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Noncommutative geometry with graded differential Lie algebras

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Starting with a Hilbert space endowed with a representation of a unitary Lie algebra and an action of a generalized Dirac operator, we develop a mathematical concept towards gauge field theories. This concept shares common features with the Connes–Lott prescription of non–commutative geometry, differs from that, however, by the implementation of unitary Lie algebras instead of associative $*$–algebras. The general scheme is presented in detail and is applied to functions $\otimes$ matrices.

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I. Introduction

We present a framework towards a construction (of the classical action) of gauge field theories out of the following input data:

1) The (Lie) group of local gauge transformations $\mathscr{G}$.
2) Chiral fermions $\boldsymbol{\psi}$ transforming under a representation $\tilde{\pi}$ of $\mathscr{G}$.
3) The fermionic mass matrix $\widetilde{\mathcal{M}}$, i.e. fermion masses plus generalized Kobayashi–Maskawa matrices.
4) Possibly the spontaneous symmetry breaking pattern of $\mathscr{G}$.

At first sight, this setting seems to be adapted to the Connes–Lott prescription[1] of non–commutative geometry (NCG). However, it was proved in Ref. 2 that only the standard model can be constructed within that scheme – out of a K–cycle[1,3] (nowadays called spectral triple) with real structure.[4] For details of this construction see Refs. 5, 6. It is certainly to early to judge from experimental results whether the standard model is correct or not. At least there exist good reasons[7] why one could be interested in Grand Unified Theories (GUT's). It is clear from Ref. 2 that the discussion of such models within NCG requires additional structures or different methods. The perhaps most successful NCG–approach towards Grand Unification was proposed in Refs. 8–10, where the K–cycle plays an auxiliary rôle.

The author of this paper has sketched in Ref. 11 a concept towards gauge field theories based upon unitary Lie algebras instead of unital associative $*$–algebras. Our concept requires the same amount of structures as the Connes–Lott prescription and is physically motivated. Starting from the above physical data 1) to 4) one obtains a K–cycle by enlarging the gauge group $\mathscr{G}$ to a unital associative $*$–algebra $\mathcal{A}$, provided that it is possible to extend the representation $\tilde{\pi}$ to a representation of $\mathcal{A}$. We shall go the opposite way: We restrict the gauge group to its infinitesimal elements, giving the Lie algebra of $\mathscr{G}$. In our case there are no obstructions for the representation, and – in principle – any physical model based upon 1) to 4) can be constructed. In this paper we present the mathematical footing of that line. We shall develop techniques adapted to this case that differ from those of Connes and Lott.

The paper is organized as follows: Sec. II contains the general construction, without any reference to a physical model. We start in Sec. II.A with basic definitions concerning L–cycles, the basic geometric object in our approach. In Sec. II.B we construct the universal graded differential Lie algebra $\Omega^*\mathfrak{g}$ and derive properties of its elements. Using the data specified in the L–cycle we define in Sec. II.C a Lie algebra representation π of $\Omega^*\mathfrak{g}$ in $\mathscr{B}(h)$. Factorization of $\pi(\Omega^*\mathfrak{g})$ with respect to the differential ideal $\pi(\mathcal{J}^*\mathfrak{g})$ yields the graded differential Lie algebra $\Omega^*_D\mathfrak{g}$. In Sec. II.D we introduce the important map σ, which enables us to give a convenient form to the ideal $\pi(\mathcal{J}^*\mathfrak{g})$. Using the language of graded Lie homomorphisms introduced in Sec. II.E we define in Sec. II.F the fundamental objects of gauge field theories: connections, curvatures, gauge transformations, bosonic and fermionic actions.

In Sec. III we apply the general scheme to L–cycles over functions $\otimes$ matrices. That class of L–cycles, which has a direct relation to physical models, is defined in Sec. III.A. For the space–time part it is convenient to redefine the exterior differential algebra Λ^*, see Sec. III.B. This enables us decompose in Sec. III.C the graded Lie algebra $\pi(\Omega^*\mathfrak{g})$ and in Sec. III.D the ideal $\pi(\mathcal{J}^*\mathfrak{g})$ into space–time part and matrix part. The decomposition of the formulae for the differential and the commutator is given in Sec. III.E. Finally, we consider in Sec. III.F local connections.

II. L–Cycles and Graded Differential Lie Algebras

A. The L–Cycle

The basic geometric object in our NCG–prescription is an L–cycle, which differs from a K–cycle[1,3] used in the Connes–Lott prescription by the implementation of unitary Lie algebras instead of unital associative $*$–algebras:

Definition 1. *An L–cycle $(\mathfrak{g}, h, D, \pi, \Gamma)$ over a unitary Lie algebra $\mathfrak{g}$ is given by*

i) an involutive representation π of $\mathfrak{g}$ in the Lie algebra $\mathscr{B}(h)$ of bounded operators on a Hilbert space h, i.e. $(\pi(a))^* = \pi(a^*) \equiv -\pi(a)$, for any $a \in \mathfrak{g}$,

ii) a (possibly unbounded) selfadjoint operator D on h such that $(\mathrm{id}_h + D^2)^{-1}$ is compact and for all $a \in \mathfrak{g}$ there is $[D, \pi(a)] \in \mathscr{B}(h)$, where id_h denotes *the identity on* h ,

iii) *a selfadjoint operator* Γ *on* h , *fulfilling* $\Gamma^2 = \mathrm{id}_h$, $\Gamma D + D\Gamma = 0$ *and* $\Gamma\pi(a) - \pi(a)\Gamma = 0$, *for all* $a \in \mathfrak{g}$.

Any Lie algebra $\mathfrak{g}$ can be embedded into its universal enveloping algebra $\mathfrak{U}(\mathfrak{g})$, and the representation $\pi : \mathfrak{g} \to \mathscr{B}(h)$ extends to a representation $\pi : \mathfrak{U}(\mathfrak{g}) \to \mathscr{B}(h)$ (Poincaré–Birkhoff–Witt theorem[12]). In this sense, any L–cycle can be embedded into its "enveloping K–cycle". However, the gauge field theory obtained by the Connes–Lott prescription[1,3] from this enveloping K–cycle differs from the gauge field theory we are going to develop for the L–cycle. Our construction follows the ideas of Connes and Lott, but the methods and results are different.

Although we do not need it, let us translate properties of a K–cycle into definitions for the L–cycle. We use the definition of the distance on a K–cycle[1,3] to define the distance between linear functionals $\mathrm{x}_1, \mathrm{x}_2 : \mathfrak{g} \to \mathbb{C}$ of the Lie algebra:

Definition 2. *Let* X *be the space of linear functionals of* $\mathfrak{g}$. *The distance* $\mathrm{dist}(\mathrm{x}_1, \mathrm{x}_2)$ *between* $\mathrm{x}_1, \mathrm{x}_2 \in \mathrm{X}$ *is given by*

$$\mathrm{dist}(\mathrm{x}_1, \mathrm{x}_2) := \sup_{a \in \mathfrak{g}} \{ \ |\mathrm{x}_1(a) - \mathrm{x}_2(a)| \ : \ \ \| [D, \pi(a)] \| \leq 1 \ \} \ .$$

This definition makes $(\mathrm{X}, \mathrm{dist})$ to a metric space, and there is no need for π being an algebra homomorphism.

Next, we can take the definition of integration on a K–cycle[1,3] to define the notion of integration on an L–cycle:

Definition 3. *Let* $\mathrm{d} \in [1, \infty)$ *be a real number. An L–cycle* $(\mathfrak{g}, h, D, \pi, \Gamma)$ *is called* d^+*–summable if the eigenvalues* E_n *of* D *– arranged in increasing order – satisfy*

$$\textstyle\sum_{n=1}^{N} E_n^{-1} = O(\sum_{n=1}^{N} n^{-1/\mathrm{d}}) \ .$$

We define the integration

$$\int_{\mathrm{X}} |a|^2 \, d\mu := const.(\mathrm{d}) \ \mathrm{Tr}_\omega((\pi(a))^2 \, |D|^{-\mathrm{d}}) \ , \quad a \in \mathfrak{a} \ ,$$

where Tr_ω *is the Dixmier trace,* $d\mu$ *is the "volume measure" on* X *and* $const.(\mathrm{d})$ *refers to a constant depending on* d .

B. The Universal Graded Differential Lie Algebra $\Omega^*\mathfrak{g}$

To construct differential algebras over a K–cycle $(\mathcal{A}, h, D)$ one starts from the universal differential algebra $\Omega^*\mathcal{A}$ over $\mathcal{A}$ and factorizes this differential algebra with respect to a differential ideal determined by the representation π of $\Omega^*\mathcal{A}$ in $\mathscr{B}(h)$. In analogy to this procedure we first define a universal differential Lie algebra $\Omega^*\mathfrak{g}$ over the Lie algebra $\mathfrak{g}$ of the L–cycle. Then we define a representation π of $\Omega^*\mathfrak{g}$ in $\mathscr{B}(h)$. Finally, we perform the factorization with respect to the differential ideal.

Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{R}$ with involution given by $a^* = -a$, for $a \in \mathfrak{g}$. The construction of the universal graded differential Lie algebra $\Omega^*\mathfrak{g}$ over the Lie algebra $\mathfrak{g}$ goes as follows: First, let $d\mathfrak{g}$ be another copy of $\mathfrak{g}$. Let $V(\mathfrak{g})$ be the free vector space generated by $\mathfrak{g}$ and let $V(d\mathfrak{g})$ be the free vector space generated by $d\mathfrak{g}$,

$$\begin{aligned} V(\mathfrak{g}) &:= \bigoplus_{a\in\mathfrak{g}} V_a \;, \qquad V_a = \mathbb{R}\ \forall a \in \mathfrak{g}\;, \\ V(d\mathfrak{g}) &:= \bigoplus_{da\in d\mathfrak{g}} V_{da} \;, \qquad V_{da} = \mathbb{R}\ \forall da \in d\mathfrak{g}\;. \end{aligned} \tag{2.1}$$

For a vector space $\mathfrak{X}$ we denote by δ_x the function on $\mathfrak{X}$, which takes the value 1 at the point $x \in \mathfrak{X}$ and the value 0 at all points $y \neq x$. Then,

$$\begin{aligned} V(\mathfrak{g}) &= \{ \textstyle\sum_\alpha \lambda_\alpha \delta_{a_\alpha} \;, \quad a_\alpha \in \mathfrak{g}\;, \quad \lambda_\alpha \in \mathbb{R} \}\;, \\ V(d\mathfrak{g}) &= \{ \textstyle\sum_\alpha \lambda_\alpha \delta_{da_\alpha} \;, \; a_\alpha \in \mathfrak{g}\;, \quad \lambda_\alpha \in \mathbb{R} \}\;, \end{aligned} \tag{2.2}$$

where the sums are finite. Let $T(\mathfrak{g})$ be the tensor algebra of $V(\mathfrak{g}) \oplus V(d\mathfrak{g})$, which carries a natural $\mathbb{N}$–grading structure. We define $\deg(v) = 0$ for $v \in V(\mathfrak{g})$ and $\deg(v) = 1$ for $v \in V(d\mathfrak{g})$. For tensor products $v_1 \otimes v_2 \otimes \ldots \otimes v_n \in T(\mathfrak{g})$, where each v_i, $i = 1, \ldots, n$, belongs either to $V(\mathfrak{g})$ or to $V(d\mathfrak{g})$, we define

$$\deg(v_1 \otimes v_2 \otimes \ldots \otimes v_n) := \textstyle\sum_{i=1}^n \deg(v_i)\;. \tag{2.3}$$

Now we have

$$T(\mathfrak{g}) = \bigoplus_{n\in\mathbb{N}} T^n(\mathfrak{g})\;, \qquad T^n(\mathfrak{g}) := \{ t \in T(\mathfrak{g})\;, \deg(t) = n \}\;. \tag{2.4}$$

In particular, we have $T^k(\mathfrak{g}) \otimes T^l(\mathfrak{g}) \subset T^{k+l}(\mathfrak{g})$.

Next, we regard $T(\mathfrak{g})$ as a graded Lie algebra with graded commutator given by

$$[t^k, \tilde{t}^l] := t^k \otimes \tilde{t}^l - (-1)^{kl}\, \tilde{t}^l \otimes t^k \;, \quad t^k \in T^k(\mathfrak{g})\;, \;\; \tilde{t}^l \in T^l(\mathfrak{g})\;. \tag{2.5}$$

Obviously, one has

$$\begin{aligned} &1) \qquad [t^k, \tilde{t}^l] = -(-1)^{kl}[\tilde{t}^l, t^k]\;, \\ &2) \qquad [t^k, \lambda\tilde{t}^l + \tilde{\lambda}\tilde{\tilde{t}}^l] = \lambda[t^k, \tilde{t}^l] + \tilde{\lambda}[t^k, \tilde{\tilde{t}}^l]\;, \\ &3) \qquad (-1)^{km}[t^k,[\tilde{t}^l,\tilde{\tilde{t}}^m]] + (-1)^{lk}[\tilde{t}^l,[\tilde{\tilde{t}}^m,t^k]] + (-1)^{ml}[\tilde{\tilde{t}}^m,[t^k,\tilde{t}^l]] = 0\;, \end{aligned} \tag{2.6}$$

for $t^k \in T^k(\mathfrak{g})$, $\tilde{t}^l, \tilde{\tilde{t}}^l \in T^l(\mathfrak{g})$, $\tilde{\tilde{t}}^m \in T^m(\mathfrak{g})$ and $\lambda, \tilde{\lambda} \in \mathbb{R}$.

Let $\tilde{\Omega}^*\mathfrak{g} = \bigoplus_{n\in\mathbb{N}} \tilde{\Omega}^n\mathfrak{g}$ be the $\mathbb{N}$–graded Lie subalgebra of $T(\mathfrak{g})$ given by the set of all repeated commutators (in the sense of (2.5)) of elements of $V(\mathfrak{g})$ and $V(d\mathfrak{g})$. Let $I'(\mathfrak{g})$ be the vector subspace of $\tilde{\Omega}^*\mathfrak{g}$ of elements of the following type:

$$\begin{array}{ll} \lambda\delta_a - \delta_{\lambda a} \;, & \lambda\delta_{da} - \delta_{d(\lambda a)} \;, \\ \delta_a + \delta_{\tilde{a}} - \delta_{a+\tilde{a}} \;, & \delta_{da} + \delta_{d\tilde{a}} - \delta_{d(a+\tilde{a})} \;, \\ [\delta_a, \delta_{\tilde{a}}] - \delta_{[a,\tilde{a}]} \;, & [\delta_{da}, \delta_{\tilde{a}}] + [\delta_a, \delta_{d\tilde{a}}] - \delta_{d[a,\tilde{a}]} \;, \end{array} \tag{2.7}$$

for $a, \tilde{a} \in \mathfrak{g}$ and $\lambda \in \mathbb{R}$. Obviously,

$$\begin{aligned} I(\mathfrak{g}) := I'(\mathfrak{g}) &+ [V(\mathfrak{g}) \oplus V(d\mathfrak{g}), I'(\mathfrak{g})] \\ &+ [V(\mathfrak{g}) \oplus V(d\mathfrak{g}), [V(\mathfrak{g}) \oplus V(d\mathfrak{g}), I'(\mathfrak{g})]] + \dots \end{aligned} \tag{2.8}$$

is an $\mathbb{N}$–graded ideal of $\tilde{\Omega}^*\mathfrak{g}$, $I(\mathfrak{g}) = \bigoplus_{n\in\mathbb{N}} I^n(\mathfrak{g})$. Then,

$$\Omega^*\mathfrak{g} := \bigoplus_{n\in\mathbb{N}} \Omega^n\mathfrak{g} \;, \qquad \Omega^n\mathfrak{g} := \tilde{\Omega}^n\mathfrak{g} \,/\, I^n(\mathfrak{g}) \;, \tag{2.9}$$

is an $\mathbb{N}$–graded Lie algebra, with commutator given by

$$[\varpi + I(\mathfrak{g}), \tilde{\varpi} + I(\mathfrak{g})] := [\varpi, \tilde{\varpi}] + I(\mathfrak{g}) \;, \quad \varpi, \tilde{\varpi} \in \tilde{\Omega}^*\mathfrak{g} \;. \tag{2.10}$$

On $T(\mathfrak{g})$ we define recursively a graded differential as an $\mathbb{R}$–linear map $d : T^n(\mathfrak{g}) \to T^{n+1}(\mathfrak{g})$ by

$$\begin{array}{rr} d(\lambda\delta_a) := \lambda\delta_{da} \;, & d(\lambda\delta_{da}) := 0 \;, \\ d(\lambda\delta_a \otimes t) := \lambda\delta_{da} \otimes t + \lambda\delta_a \otimes dt \;, & d(\lambda\delta_{da} \otimes t) := -\lambda\delta_{da} \otimes dt \;, \end{array} \tag{2.11}$$

for $a \in \mathfrak{g}$, $t \in T(\mathfrak{g})$ and $\lambda \in \mathbb{R}$. From this definition we get

$$\begin{aligned} d^2(\lambda\delta_a) &= d(\lambda\delta_{da}) = 0 \;, \qquad\qquad d^2(\lambda\delta_{da}) = 0 \;, \\ d^2(\lambda\delta_a \otimes t) &= d(\lambda\delta_{da} \otimes t) + d(\lambda\delta_a \otimes dt) \\ &= -\lambda\delta_{da} \otimes dt + \lambda\delta_{da} \otimes dt + \lambda\delta_a \otimes d^2t = \lambda\delta_a \otimes d^2t \;, \\ d^2(\lambda\delta_{da} \otimes t) &= \lambda\delta_{da} \otimes d^2t \;, \end{aligned} \tag{2.12}$$

therefore, by induction, $d^2 \equiv 0$ on $T(\mathfrak{g})$. In order to show that d is a graded differential we use the following equivalent characterization of (2.11):

$$d(v_1 \otimes \ldots \otimes v_n) = \sum_{i=1}^n (-1)^{\sum_{j=1}^{i-1} \deg(v_j)} v_1 \otimes \ldots \otimes v_{i-1} \otimes dv_i \otimes v_{i+1} \otimes \ldots \otimes v_n \;. \tag{2.13}$$

For $t^k = v_1 \otimes \ldots \otimes v_n \in T^k(\mathfrak{g})$, $k = \sum_{i=1}^n \deg(v_i)$, and $\tilde{t}^l \in T^l(\mathfrak{g})$ we get from (2.13)

$$d(t^k \otimes \tilde{t}^l) = d(t^k) \otimes \tilde{t}^l + (-1)^k t^k \otimes d\tilde{t}^l \;. \tag{2.14}$$

Thus, d defined by (2.11) is a graded differential of the tensor algebra $T(\mathfrak{g})$. Moreover, d is also a graded differential of the graded Lie algebra $T(\mathfrak{g})$:

$$\begin{aligned} d[t^k,\tilde t^l] &= d(t^k\otimes\tilde t^l-(-1)^{kl}\tilde t^l\otimes t^k) \\ &= (d(t^k)\otimes\tilde t^l-(-1)^{(k+1)l}\tilde t^l\otimes dt^k)+(-1)^k(t^k\otimes d\tilde t^l-(-1)^{k(l+1)}d(\tilde t^l)\otimes t^k) \\ &= [dt^k,\tilde t^l]+(-1)^k[t^k,d\tilde t^l]\ . \end{aligned}$$

Now, from $d(V(\mathfrak{g})\oplus V(d\mathfrak{g}))\subset V(\mathfrak{g})\oplus V(d\mathfrak{g})$ we conclude that d is also a graded differential of the graded Lie subalgebra $\tilde\Omega^*\mathfrak{g}\subset T(\mathfrak{g})$.

