\delta J(\xi_1) \dots \delta J(\xi_N)} \Big|_{J=0}$ with the standard convention. It is important to note that these field redefinitions are neutral with respect to the number B of boundary components.

The evaluation of (6.5) follows the same lines as in [16]. To keep this paper self-contained, we outline the steps until the technical lemma proved in [16, Lemma 4+Corollary 5] can be used. We collect the indices $\underline{q}^\beta := (q_1^\beta, \dots, q_{N_\beta}^\beta)$ and define $|\underline{q}^\beta| := q_1^\beta + \dots + q_{N_\beta}^\beta$ and $\langle \underline{\omega}^\beta, \underline{q}^\beta \rangle := \sum_{i=1}^{N_\beta-1} \omega_i^\beta (q_i^\beta - q_{i+1}^\beta)$ for $\underline{\omega}^\beta = (\omega_1^\beta, \dots, \omega_{N_\beta-1}^\beta)$. We assume that the matrix functions G have a representation as Laplace–Fourier transform,

$$\frac{G_{|q_1^1| \dots |q_{N_1}^1| \dots |q_1^B| \dots |q_{N_B}^B|}}{\mu^{2(2-B-N)}} = \int_{\mathbb{R}_+^B} d(t^1, \dots, t^B) \int_{\mathbb{R}^{N-B}} d(\underline{\omega}^1, \dots, \underline{\omega}^B) \mathcal{G}_{\mathcal{N}, V}(t^1, \underline{\omega}^1 | \dots | t^B, \underline{\omega}^B) \\ \times \exp \left(- \frac{1}{V \mu^2} \sum_{\beta=1}^B (t^\beta |\underline{q}^\beta| - i \langle \underline{\omega}^\beta, \underline{q}^\beta \rangle) \right). \tag{6.7}$$

The inverse Laplace–Fourier transforms $\mathcal{G}_{\mathcal{N}, V}(t^1, \underline{\omega}^1 | \dots | t^B, \underline{\omega}^B)$ depend on \mathcal{N}, V but have a limit $\mathcal{G}(t^1, \underline{\omega}^1 | \dots | t^B, \underline{\omega}^B) = \lim_{\mathcal{N}, V \rightarrow \infty} \mathcal{G}_{\mathcal{N}, V}(t^1, \underline{\omega}^1 | \dots | t^B, \underline{\omega}^B)$ satisfying

$$\tilde{G}(\underline{x}^1 | \dots | \underline{x}^B) = \int_{\mathbb{R}_+^B} d(t^1, \dots, t^B) \int_{\mathbb{R}^{N-B}} d(\underline{\omega}^1, \dots, \underline{\omega}^B) \mathcal{G}(t^1, \underline{\omega}^1 | \dots | t^B, \underline{\omega}^B) \times \exp\left(-\sum_{\beta=1}^B (t^\beta |\underline{x}^\beta| - i \langle \underline{\omega}^\beta, \underline{x}^\beta \rangle)\right). \tag{6.8}$$

Inserting (6.7) into (6.5) gives, besides $\mathcal{G}_{\mathcal{N},V}(t^1, \underline{\omega}^1 | \dots | t^B, \underline{\omega}^B)$, the following type of factors (for each $\beta = 1, \dots, N_\beta$ omitted below) under the Laplace–Fourier integral and the sum over permutations and partitions of N :

$$\sum_{q_1, \dots, q_N=0}^{\mathcal{N}} \frac{f_{q_1 q_2}(\xi_{\sigma(s+1)}) \cdots f_{q_N q_1}(\xi_{\sigma(s+N)})}{V \mu^2 N} z_1^{q_1}(t, \underline{\omega}) \cdots z_N^{q_N}(t, \underline{\omega}), \tag{6.9}$$

$$z_1 = e^{-\frac{t}{V \mu^2} + i \frac{\omega_1}{V \mu^2}}, \quad z_i = e^{-\frac{t}{V \mu^2} + i \frac{\omega_i - \omega_{i-1}}{V \mu^2}} \text{ for } i = 2, \dots, N-1, \quad z_N = e^{-\frac{t}{V \mu^2} - i \frac{\omega_{N-1}}{V \mu^2}}.$$

For $\mathcal{N} \rightarrow \infty$ but fixed V , the index sum was evaluated in [16]:

Lemma 6.1 ([16, Lemma 4+Corollary 5]). *Let $\langle \xi, \eta \rangle$, $\|\xi\|$ and $\xi \times \eta = \det(\xi, \eta)$ be scalar product, norm and (third component of) vector product of $\xi, \eta \in \mathbb{R}^2$. Then for $\xi_i \in \mathbb{R}^2$ and $z_i \in \mathbb{C}$ with $|z_i| < 1$ one has (with cyclic identification $N + i \equiv i$ where necessary)*

$$\sum_{q_1, \dots, q_N=0}^{\infty} \frac{1}{V \mu^2} \prod_{i=1}^N f_{q_i q_{i+1}}(\xi_i) z_i^{q_i} \tag{6.10}$$

$$= \frac{2^N}{V \mu^2 (1 - \prod_{i=1}^N (-z_i))} \exp\left(-\frac{\sum_{i=1}^N \|\xi_i\|^2}{4V} \frac{1 + \prod_{i=1}^N (-z_i)}{1 - \prod_{i=1}^N (-z_i)}\right)$$

$$\times \exp\left(-\sum_{1 \leq k < l \leq N} \left(\frac{(\langle \xi_k, \xi_l \rangle - i \xi_k \times \xi_l)}{2V} \frac{\prod_{j=k+1}^l (-z_j)}{1 - \prod_{i=1}^N (-z_i)} + \frac{(\langle \xi_k, \xi_l \rangle + i \xi_k \times \xi_l)}{2V} \frac{\prod_{j=l+1}^{N+k} (-z_j)}{1 - \prod_{i=1}^N (-z_i)} \right)\right).$$

That the result can be applied to the combined limit $\mathcal{N}, V \rightarrow \infty$ with $\mathcal{N} = \Lambda^2 V \mu^2$, where $|z_i| = 1$ becomes critical, needs some explanation. It is uncritical to move the convergent $\mathcal{G}(t^1, \underline{\omega}^1 | \dots | t^B, \underline{\omega}^B)$ in front of the limit. The result (6.10) relies on the generating function $\sum_{n=0}^{\infty} L_n^{\alpha-n}(t) z^n = e^{-zt} (1+z)^\alpha$ which precisely for $\alpha \in \mathbb{N}$ is absolutely convergent for any $z \in \mathbb{C}$. The only place where $|z| < 1$ matters is a final sum $\sum_{q=0}^{\infty} \frac{(q+k)!}{q!k!} ((-z_1) \cdots (-z_N))^q = \frac{1}{(1-(-z_1) \cdots (-z_N))^{1+k}}$. Restricting this sum to $1 \leq \mathcal{N}$ gives (for N being even) instead

$$\sum_{q=0}^{\mathcal{N}} \frac{(q+k)!}{q!k!} (z_1 \cdots z_N)^q = \frac{1 - (z_1 \cdots z_N)^{\mathcal{N}+1} P_k(z_1 \cdots z_N)}{(1 - z_1 \cdots z_N)^{1+k}},$$

where $P_k(z)$ is a polynomial of degree k with $P_k(1) = 1$. Since $(z_1 \cdots z_N)^{\mathcal{N}} = e^{-\Lambda^2 N t}$, there is a V -uniform multiplicative error of $1 + \mathcal{O}(1)e^{-\Lambda^2 N t}$ if we restrict in (6.10) the sum to $q_i \leq \mathcal{N}$. Therefore, the limit $\lim_{V \rightarrow \infty}$ of (6.10) agrees with the scaling limit $\mathcal{N}, V \rightarrow \infty$ with $\frac{\mathcal{N}}{V \mu^2} = \Lambda^2$ fixed of (6.9) followed by sending $\Lambda \rightarrow \infty$. We thus have

$$\lim_{\Lambda \rightarrow \infty} \left(\lim_{\substack{N, V \rightarrow \infty \\ \frac{N}{V\mu^2} = \Lambda^2}} \sum_{q_1, \dots, q_N=0}^N \frac{f_{q_1 q_2}(\xi_{\sigma(s+1)}) \cdots f_{q_N q_1}(\xi_{\sigma(s+N)})}{V\mu^2 N} z_1^{q_1}(t, \underline{\omega}) \cdots z_N^{q_N}(t, \underline{\omega}) \right) = \begin{cases} 0 & \text{for } N \text{ odd,} \\ \frac{2^N}{N^2 t} \exp\left(-\frac{\mu^2}{2Nt} \|\xi_{\sigma(s+1)} - \xi_{\sigma(s+2)} + \dots - \xi_{\sigma(s+N)}\|^2\right) & \text{for } N \text{ even.} \end{cases} \tag{6.11}$$