Next, we show that $dI'(\mathfrak{g})\subset I'(\mathfrak{g})$:

$$\begin{aligned} &d(\lambda\delta_a-\delta_{\lambda a})=\lambda\delta_{da}-\delta_{d(\lambda a)}\ , \qquad d(\lambda\delta_{da}-\delta_{d(\lambda a)})=0\ , \\ &d(\delta_a+\delta_{\tilde a}-\delta_{a+\tilde a})=\delta_{da}+\delta_{d\tilde a}-\delta_{d(a+\tilde a)}\ , \qquad d(\delta_{da}+\delta_{d\tilde a}-\delta_{d(a+\tilde a)})=0\ , \\ &d([\delta_a,\delta_{\tilde a}]-\delta_{[a,\tilde a]})=[\delta_{da},\delta_{\tilde a}]+[\delta_a,\delta_{d\tilde a}]-\delta_{d[a,\tilde a]}\ , \\ &d([\delta_{da},\delta_{\tilde a}]+[\delta_a,\delta_{d\tilde a}]-\delta_{d[a,\tilde a]})=-[\delta_{da},\delta_{d\tilde a}]+[\delta_{da},\delta_{d\tilde a}]=0\ . \end{aligned} \tag{2.15}$$

Since $d(V(\mathfrak{g})\oplus V(d\mathfrak{g}))\subset V(\mathfrak{g})\oplus V(d\mathfrak{g})$, we get from (2.8)

$$dI(\mathfrak{g})\subset I(\mathfrak{g})\ . \tag{2.16}$$

Therefore, the graded differential d on $\tilde\Omega^*\mathfrak{g}$ induces a graded differential on $\Omega^*\mathfrak{g}$ denoted by the same symbol:

$$d(\varpi+I(\mathfrak{g})):=d\varpi+I(\mathfrak{g})\ ,\quad \varpi\in\tilde\Omega^*\mathfrak{g}\ . \tag{2.17}$$

Hence, $(\Omega^*\mathfrak{g}\,,[\ ,\ ]\,,d)$ is a graded differential Lie algebra.

We extend the involution $*:a\mapsto -a$ on $\mathfrak{g}$ to an involution of the free vector spaces $V(\mathfrak{g})$ and $V(d\mathfrak{g})$ by

$$(\lambda\delta_a)^*:=-\lambda\delta_a\ ,\qquad (\lambda\delta_{da})^*:=-\lambda\delta_{da}\ . \tag{2.18}$$

We obtain an involution of $T(\mathfrak{g})$ by

$$(v_1\otimes v_2\otimes\ldots\otimes v_n)^*:=v_n^*\otimes\ldots\otimes v_2^*\otimes v_1^*\ , \tag{2.19}$$

fulfilling

$$(t\otimes\tilde t)^*=\tilde t^*\otimes t^*\ . \tag{2.20}$$

Formula (2.20) induces the following property of the Lie bracket (2.5):

$$[t^k,\tilde t^l]^*=-(-1)^{kl}[t^{k*},\tilde t^{l*}]\ . \tag{2.21}$$

Because of $(V(\mathfrak{g})\oplus V(d\mathfrak{g}))^*=V(\mathfrak{g})\oplus V(d\mathfrak{g})$ we get an involution on $\tilde\Omega^*\mathfrak{g}$ by restricting the involution on $T(V)$ to its graded Lie subalgebra $\tilde\Omega^*\mathfrak{g}$. Obviously, we have $I'(\mathfrak{g})^*=I'(\mathfrak{g})$, giving $I(\mathfrak{g})^*=I(\mathfrak{g})$. Therefore, we obtain an involution on $\Omega^*\mathfrak{g}$ by

$$(\varpi+I(\mathfrak{g}))^*:=\varpi^*+I(\mathfrak{g})\ ,\quad \varpi\in\tilde\Omega^*\mathfrak{g}\ . \tag{2.22}$$

The graded differential Lie algebra $\Omega^*\mathfrak{g}$ is universal in the following sense:

Proposition 4. *Let* $\Lambda^*\mathfrak{g} = \bigoplus_{n\in\mathbb{N}} \Lambda^n\mathfrak{g}$ *be an* $\mathbb{N}$*–graded Lie algebra with graded differential* $\mathrm{d} : \Lambda^n\mathfrak{g} \to \Lambda^{n+1}\mathfrak{g}$ *such that*
i) $\Lambda^0\mathfrak{g} = \pi(\mathfrak{g})$ *for a surjective homomorphism* $\pi : \mathfrak{g} \to \pi(\mathfrak{g})$ *of Lie algebras,*
ii) $\Lambda^*\mathfrak{g}$ *is generated by* $\pi(\mathfrak{g})$ *and* $\mathrm{d}\pi(\mathfrak{g})$ *as the set of repeated commutators.*
Then there exists a differential ideal $I_\Lambda \subset \Omega^*\mathfrak{g}$ *such that* $\Lambda^*\mathfrak{g} \cong \Omega^*\mathfrak{g}/I_\Lambda$ *.*

Proof: We define a surjective mapping $\tilde{p} : \tilde{\Omega}^*\mathfrak{g} \to \Lambda^*\mathfrak{g}$ by

$$\begin{aligned} \tilde{p}(\lambda\delta_a) &:= \pi(\lambda a) \;, \\ \tilde{p}(d\varpi) &:= \mathrm{d}(\tilde{p}(\varpi)) \;, \\ \tilde{p}([\varpi,\tilde{\varpi}]) &:= [\tilde{p}(\varpi),\tilde{p}(\tilde{\varpi})] \;, \end{aligned}$$

for $a \in \mathfrak{g}$, $\varpi, \tilde{\varpi} \in \tilde{\Omega}^*\mathfrak{g}$ and $\lambda \in \mathbb{R}$. Obviously, $\tilde{p}(I(\mathfrak{g})) = 0$. Therefore, by factorization with respect to $I(\mathfrak{g})$ we get a surjection $p : \Omega^*\mathfrak{g} \to \Lambda^*\mathfrak{g}$ by $p(\varpi + I(\mathfrak{g})) := \tilde{p}(\varpi)$, for $\varpi \in \tilde{\Omega}^*\mathfrak{g}$. We have $p(d\ker p) = 0$, therefore, $I_\Lambda = \ker p$ is the desired differential ideal of $\Omega^*\mathfrak{g}$:

$$\Lambda^*\mathfrak{g} \cong \Omega^*\mathfrak{g} \,/\, \ker p \;. \qquad \square$$

Proposition 4 tells us that each graded differential Lie algebra generated by $\pi(\mathfrak{g})$ and its differential is obtained by factorizing $\Omega^*\mathfrak{g}$ with respect to a differential ideal. For the setting described by an L–cycle, such a differential ideal is canonically given. This leads to a canonical graded differential Lie algebra, see Sec. II.C.

To summarize: We have defined a universal graded differential Lie algebra $\Omega^*\mathfrak{g} = \bigoplus_{n=0}^\infty \Omega^n\mathfrak{g}$ over a Lie algebra $\mathfrak{g}$, with:
- graded commutator $[\;,\;] : \Omega^k\mathfrak{g} \times \Omega^l\mathfrak{g} \to \Omega^{k+l}\mathfrak{g}$,
- universal differential $d : \Omega^k\mathfrak{g} \to \Omega^{k+1}\mathfrak{g}$, which is linear, nilpotent and obeys the graded Leibniz rule.
- involution $* : \Omega^k\mathfrak{g} \to \Omega^k\mathfrak{g}$.

Explicitly, we have the following properties:

1) $[\omega^k, \tilde{\omega}^l] = -(-1)^{kl}[\tilde{\omega}^l, \omega^k]$, (2.23a)
2) $[\omega^k, \lambda\tilde{\omega}^l + \tilde{\lambda}\tilde{\tilde{\omega}}^l] = \lambda[\omega^k, \tilde{\omega}^l] + \tilde{\lambda}[\omega^k, \tilde{\tilde{\omega}}^l]$, (2.23b)
3) $(-1)^{km}[\omega^k, [\tilde{\omega}^l, \tilde{\tilde{\omega}}^m]] + (-1)^{lk}[\tilde{\omega}^l, [\tilde{\tilde{\omega}}^m, \omega^k]] + (-1)^{ml}[\tilde{\tilde{\omega}}^m, [\omega^k, \tilde{\omega}^l]] = 0$, (2.23c)
4) $d[\omega^k, \tilde{\omega}^l] = [d\omega^k, \tilde{\omega}^l] + (-1)^k[\omega^k, d\tilde{\omega}^l]$, (2.23d)
5) $d^2\omega^k = 0$, (2.23e)
6) $[\omega^k, \tilde{\omega}^l]^* = -(-1)^{kl}[\omega^{k*}, \tilde{\omega}^{l*}]$, (2.23f)

for $\omega^k \in \Omega^k\mathfrak{g}$, $\tilde{\omega}^l, \tilde{\tilde{\omega}}^l \in \Omega^l\mathfrak{g}$, $\tilde{\tilde{\omega}}^m \in \Omega^m\mathfrak{g}$ and $\lambda, \tilde{\lambda} \in \mathbb{R}$.

It is convenient to fix a canonical ordering in elements of $\Omega^k\mathfrak{g}$, $k \geq 1$. First, let

$$\iota(a) := \delta_a + I(\mathfrak{g}) \;, \qquad\qquad \iota(da) := \delta_{da} + I(\mathfrak{g}) \;, \tag{2.24}$$

for $a \in \mathfrak{g}$. The first equation establishes an isomorphism $\Omega^0\mathfrak{g} \cong \mathfrak{g}$. We shall represent elements $\omega^1 \in \Omega^1\mathfrak{g}$ as

$$\begin{aligned}\omega^1 &= \iota(d\tilde{a}) + \textstyle\sum_{\alpha,z\geq 1}[\iota(a^z_\alpha),[\dots[\iota(a^2_\alpha),[\iota(a^1_\alpha),\iota(da^0_\alpha)]]\dots]] \\ &\equiv \textstyle\sum_{\alpha,z\geq 0}[\iota(a^z_\alpha),[\dots[\iota(a^2_\alpha),[\iota(a^1_\alpha),\iota(da^0_\alpha)]]\dots]]\;, \end{aligned} \tag{2.25}$$

where $\tilde{a}, a^i_\alpha \in \mathfrak{g}$ and the sums are finite. To avoid possible misunderstandings concerning this notation we fix throughout this paper the following convention: Beginning with $z = 1$, the index α first runs from 1 to $\alpha_1 > 0$ and labels the terms

$[\iota(a^1_1),\iota(da^0_1)],\dots,[\iota(a^1_{\alpha_1}),\iota(da^0_{\alpha_1})]$

in (2.25). Then, for $z = 2$, the index α runs from $\alpha_1 + 1$ to $\alpha_2 > \alpha_1$ and labels the commutators

$[\iota(a^2_{\alpha_1+1}),[\iota(a^1_{\alpha_1+1}),\iota(da^0_{\alpha_1+1})]],\dots,[\iota(a^2_{\alpha_2}),[\iota(a^1_{\alpha_2}),\iota(da^0_{\alpha_2})]]$

in (2.25), and so on. Therefore, the pair (i,β) of indices labelling an element $a^i_\beta \in \mathfrak{g}$ does never occur more than once in the sum (2.25). Moreover, we identify the term belonging to the pair $(\alpha = 0, z = 0)$ of indices with $\iota(d\tilde{a})$, as already indicated in (2.25).

Now, we write down elements $\omega^k \in \Omega^k\mathfrak{g}$, $k \geq 2$, recursively as

$$\omega^k = \textstyle\sum_\alpha[\omega^1_\alpha,\tilde{\omega}^{k-1}_\alpha]\;,\quad \omega^1_\alpha \in \Omega^1\mathfrak{g}\;,\quad \tilde{\omega}^{k-1}_\alpha \in \Omega^{k-1}\mathfrak{g}\;,\quad \text{finite sum}\;. \tag{2.26}$$

There are two things to check concerning (2.26). First, for $\tilde{\omega}^n \equiv \sum_\alpha[\tilde{\omega}^1_\alpha,\tilde{\tilde{\omega}}^{n-1}_\alpha] \in \Omega^n\mathfrak{g}$, with $\tilde{\omega}^1_\alpha \in \Omega^1\mathfrak{g}$ and $\tilde{\tilde{\omega}}^{n-1}_\alpha \in \Omega^{n-1}\mathfrak{g}$, we must show that also $[\omega^0,\tilde{\omega}^n] \in \Omega^n\mathfrak{g}$ can be represented in the standard form (2.26), for any $\omega^0 \in \Omega^0\mathfrak{g}$. But this follows from the graded Jacobi identity (2.23c):

$$\begin{aligned}[\omega^0,\tilde{\omega}^n] &= [\omega^0,\textstyle\sum_\alpha[\tilde{\omega}^1_\alpha,\tilde{\tilde{\omega}}^{n-1}_\alpha]] \\ &= -\textstyle\sum_\alpha[\tilde{\omega}^1_\alpha,[\tilde{\tilde{\omega}}^{n-1}_\alpha,\omega^0]] - (-1)^{n-1}\sum_\alpha[\tilde{\tilde{\omega}}^{n-1}_\alpha,[\omega^0,\tilde{\omega}^1_\alpha]] \\ &= \textstyle\sum_\alpha\big([\tilde{\omega}^1_\alpha,[\omega^0,\tilde{\tilde{\omega}}^{n-1}_\alpha]] + [[\omega^0,\tilde{\omega}^1_\alpha],\tilde{\tilde{\omega}}^{n-1}_\alpha]\big)\;.\end{aligned}$$

Second, we must show that the commutator $[\omega^k,\tilde{\omega}^l] \in \Omega^{k+l}\mathfrak{g}$, for $2 \leq k \leq l$, can be represented in the standard form (2.26) of an element of $\Omega^{k+l}\mathfrak{g}$, provided that both $\omega^k \in \Omega^k\mathfrak{g}$ and $\tilde{\omega}^l \in \Omega^l\mathfrak{g}$ are written down recursively in the form (2.26). Using again (2.23b) and (2.23c) we get for $\omega^k = \sum_\alpha[\omega^1_\alpha,\tilde{\tilde{\omega}}^{k-1}_\alpha]$

$$\begin{aligned}[\omega^k,\tilde{\omega}^l] &= -(-1)^{lk}\textstyle\sum_\alpha[\tilde{\omega}^l,[\omega^1_\alpha,\tilde{\tilde{\omega}}^{k-1}_\alpha]] \\ &= \textstyle\sum_\alpha\big([\omega^1_\alpha,[\tilde{\tilde{\omega}}^{k-1}_\alpha,\tilde{\omega}^l]] + (-1)^k[\tilde{\tilde{\omega}}^{k-1}_\alpha,[\omega^1_\alpha,\tilde{\omega}^l]]\big)\;.\end{aligned}$$

Repeating this calculation for the commutators $[\tilde{\tilde{\omega}}^{k-1}_\alpha,\tilde{\omega}^l]$ and $[\tilde{\tilde{\omega}}^{k-1}_\alpha,[\omega^1_\alpha,\tilde{\omega}^l]]$, we can recursively decrease the degree k until we arrive at degree 1.

Now we can easily prove

$$(\omega^k)^* = -(-1)^{k(k-1)/2}\omega^k \;, \quad \omega^k \in \Omega^k\mathfrak{g} \;. \tag{2.27}$$

By definition, (2.27) holds for $k=0$. From (2.25) and (2.23f) we get for $\omega^1 \in \Omega^1\mathfrak{g}$

$$\begin{aligned}\omega^{1*} &= \textstyle\sum_{\alpha,z\geq 0}[\iota(a^z_\alpha),[\dots,[\iota(a^1_\alpha),\iota(da^0_\alpha)]\dots]]^* \\ &= \textstyle\sum_{\alpha,z\geq 0}[\iota(a^z_\alpha),([\dots,[\iota(a^1_\alpha),\iota(da^0_\alpha)]\dots])^*] \\ &= \dots = \textstyle\sum_{\alpha,z\geq 0}[\iota(a^z_\alpha),[\dots,[\iota(a^1_\alpha),(\iota(da^0_\alpha))^*]\dots]] = -\omega^1 \;.\end{aligned}$$

In the same way we get from (2.26) and (2.23f) for $\omega^k \in \Omega^k\mathfrak{g}$

$$\begin{aligned}\omega^{k*} &= \textstyle\sum_\alpha[\omega^1_\alpha,\tilde{\omega}^{k-1}_\alpha]^* = (-1)^{k-1}\sum_\alpha[\omega^1_\alpha,(\tilde{\omega}^{k-1}_\alpha)^*] = (-1)^{(\sum_{i=2}^{k-1} i)}\omega^k \\ &= -(-1)^{k(k-1)/2}\omega^k \;.\end{aligned}$$

C. The Graded Differential Lie Algebra $\Omega^*_D\mathfrak{g}$

Following the procedure for K–cycles we define an involutive representation π of the universal differential Lie algebra $\Omega^*\mathfrak{g}$ introduced in Sec. II.B in the graded Lie algebra $\mathscr{B}(h)$ of bounded operators on h, where h is the Hilbert space of the L–cycle given in Definition 1. We underline that π will not be a representation of graded Lie algebras with differential. The definition of π uses almost the whole input contained in the L–cycle. First, using the grading operator Γ, we define a $\mathbb{Z}_2$–grading structure on the vector space $\mathscr{O}(h)$ of linear operators on the Hilbert space h, $\mathscr{O}(h) = \mathscr{O}_0(h) \oplus \mathscr{O}_1(h)$, by

$$\mathscr{O}_0(h)\Gamma = \Gamma\mathscr{O}_0(h) \;, \qquad\qquad \mathscr{O}_1(h)\Gamma = -\Gamma\mathscr{O}_1(h) \;. \tag{2.28}$$

This enables us to introduce the graded commutator for $\mathbb{Z}_2$–graded linear operators on h : For $A_i \in \mathscr{O}_i(h)$ and $B_j \in \mathscr{O}_j(h) \cap \mathscr{B}(h)$, where both A_i, B_j are selfadjoint or skew–adjoint on h, we define

$$[A_i,B_j]_g := A_i \circ B_j - (-1)^{ij} B_j \circ A_i \equiv -(-1)^{ij}[B_j,A_i]_g \tag{2.29}$$

on the subset $h' = \mathrm{domain}(A_i) \cap \{\boldsymbol{\psi} \in h \;,\;\; B_j\boldsymbol{\psi} \in \mathrm{domain}(A_i)\}$ of h. In certain cases it may be possible to extend h'. One has $A_j \in \mathscr{B}(h)$ iff $h' = h$.

Let us define a linear mapping $\tilde{\pi} : \tilde{\Omega}^*\mathfrak{g} \to \mathscr{B}(h)$ by

$$\begin{aligned}\tilde{\pi}(\lambda\delta_a) &:= \pi(\lambda a) \;, & \text{(2.30a)}\\ \tilde{\pi}(\lambda\delta_{da}) &:= [-\mathrm{i}D,\pi(\lambda a)]_g \equiv [-\mathrm{i}D,\pi(\lambda a)] \;, & \text{(2.30b)}\\ \tilde{\pi}([\varpi^k,\tilde{\varpi}^l]) &:= [\tilde{\pi}(\varpi^k),\tilde{\pi}(\tilde{\varpi}^l)]_g \;, & \text{(2.30c)}\end{aligned}$$

for $a \in \mathfrak{g}$, $\varpi^k \in \tilde{\Omega}^k\mathfrak{g}$, $\tilde{\varpi}^l \in \tilde{\Omega}^l\mathfrak{g}$ and $\lambda \in \mathbb{R}$. Note that $\pi(a)$ and $[D,\pi(a)]$ are bounded due to Definition 1 so that the r.h.s. of equations (2.30a) and (2.30b) belong to $\mathscr{B}(h)$. Now, due to $\pi(\mathfrak{g}) \subset \mathscr{O}_0(h)$ and $D \in \mathscr{O}_1(h)$, we get from (2.30)

$$\tilde{\pi}(\tilde{\Omega}^{2k}\mathfrak{g}) \subset \mathscr{O}_0(h) \;, \qquad\qquad \tilde{\pi}(\tilde{\Omega}^{2k+1}\mathfrak{g}) \subset \mathscr{O}_1(h) \;. \tag{2.31}$$

Next, we show that $\tilde{\pi} : \tilde{\Omega}^*\mathfrak{g} \to \mathscr{B}(h)$ is an involutive representation, where we recall that the involution in $\mathscr{B}(h)$ is defined as usual by means of the scalar product $\langle\ ,\ \rangle_h$ on h :

$$\langle \boldsymbol{\psi}, \tau^* \tilde{\boldsymbol{\psi}} \rangle_h := \langle \tau \boldsymbol{\psi}, \tilde{\boldsymbol{\psi}} \rangle_h \;, \quad \forall \boldsymbol{\psi}, \tilde{\boldsymbol{\psi}} \in h \;, \quad \tau \in \mathscr{B}(h) \;. \tag{2.32}$$

First, from (2.18), (2.30a) and the fact that $\pi : \mathfrak{g} \to \mathscr{B}(h)$ is an involutive representation we get

$$\tilde{\pi}((\lambda\delta_a)^*) = -\tilde{\pi}(\lambda\delta_a) = -\pi(\lambda a) = (\pi(\lambda a))^* = (\tilde{\pi}(\lambda\delta_a))^* \;.$$

Second, from (2.18), (2.30b) and the selfadjointness of D we obtain

$$\begin{aligned} \pi((\lambda\delta_{da})^*) &= -\pi(\lambda\delta_{da}) = \mathrm{i}(D \circ \pi(\lambda a) - \pi(\lambda a) \circ D) \\ &= -(-\mathrm{i})^*(D^* \circ (\pi(\lambda a))^* - (\pi(\lambda a))^* \circ D^*) \\ &= -\{-\mathrm{i}(\pi(\lambda a) \circ D - D \circ \pi(\lambda a))\}^* = (\pi(\lambda\delta_{da}))^* \;. \end{aligned}$$

Now we get by induction that $\tilde{\pi}$ is an involutive representation on $\tilde{\Omega}^*\mathfrak{g}$.