Now write

$$\frac{2^N}{N^2 t} e^{-\frac{\mu^2}{2Nt} \|\xi\|^2} = \frac{2^N}{2\pi N} \int_{\mathbb{R}^2} dp e^{-\frac{N}{2\mu^2} \|p\|^2 t + i(p, \xi)} \tag{6.12}$$

and recall that the z_i factors of (6.11) were introduced via the Laplace–Fourier transform (6.7) to be inserted into (6.5). Combining all these steps and limits, we can immediately perform the Laplace–Fourier transform (6.8) to a function with arguments $x_i^\beta = \frac{\|p_i\|^2}{2\mu^2}$ for all $i = 1, \dots, N_\beta$. The final result reads

$$S_c(\mu\xi_1, \dots, \mu\xi_N) = \sum_{\substack{N_1 + \dots + N_B = N \\ N_\beta \text{ even}}} \sum_{\sigma \in S_N} \prod_{\beta=1}^B \left(\frac{2^{N_\beta}}{N_B} \int_{\mathbb{R}^2} \frac{dp^\beta}{2\pi \mu^2} e^{i(p^\beta, \xi_{\sigma(s_\beta+1)} - \xi_{\sigma(s_\beta+2)} + \dots - \xi_{\sigma(s_\beta+N_\beta)})} \right) \times \frac{1}{8\pi S_{(N_1, \dots, N_B)}} \tilde{G}\left(\underbrace{\frac{\|p^1\|^2}{2\mu^2}, \dots, \frac{\|p^1\|^2}{2\mu^2}}_{N_1} \mid \dots \mid \underbrace{\frac{\|p^B\|^2}{2\mu^2}, \dots, \frac{\|p^B\|^2}{2\mu^2}}_{N_B}\right). \tag{6.13}$$

For $N = 2$ the formula specifies with (4.21) and (4.13) to

$$S_c(\mu\xi_1, \mu\xi_2) = \int_{\mathbb{R}^2} \frac{dp}{4\pi^2 \mu^2} e^{i(p, \xi_1 - \xi_2)} \hat{S}_2(p), \tag{6.14}$$

$$\hat{S}_2(p) = 2W'\left(\left(\frac{\|p\|^2}{\mu^2} + 1\right)^2\right) = \frac{1 - \tilde{\lambda}^2 \int_1^\infty \frac{dT}{\sqrt{T} \sqrt{T+c}} \frac{\mu^4}{(\sqrt{(\|p\|^2 + \mu^2)^2 + c\mu^4} + \mu^2 \sqrt{T+c})^2}}{\sqrt{(\|p\|^2 + \mu^2)^2 + c\mu^4}}.$$

It was also pointed out in [16] and [15] that the Schwinger 2-point function is reflection positive iff the function $\|p\|^2 \mapsto \hat{S}_2(p)$ is a Stieltjes function. This is not the case, neither for real nor purely imaginary non-vanishing $\tilde{\lambda}$! For $c > 0$ and thus $\tilde{\lambda} \in i\mathbb{R}$, the integrand has a pole (or end point of a branch cut) in the complex plane at $\|p\|^2 = \mu^2(-1 \pm i\sqrt{c})$, contradicting holomorphicity in $\mathbb{C} \setminus \mathbb{R}_-$. For $-1 < c < 0$ and thus $\tilde{\lambda} \in \mathbb{R}$ one finds that the imaginary part of $\hat{S}_2(p)$ at $\|p\|^2 = (-3 - i\frac{|c|}{10})\mu^2$ is negative.⁹ This contradicts the anti-Herglotz property of Stieltjes functions. A rigorous proof that the 2-point function of Φ_2^3 is not reflection positive will be given in [17].

⁹ Here one should write $\sqrt{(\|p\|^2 + \mu^2)^2 + c\mu^4} \mapsto \sqrt{\|p\|^2 + (1 - \sqrt{-c})\mu^2} \sqrt{\|p\|^2 + (1 + \sqrt{-c})\mu^2}$ for a well-defined holomorphic extension of (6.14).

7. Summary

We have given an alternative solution strategy for the large- \mathcal{N} limit of the Φ_2^3 matrix model (= renormalised Kontsevich model). This limit suppresses non-planar graphs. In principle, punctures (or boundary components) are also suppressed, but special limits of noncommutative field theory amplify them to the same level as the disk topology. We have established exact formulae, analytic in the (squared) coupling constant, for all these correlation functions. Correlation functions of disk topology (single puncture) can certainly be derived from previous results on the Kontsevich model. The complete treatment of the multi-punctured cases is new (to the best of our knowledge).

In our subsequent paper [17] we extend this work to the Φ_4^3 and Φ_6^3 models. There the renormalisation is much more involved, whereas the solution of Schwinger–Dyson equations is easily adapted from Φ_2^3 . We will discuss the issue of overlapping divergences and renormalons in Φ_6^3 . The main result will be the proof that Φ_4^3 and Φ_6^3 , but not Φ_2^3 , have reflection positive 2-point functions.

Reflection positivity of higher correlation functions is work in progress. Another interesting question concerns the identification of the KdV hierarchy in the solution we found.

We also hope that these investigations provide new ideas for attacking the more difficult equations of the Φ_4^4 model.

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