Observe that

$$\tilde{\pi}(I(\mathfrak{g})) \equiv 0 \;. \tag{2.33}$$

Therefore, the involutive representation $\tilde{\pi} : \tilde{\Omega}^*\mathfrak{g} \to \mathscr{B}(h)$ induces an involutive representation $\pi : \Omega^*\mathfrak{g} \to \mathscr{B}(h)$ by (the symbol π is already used but there is no danger of confusion)

$$\pi(\varpi + I(\mathfrak{g})) := \tilde{\pi}(\varpi) \;, \quad \varpi \in \tilde{\Omega}^*\mathfrak{g} \;. \tag{2.34}$$

In the same way as for K–cycles there may exist $\omega \in \Omega^*\mathfrak{g}$, fulfilling $\pi(\omega) = 0$ but not $\pi(d\omega) = 0$. Therefore, $\pi(\Omega^*\mathfrak{g})$ is not a differential Lie algebra. But there is a canonical construction towards such an object. Let us define

$$\mathcal{J}^*\mathfrak{g} = \ker\pi + d\ker\pi = \bigoplus_{k=0}^{\infty} \mathcal{J}^k\mathfrak{g} \;, \qquad \mathcal{J}^k\mathfrak{g} = \mathcal{J}^*\mathfrak{g} \cap \Omega^k\mathfrak{g} \;. \tag{2.35}$$

To obtain a differential Lie algebra we first prove:

Lemma 5. *$\mathcal{J}^*\mathfrak{g}$ is a graded differential ideal of the graded Lie algebra $\Omega^*\mathfrak{g}$.*

Proof: It is clear that $\ker\pi$ is an ideal of $\Omega^*\mathfrak{g}$. Then, for $j^k \in \ker\pi \cap \Omega^k\mathfrak{g}$ and $\omega \in \Omega^*\mathfrak{g}$ we have, see (2.23d),

$$[dj^k, \omega] = d([j^k, \omega]) - (-1)^k [j^k, d\omega] \;.$$

Because of $[j^k, d\omega] \in \ker\pi$ and $d([j^k, \omega]) \in d\ker\pi$, $\mathcal{J}^*\mathfrak{g}$ is an ideal of $\Omega^*\mathfrak{g}$. Moreover, it is obviously a differential ideal: $d\mathcal{J}^*\mathfrak{g} \subset \mathcal{J}^*\mathfrak{g}$, due to $d^2 = 0$. □

By virtue of Proposition 4, the canonical differential ideal (2.35) gives rise to a graded differential Lie algebra $\Omega^*_D\mathfrak{g}$:

$$\Omega^*_D\mathfrak{g} = \bigoplus_{k=0}^{\infty} \Omega^k_D\mathfrak{g} \;, \qquad\qquad \Omega^k_D\mathfrak{g} := \Omega^k\mathfrak{g}/\mathcal{J}^k\mathfrak{g} \;. \tag{2.36a}$$

There is a canonical isomorphism

$$\frac{\Omega^k\mathfrak{g}}{\mathcal{J}^k\mathfrak{g}} \cong \frac{\Omega^k\mathfrak{g}/(\ker\pi \cap \Omega^k\mathfrak{g})}{\mathcal{J}^k\mathfrak{g}/(\ker\pi \cap \Omega^k\mathfrak{g})} \;, \tag{2.36b}$$

establishing the isomorphism

$$\Omega^k_D\mathfrak{g} \cong \pi(\Omega^k\mathfrak{g})/\pi(\mathcal{J}^k\mathfrak{g}) \;. \tag{2.36c}$$

In particular, one has

$$\Omega^0_D\mathfrak{g} \cong \pi(\Omega^0\mathfrak{g}) \equiv \pi(\mathfrak{g}) \;, \qquad\qquad \Omega^1_D\mathfrak{g} \cong \pi(\Omega^1\mathfrak{g}) \;. \tag{2.36d}$$

Let ς denote the projection onto equivalence classes, $\varsigma : \pi(\Omega^k\mathfrak{g}) \to \Omega^k_D\mathfrak{g}$. In this notation, the commutator and the differential on $\Omega^*_D\mathfrak{g}$ are defined as

$$[\varsigma\circ\pi(\omega^k), \varsigma\circ\pi(\tilde{\omega}^l)]_g := \varsigma([\pi(\omega^k), \pi(\tilde{\omega}^l)]_g) \;, \tag{2.37a}$$

$$d(\varsigma\circ\pi(\omega^k)) := \varsigma\circ\pi(d\omega^k) \;, \tag{2.37b}$$

for $\omega^k \in \Omega^k\mathfrak{g}$ and $\tilde{\omega}^l \in \Omega^l\mathfrak{g}$. From (2.37a) there follows that $\Omega^*_D\mathfrak{g}$ is a graded Lie algebra, and the bracket $[\;,\;]_g : \Omega^*_D\mathfrak{g} \times \Omega^*_D\mathfrak{g} \to \Omega^*_D\mathfrak{g}$ has properties analogous to (2.23). For $\varrho^k = \varsigma\circ\pi(\omega^k)$ and $\tilde{\varrho}^l = \varsigma\circ\pi(\tilde{\omega}^l)$ we have with (2.37a) and (2.37b)

$$\begin{aligned} d[\varrho^k, \tilde{\varrho}^l]_g &= \varsigma\circ\pi(d[\omega^k, \tilde{\omega}^l]) = \varsigma\circ\pi([d\omega^k, \tilde{\omega}^l] + (-1)^k[\omega^k, d\tilde{\omega}^l]) \\ &= [d\varrho^k, \tilde{\varrho}^l]_g + (-1)^k[\varrho^k, d\tilde{\varrho}^l]_g \;. \end{aligned} \tag{2.37c}$$

Obviously, $d^2 \equiv 0$ on $\Omega^*_D\mathfrak{g}$. This means that d is a graded differential on $\Omega_D\mathfrak{g}$. Moreover, we have

$$(\varsigma\circ\pi(\omega^k))^* = \varsigma\circ\pi((\omega^k)^*) \;, \quad \omega^k \in \Omega^k_D\mathfrak{g} \;, \tag{2.38}$$

because π is an involutive representation and $\pi(\mathcal{J}^*\mathfrak{g})$ is invariant under the involution. From (2.27) we get

$$\varrho^{n\,*} = -(-1)^{n(n-1)/2}\varrho^n \;, \quad \varrho^n \in \Omega^n_D\mathfrak{g} \;. \tag{2.39}$$

D. Towards the Analysis of the Differential Ideal

Our goal is the analysis of the ideal $\pi(\mathcal{J}^*\mathfrak{g})$. For this purpose we define

$$\sigma\Big(\sum_{\alpha,z\geq 0}[\iota(a^z_\alpha),[\dots[\iota(a^1_\alpha),\iota(da^0_\alpha)]\dots]]\Big) := \sum_{\alpha,z\geq 0}[\pi(a^z_\alpha),[\dots[\pi(a^1_\alpha),[D^2,\pi(a^0_\alpha)]]\dots]]\;, \tag{2.40}$$

where $a^i_\alpha \in \mathfrak{g}$. In particular, from (2.40) we get

$$\sigma(\iota(da)) = [D^2,\pi(a)]\;, \qquad \sigma([\iota(a),\omega^1]) = [\pi(a),\sigma(\omega^1)]\;, \tag{2.41}$$

for $a\in\mathfrak{g}$ and $\omega^1\in\Omega^1\mathfrak{g}$. We extend σ to $\Omega^*\mathfrak{g}$, putting $\sigma(\Omega^0\mathfrak{g})\equiv 0$ and

$$\sigma(\sum_\alpha[\omega^k_\alpha,\tilde\omega^l_\alpha]) := \sum_\alpha\big([\sigma(\omega^k_\alpha),\pi(\tilde\omega^l_\alpha)]_g + (-1)^k[\pi(\omega^k_\alpha),\sigma(\tilde\omega^l_\alpha)]_g\big)\;, \tag{2.42}$$

for $\omega^k_\alpha\in\Omega^k\mathfrak{g}$ and $\tilde\omega^l_\alpha\in\Omega^l\mathfrak{g}$. Note that $\sigma(\omega^k)\in\mathscr{O}_{z_{k+1}}(h)$ if $\pi(\omega^k)\in\mathscr{O}_{z_k}(h)$, where $z_n = n \mod 2$. We do not necessarily have $\sigma(\omega^k)\in\mathscr{B}(h)$. Now we prove:

Proposition 6. *We have $\pi(d\omega^k) = [-\mathrm{i}D,\pi(\omega^k)]_g + \sigma(\omega^k)$, for $\omega^k\in\Omega^k\mathfrak{g}$.*

Proof: The Proposition is clearly true for $k=0$. To prove the Proposition for $k=1$ we first consider the case $\omega^1=\iota(da)\in\Omega^1\mathfrak{g}$. Then we have

$$[-\mathrm{i}D,\pi(\omega^1)]_g = [-\mathrm{i}D,[-\mathrm{i}D,\pi(a)]_g]_g = [(-\mathrm{i}D)^2,\pi(a)] = -\sigma(\iota(da))$$

so that $\pi(d\omega^1)=0$. But this is consistent with $d\omega^1 = d^2(\iota(a)) = 0$. Now we prove the Proposition for $k=1$ by induction. Because of (2.41), the linearity of π and the structure of elements of $\Omega^1\mathfrak{g}$, see (2.25), it suffices to assume that the Proposition is true for all $\omega^1\in\Omega^1\mathfrak{g}$ and to show that from this assumption there follows

$$\pi(d[\iota(a),\omega^1]) = [-\mathrm{i}D,\pi([\iota(a),\omega^1])]_g + \sigma([\iota(a),\omega^1])\;,$$

for all $a\in\mathfrak{g}$. We calculate

$$\begin{aligned}\pi(d[\iota(a),\omega^1]) &= [\pi(\iota(da)),\pi(\omega^1)]_g + [\pi(\iota(a)),\pi(d\omega^1)]_g\\ &= [[-\mathrm{i}D,\pi(a)]_g,\pi(\omega^1)]_g + [\pi(a),[-\mathrm{i}D,\pi(\omega^1)]_g + \sigma(\omega^1)]_g\\ &= [-\mathrm{i}D,[\pi(a),\pi(\omega^1)]_g]_g + \sigma([\iota(a),\omega^1])\\ &= [-\mathrm{i}D,\pi([\iota(a),\omega^1])]_g + \sigma([\iota(a),\omega^1])\;.\end{aligned}$$

Finally, we extend the proof to any k by induction. For that purpose let us assume that the Proposition holds for $k-1$. Due to linearity we can restrict ourselves to elements $\omega^k = [\omega^1,\tilde\omega^{k-1}]\in\Omega^k\mathfrak{g}$. Using (2.42) and the graded Jacobi identity we calculate

$$\begin{aligned}\pi(d[\omega^1,\tilde\omega^{k-1}]) &= [\pi(d\omega^1),\pi(\tilde\omega^{k-1})]_g - [\pi(\omega^1),\pi(d\tilde\omega^{k-1})]_g\\ &= [[-\mathrm{i}D,\pi(\omega^1)]_g + \sigma(\omega^1),\pi(\tilde\omega^{k-1})]_g - [\pi(\omega^1),[-\mathrm{i}D,\pi(\tilde\omega^{k-1})]_g + \sigma(\tilde\omega^{k-1})]_g\\ &= -[\pi(\tilde\omega^{k-1}),[-\mathrm{i}D,\pi(\omega^1)]_g]_g - (-1)^k[\pi(\omega^1),[\pi(\tilde\omega^{k-1}),-\mathrm{i}D]_g]_g + \sigma([\omega^1,\tilde\omega^{k-1}])\\ &= [-\mathrm{i}D,[\pi(\omega^1),\pi(\tilde\omega^{k-1})]_g]_g + \sigma([\omega^1,\tilde\omega^{k-1}])\;.\end{aligned}$$

□

We recall that

$$\pi(\mathcal{J}^k\mathfrak{g}) = \{\pi(d\omega^{k-1})\ ,\quad \omega^{k-1} \in \Omega^{k-1}\mathfrak{g} \cap \ker\pi\ \}\ . \tag{2.43}$$

From Proposition 6 we get the following equivalent characterization:

$$\pi(\mathcal{J}^k\mathfrak{g}) = \{\sigma(\omega^{k-1})\ ,\quad \omega^{k-1} \in \Omega^{k-1}\mathfrak{g} \cap \ker\pi\ \}\ . \tag{2.44}$$

Obviously, $\sigma(\omega^{k-1})$ is bounded if $\pi(\omega^{k-1}) = 0$. Of course, (2.44) is only a rewriting of (2.43), but it is a convenient starting point for the analysis of $\pi(\mathcal{J}^*\mathfrak{g})$.

E. Graded Lie Homomorphisms

In this subsection we provide the framework for the formulation of connections and gauge transformations. Let

$$\begin{aligned}\mathcal{H}^n\mathfrak{g} := \{\ & \eta^n \in \mathscr{O}_{z_n}(h)\ ,\quad z_n = n \mod 2\ ,\quad \eta^{n*} = -(-1)^{n(n-1)/2}\,\eta^n\ ,\\ & [\eta^n, \pi(\Omega^k\mathfrak{g})]_g \subset \pi(\Omega^{k+n}\mathfrak{g})\ ,\quad [\eta^n, \pi(\mathcal{J}^k\mathfrak{g})]_g \subset \pi(\mathcal{J}^{k+n}\mathfrak{g})\ \}\end{aligned} \tag{2.45}$$

be the set of graded Lie homomorphisms of $\pi(\Omega^*\mathfrak{g})$ of n^{th} degree. Note that $\mathcal{H}^n\mathfrak{g}$ may contain unbounded operators η on h , but such that

$$h' = \text{domain}(\eta) \cap \{\boldsymbol{\psi} \in h\ ,\quad \pi(\Omega^*\mathfrak{g})\boldsymbol{\psi} \subset \text{domain}(\eta)\}$$

is dense in h . This is necessary to ensure that the sequence $\{\ [\eta, \pi(\omega)]_g \boldsymbol{\psi}_n\ \}_n$ of elements of h , for $\boldsymbol{\psi}_n \in h'$ and any $\omega \in \Omega^*\mathfrak{g}$, converges to $\pi(\tilde{\omega})\boldsymbol{\psi}$ if $\boldsymbol{\psi}_n$ tends to $\boldsymbol{\psi} \in h$, where $\pi(\tilde{\omega}) \in \pi(\Omega^*\mathfrak{g})$ is independent of $\boldsymbol{\psi}_n$. Let

$$\tilde{\mathfrak{c}}^n\mathfrak{a} := \{j^n \in \mathcal{H}^n\mathfrak{g}\ ,\quad [j^n, \pi(\Omega^*\mathfrak{g})]_g = 0\ \} \tag{2.46}$$

be the graded centre of $\pi(\Omega^*\mathfrak{g})$ of n^{th} degree. Then, the factor space

$$\tilde{\mathcal{H}}^*\mathfrak{g} := \bigoplus_{n\in\mathbb{N}_0} \tilde{\mathcal{H}}^n\mathfrak{g}\ , \qquad\qquad \tilde{\mathcal{H}}^n\mathfrak{g} := \mathcal{H}^n\mathfrak{g}\,/\,\tilde{\mathfrak{c}}^n\mathfrak{a}\ , \tag{2.47a}$$

is a graded Lie algebra, with the graded commutator given by

$$\begin{aligned}&[[\eta^k + \tilde{\mathfrak{c}}^k\mathfrak{a}, \tilde{\eta}^l + \tilde{\mathfrak{c}}^l\mathfrak{a}]_g, \pi(\omega^n)]_g\\ &\qquad := [\eta^k, [\tilde{\eta}^l, \pi(\omega^n)]_g]_g - (-1)^{kl}[\tilde{\eta}^l, [\eta^k, \pi(\omega^n)]_g]_g\ ,\end{aligned} \tag{2.47b}$$

for $\eta^k \in \mathcal{H}^k\mathfrak{g}$, $\tilde{\eta}^l \in \mathcal{H}^l\mathfrak{g}$ and $\omega^n \in \Omega^n\mathfrak{g}$. It is clear that this equation is well–defined. Obviously, $\pi(\Omega^*\mathfrak{g})$ is a graded Lie subalgebra of $\tilde{\mathcal{H}}^*\mathfrak{g}$.

It is clear that the graded ideal $\pi(\mathcal{J}^*\mathfrak{g})$ of $\pi(\Omega^*\mathfrak{g})$ yields a graded ideal $\pi(\mathcal{J}^*\mathfrak{g}) + \tilde{\mathfrak{c}}^*\mathfrak{a}$ of $\mathcal{H}^*\mathfrak{g}$, see (2.45). Therefore,

$$\hat{\mathcal{H}}^*\mathfrak{g} := \bigoplus_{n\in\mathbb{N}_0} \hat{\mathcal{H}}^n\mathfrak{g}\ , \qquad \hat{\mathcal{H}}^n\mathfrak{g} = \mathcal{H}^n\mathfrak{g}\,/\,\mathbb{J}^n\mathfrak{g}\ , \qquad \mathbb{J}^n\mathfrak{g} = \tilde{\mathfrak{c}}^n\mathfrak{a} + \pi(\mathcal{J}^n\mathfrak{g})\ , \tag{2.48a}$$

is a graded Lie algebra. Moreover, it is a graded differential Lie algebra, too, where the graded differential is defined by

$$\begin{aligned}&[d(\eta^k+\pi(\mathcal{J}^k\mathfrak{g})+\tilde{\mathfrak{c}}^k\mathfrak{a}),\pi(\omega^n)+\pi(\mathcal{J}^n\mathfrak{g})]_g \\ &\qquad := \pi\circ d\circ\pi^{-1}([\eta^k,\pi(\omega^n)]_g)-(-1)^k[\eta^k,\pi(d\omega^n)]_g+\pi(\mathcal{J}^{k+n+1}\mathfrak{g})\;,\end{aligned} \tag{2.48b}$$

for $\eta^k\in\mathcal{H}^k\mathfrak{g}$ and $\omega^n\in\Omega^n\mathfrak{g}$. It is obvious that this equation is well–defined and that $\Omega^*_D\mathfrak{g}$ is a graded Lie subalgebra of $\hat{\mathcal{H}}^*\mathfrak{g}$.

Let

$$\begin{aligned}\mathbb{u}(\mathfrak{g}) := \{\,&\eta^0\in\mathcal{H}^0\mathfrak{g}\cap\mathscr{B}(h)\;,\\ &\sigma\circ\pi^{-1}([\eta^0,\pi(\omega^k)]_g)-[\eta^0,\sigma(\omega^k)]_g\in\pi(\mathcal{J}^{k+1}\mathfrak{g})\;,\quad\forall\omega^k\in\Omega^k\mathfrak{g}\;\}\;.\end{aligned} \tag{2.49}$$

Obviously, $\pi(\mathfrak{g})\subset\mathbb{u}(\mathfrak{g})$. Let $\mathcal{O}_0\subset\mathbb{u}(\mathfrak{g})$ be an open neighbourhood of the zero element of $\mathbb{u}(\mathfrak{g})$ and $\mathcal{O}_1\subset\mathscr{B}(h)$ be an open neighbourhood of $\mathbb{1}_{\mathscr{B}(h)}$. For an appropriate choice of $\mathcal{O}_0$ and $\mathcal{O}_1$ we define the exponential mapping

$$\exp:\mathcal{O}_0\to\mathcal{O}_1\;,\qquad \exp(\eta):=\mathbb{1}_{\mathscr{B}(h)}+\sum_{k=1}^{\infty}\frac{1}{k!}(\eta)^k\;,\quad\eta\in\mathcal{O}_0\;. \tag{2.50}$$

The Baker–Campbell–Hausdorff formula for $\eta_\alpha,\eta_\beta\in\mathcal{O}_0$,

$$\begin{aligned}&\exp(\eta_\alpha)\exp(\eta_\beta)=\exp(\eta_\gamma)\;,\\ &\eta_\gamma=\eta_\alpha+\eta_\beta+\tfrac{1}{2}[\eta_\alpha,\eta_\beta]+\tfrac{1}{12}([\eta_\alpha,[\eta_\alpha,\eta_\beta]]-[\eta_\beta,[\eta_\alpha,\eta_\beta]])+\dots\in\mathbb{u}(\mathfrak{g})\;,\end{aligned} \tag{2.51}$$

implies that we have a multiplication in $\exp(\mathcal{O}_0)$. In particular, for η_β proportional to η_α we get

$$\exp(\lambda_1\eta)\exp(\lambda_2\eta)=\exp((\lambda_1+\lambda_2)\eta)=\exp(\lambda_2\eta)\exp(\lambda_1\eta)\;, \tag{2.52}$$

for $\lambda_1,\lambda_2\in\mathbb{R}$ and $\eta\in\mathcal{O}_0$. Thus, $\exp(\eta)$ is invertible in $\mathscr{B}(h)$ for each $\eta\in\mathcal{O}_0$, and the inverse is given by

$$(\exp(\eta))^{-1}=\exp(-\eta)=\exp(\eta^*)=(\exp(\eta))^*\;. \tag{2.53}$$

Therefore, all elements $\exp(\eta)$ are unitary. Since $\mathscr{B}(h)$ is a C^*–algebra we conclude that for all $\eta\in\mathbb{u}(\mathfrak{g})$ we have

$$\|\exp(\eta)\|=\|\exp(\eta)^*\exp(\eta)\|^{1/2}=\|\mathbb{1}_{\mathscr{B}(h)}\|^{1/2}=1\;. \tag{2.54}$$

Hence, our construction leads to the subgroup

$$\exp(\mathbb{u}(\mathfrak{g})):=\{\;\textstyle\prod_{\alpha=1}^N\exp(\eta_\alpha)\;,\quad\eta_\alpha\in\mathcal{O}_0\;,\quad N\text{ finite }\;\} \tag{2.55}$$

of the group of unitary elements of $\mathscr{B}(h)$.

For A being a linear operator on h and $\eta \in \mathcal{O}_0$ we have

$$\exp(\eta) A \exp(-\eta) = A + \sum_{k=1}^{\infty} \frac{1}{k!} [\underbrace{\eta, [\eta, \dots, [\eta}_{k}, A] \dots]] \,. \tag{2.56}$$

For $A = \pi(a) \in \pi(\mathfrak{g})$ and $\exp(\eta) = u \in \mathcal{O}_1$ we get $u\pi(a)u^* \in \pi(\mathfrak{g})$. For $A = -\mathrm{i}D$ we get $u[-\mathrm{i}D, u^*] = -\mathrm{i}(uDu^* - D) \equiv ud(u^*) \in \hat{\mathcal{H}}^1\mathfrak{g}$, because with (2.49) and (2.48b) we have

$$\begin{aligned}
&[[-\mathrm{i}D, \eta], \pi(\omega^k)]_g + \pi(\mathcal{J}^{k+1}\mathfrak{g}) = [-\mathrm{i}D, [\eta, \pi(\omega^k)]]_g - [\eta, [-\mathrm{i}D, \pi(\omega^k)]_g] + \pi(\mathcal{J}^{k+1}\mathfrak{g}) \\
&\quad = \pi \circ d \circ \pi^{-1}([\eta, \pi(\omega^k)]) - \sigma \circ \pi^{-1}([\eta, \pi(\omega^k)]) \\
&\qquad - [\eta, \pi(d\omega^k)] + [\eta, \sigma(\omega^k)] + \pi(\mathcal{J}^{k+1}\mathfrak{g}) \\
&\quad = [d\eta, \pi(\omega^k)]_g + \pi(\mathcal{J}^{k+1}\mathfrak{g}) \;.
\end{aligned}$$

If $\pi(\omega^k) \in \pi(\mathcal{J}^k\mathfrak{g})$ then $[[-\mathrm{i}D, \eta], \pi(\omega^k)]_g \in \pi(\mathcal{J}^{k+1}\mathfrak{g})$. Therefore, there is a natural degree–preserving representation Ad of $\exp(\mathfrak{u}(\mathfrak{g}))$ in $\Omega^*_D\mathfrak{g}$ defined by

$$\begin{aligned}
\mathrm{Ad}_u\, \pi(a) &:= u\pi(a)u^* \;, \\
\mathrm{Ad}_u\, [-\mathrm{i}D, \pi(a)] &:= [-\mathrm{i}D, \mathrm{Ad}_u\, \pi(a)] + [u[-\mathrm{i}D, u^*], \mathrm{Ad}_u\, \pi(a)] \;, \\
\mathrm{Ad}_u\, (\pi(\omega^k) + \pi(\mathcal{J}^k\mathfrak{g})) &:= (\mathrm{Ad}_u\, \pi(\omega^k)) + \pi(\mathcal{J}^k\mathfrak{g}) \;, \\
\mathrm{Ad}_u\, [\varrho, \tilde{\varrho}]_g &:= [\mathrm{Ad}_u\, \varrho, \mathrm{Ad}_u\, \tilde{\varrho}]_g \;,
\end{aligned} \tag{2.57}$$

for $u \in \exp(\mathfrak{u}(\mathfrak{g}))$, $a \in \mathfrak{g}$, $\omega^k \in \Omega^k\mathfrak{g}$ and $\varrho, \tilde{\varrho} \in \Omega^*_D\mathfrak{g}$. Note that due to (2.56) we have $\mathrm{Ad}_u\, \pi(\mathcal{J}^k\mathfrak{g}) \subset \pi(\mathcal{J}^k\mathfrak{g})$.

F. Connections and Gauge Transformations

In this subsection we define the notion of a connection, of its curvature, of gauge transformations and of bosonic and fermionic actions.

Definition 7. *A connection on an L–cycle is a pair (∇, ∇_h) , where*

i) $\nabla_h : h \to h$ *is linear, odd and skew–adjoint,*
$\nabla_h \in \mathscr{O}_1(h)$, $\langle \boldsymbol{\psi}, \nabla_h \tilde{\boldsymbol{\psi}} \rangle_h = -\langle \nabla_h \boldsymbol{\psi}, \tilde{\boldsymbol{\psi}} \rangle_h$, $\forall \boldsymbol{\psi}, \tilde{\boldsymbol{\psi}} \in h$,
ii) $\nabla : \Omega^n_D\mathfrak{g} \to \Omega^{n+1}_D\mathfrak{g}$ *is linear,*
iii) $\nabla(\pi(\omega^n) + \pi(\mathcal{J}^n\mathfrak{g})) = [\nabla_h, \pi(\omega^n)]_g + \sigma(\omega^n) + \pi(\mathcal{J}^{n+1}\mathfrak{g})$, $\omega^n \in \Omega^n\mathfrak{g}$.

The operator $\nabla^2 : \Omega^n_D\mathfrak{g} \to \Omega^{n+2}_D\mathfrak{g}$ is called the curvature of the connection.

As a consequence of iii) we get with (2.42)

$$\nabla([\varrho^k, \tilde{\varrho}^l]_g) = [\nabla(\varrho^k), \tilde{\varrho}^l]_g + (-1)^k[\varrho^k, \nabla(\tilde{\varrho}^l)] \;, \quad \varrho^k \in \Omega^k_D\mathfrak{g} \;, \quad \tilde{\varrho}^l \in \Omega^l_D\mathfrak{g} \;. \tag{2.58}$$

Proposition 8. *Any connection has the form $\big(\nabla = d + [\tilde{\rho}\,,\;.\,]_g\,,\; \nabla_h = -\mathrm{i}D + \rho\big)$, for $\rho \in \mathcal{H}^1\mathfrak{g}$ and $\tilde{\rho} := \rho + \tilde{\mathfrak{c}}^1\mathfrak{a} \in \hat{\mathcal{H}}^1\mathfrak{g}$. Its curvature is $\nabla^2 = [\theta,\;.\;]$, with $\theta = d\tilde{\rho} + \frac{1}{2}[\tilde{\rho}, \tilde{\rho}]_g \in \hat{\mathcal{H}}^2\mathfrak{g}$.*

Proof: There is a canonical connection given by $(\nabla = d, \nabla_h = -\mathrm{i}D)$. Items i) and ii) of Definition 7 are obvious. For iii) we find with Proposition 6

$$[-\mathrm{i}D, \pi(\omega^k)]_g + \sigma(\omega^k) = \pi(d\omega^k) \;. \tag{2.59}$$

Taking $\omega \in \ker\pi$ we see that iii) is well–defined. Let $(\nabla^{(1)}, \nabla_h^{(1)})$ and $(\nabla^{(2)}, \nabla_h^{(2)})$ be two connections. Then we get from iii) of Definition 7

$$(\nabla^{(1)} - \nabla^{(2)})(\pi(\omega^k) + \pi(\mathcal{J}^k\mathfrak{g})) = [\nabla_h^{(1)} - \nabla_h^{(2)}, \pi(\omega^k)]_g + \pi(\mathcal{J}^{k+1}\mathfrak{g}) \;, \tag{2.60}$$

for $\omega^k \in \Omega^k\mathfrak{g}$. Now, item ii) yields $\rho := \nabla_h^{(1)} - \nabla_h^{(2)} \in \mathcal{H}^1\mathfrak{g}$. Since a modification of ρ by an element of $\tilde{\mathbb{c}}^1\mathfrak{a} \equiv \mathbb{J}^1\mathfrak{g}$ does not change formula (2.60), we get $\nabla^{(1)} - \nabla^{(2)} = [\tilde\rho\,,\ .\ ]$, where $\tilde\rho := \rho + \tilde{\mathbb{c}}^1\mathfrak{a} \in \hat{\mathcal{H}}^1\mathfrak{g}$. Taking $(\nabla^{(2)}, \nabla_h^{(2)}) = (d, -\mathrm{i}D)$ we obtain $(\nabla^{(1)}, \nabla_h^{(1)}) = (d + [\tilde\rho\,,\ .\ ]_g\,, -\mathrm{i}D + \rho)$.

Note that if $\sigma(\omega^k) \subset \pi(\mathcal{J}^{k+1}\mathfrak{g})$ for all $\omega^k \in \pi(\Omega^k\mathfrak{g})$ then there is $-\mathrm{i}D \in \mathcal{H}^1\mathfrak{g}$. Thus, the assertion remains true although the connection $(\nabla = d, \nabla_h = -\mathrm{i}D)$ is not distinguished in this case.

Finally, we compute the curvature ∇^2. For $\omega^k \in \Omega^k\mathfrak{g}$ we have with (2.47)

$$\begin{aligned}\nabla^2(\pi(\omega^k) + \pi(\mathcal{J}^k\mathfrak{g})) &= \nabla(\pi(d\omega^k) + [\tilde\rho, \pi(\omega^k)]_g + \pi(\mathcal{J}^{k+1}\mathfrak{g}))\\ &= [\tilde\rho, \pi(d\omega^k)]_g + \pi\circ d\circ\pi^{-1}([\tilde\rho, \pi(\omega^k)]_g) + [\tilde\rho, [\tilde\rho, \pi(\omega^k)]_g]_g + \pi(\mathcal{J}^{k+2}\mathfrak{g})\\ &\equiv [d\tilde\rho + \tfrac{1}{2}[\tilde\rho, \tilde\rho]_g, \pi(\omega^k) + \pi(\mathcal{J}^k\mathfrak{g})]_g =: [\theta, \pi(\omega^k) + \pi(\mathcal{J}^k\mathfrak{g})] \;. \qquad \square\end{aligned}$$

Note that the relation between $\rho \in \mathcal{H}^1\mathfrak{g}$ and $\rho' \in \mathcal{H}^1\mathfrak{g}$ in (2.60),

$$[\rho, \pi(\omega^k)]_g + \pi(\mathcal{J}^{k+1}\mathfrak{g}) = [\rho', \pi(\omega^k)]_g + \pi(\mathcal{J}^{k+1}\mathfrak{g}) \;,$$

may have more solutions than $\rho' = \rho + \tilde{\mathbb{c}}^1\mathfrak{a}$. However, we shall regard ρ and ρ' as different connection forms if $\rho - \rho' \notin \tilde{\mathbb{c}}^1\mathfrak{a}$. Analogously, the determining equation for $\theta' \in \hat{\mathcal{H}}^2\mathfrak{g}$,

$$[\theta', \varrho]_g = [\theta, \varrho]_g \quad \text{for all } \varrho \in \Omega^*_D\mathfrak{g} \;,$$

may have more solutions than $\theta' = \theta$. However, we shall select always the canonical representative $\theta = d\tilde\rho + \frac{1}{2}[\tilde\rho, \tilde\rho]_g$ in the curvature form of the connection ∇^2. Often we shall denote $\theta \in \hat{\mathcal{H}}^2\mathfrak{g}$ itself instead of ∇^2 the curvature of the connection (∇, ∇_h).

Definition 9. *The gauge group of the L–cycle is the group* $\mathcal{U}(\mathfrak{g}) := \exp(\mathbb{u}(\mathfrak{g}))$ *defined in* (2.55)*. Gauge transformations of the connection are given by*

$$(\nabla, \nabla_h) \longmapsto (\nabla', \nabla'_h) := (\mathrm{Ad}_u\,\nabla\mathrm{Ad}_{u^*}\;,\; u\nabla_h u^*) \;, \quad u \in \mathcal{U}(\mathfrak{g}) \,.$$

We must check that the definition of gauge transformations of a connection is compatible with Definition 7:

$$\begin{aligned}[\nabla'_h, \pi(\omega^n)]_g + \pi(\mathcal{J}^{n+1}\mathfrak{g}) &= u[\nabla_h, u^*\pi(\omega^n)u]_g u^* + \pi(\mathcal{J}^{n+1}\mathfrak{g}) \\ &= \mathrm{Ad}_u \big(\nabla(\mathrm{Ad}_{u^*}(\pi(\omega^n)+\pi(\mathcal{J}^n\mathfrak{g}))) - \sigma(\pi^{-1}\circ \mathrm{Ad}_{u^*}\circ \pi(\omega^n)) + \pi(\mathcal{J}^{n+1}\mathfrak{g})\big) \\ &= \nabla'(\pi(\omega^n)+\pi(\mathcal{J}^n\mathfrak{g})) - \mathrm{Ad}_u(\sigma(\pi^{-1}\circ \mathrm{Ad}_{u^*}\circ \pi(\omega^n))) + \pi(\mathcal{J}^{n+1}\mathfrak{g}) \;.\end{aligned}$$

Thus, the definition is consistent iff $\sigma(\pi^{-1}\circ \mathrm{Ad}_u \circ \pi(\omega^n)) + \pi(\mathcal{J}^{n+1}\mathfrak{g}) = \mathrm{Ad}_u(\sigma(\omega^n)) + \pi(\mathcal{J}^{n+1}\mathfrak{g})$. But this equation is satisfied due to (2.49).

The gauge transformation of the connection form ρ occurring in the connection $\nabla_h = -\mathrm{i}D + \rho$ is defined by

$$\nabla'_h =: -\mathrm{i}D + \gamma_u(\rho) \;. \tag{2.61}$$

From $\nabla'_h\boldsymbol{\psi} = u(-\mathrm{i}D+\rho)u^*\boldsymbol{\psi} = (-\mathrm{i}D + u[-\mathrm{i}D, u^*] + u\rho u^*)\boldsymbol{\psi}$ one finds

$$\gamma_u(\rho) = udu^* + u\rho u^* \;. \tag{2.62}$$

The gauge transformation of the curvature is due to

$$(\mathrm{Ad}_u \nabla \mathrm{Ad}_{u^*})^2(\varrho^k) = \mathrm{Ad}_u \nabla^2 \mathrm{Ad}_{u^*}\varrho^k = u[\theta, u^*\varrho^k u]u^*$$

given by

$$\gamma_u(\theta) = \mathrm{Ad}_u\,\theta = u\theta u^* \;. \tag{2.63}$$

The Dixmier trace[1] provides a canonical scalar product $\langle\;,\;\rangle$ on $\mathscr{B}(h)$. If the L–cycle is d^+–summable (see Definition 3) we define for $\tau, \tilde\tau \in \mathscr{B}(h)$

$$\langle \tau, \tilde\tau\rangle := \mathrm{Tr}_\omega(\tau^*\tilde\tau\,|D|^{-\mathrm{d}}) \;. \tag{2.64}$$

We assume that in some sense there exists an extension of this formula to linear operators on h belonging to $\mathcal{H}^2\mathfrak{g}$ (recall that $\mathcal{H}^2\mathfrak{g}$ is bounded on a dense subset of h).

Definition 10. *The bosonic action S_B and the fermionic action S_F of the connection (∇, ∇_h) are given by*

$$S_B(\nabla) = \langle\theta,\theta\rangle_{\hat{\mathcal{H}}^2\mathfrak{g}} := \min_{j^2\in\mathbb{J}^2\mathfrak{g}} \mathrm{Tr}_\omega((\theta_0 + j^2)^2\,|D|^{-\mathrm{d}}) \;, \tag{2.65a}$$

$$S_F(\boldsymbol{\psi}, \nabla_h) := \langle\boldsymbol{\psi}, \mathrm{i}\nabla_h\boldsymbol{\psi}\rangle_h \;, \quad \boldsymbol{\psi}\in h \;, \tag{2.65b}$$

where Tr_ω *is the Dixmier trace,* $\langle\;,\;\rangle_h$ *the scalar product on h and $\theta_0 \in \mathcal{H}^2\mathfrak{g}$ any representative of the curvature of ∇.*

Since both $\langle\ ,\ \rangle_{\hat{\mathcal{H}}^2\mathfrak{g}}$ and $\langle\ ,\ \rangle_h$ are invariant[1] under unitary transformations we get from (2.63) and Definition 9 that the action (2.65) is invariant under gauge transformations

$$(\nabla,\nabla_h)\longmapsto(\mathrm{Ad}_u\,\nabla\mathrm{Ad}_{u^*}\,,u\nabla_h u^*)\;,\quad \boldsymbol{\psi}\longmapsto u\boldsymbol{\psi}\;,\quad u\in\mathcal{U}(\mathfrak{g})\;. \tag{2.66}$$

There is an equivalent formulation of (2.65a). Let $\mathfrak{e}(\theta_0+j^2)\in\mathcal{H}^2\mathfrak{g}$ be those representative of $\theta\in\hat{\mathcal{H}}^2\mathfrak{g}$, for which the minimum in (2.65a) is attained. Let $j^2=\sum_\alpha\lambda_\alpha j^2_\alpha$, for $\lambda_\alpha\in\mathbb{R}$, be a parameterization of $j^2\in\mathbb{J}^2\mathfrak{g}$. Then,

$$0=\frac{d}{d\lambda_\alpha}\,\mathrm{Tr}_\omega((\theta_0+j^2)^2\,|D|^{-\mathrm{d}})=2\ \mathrm{Tr}_\omega((\theta_0+j^2)j^2_\alpha)\,|D|^{-\mathrm{d}})\;.$$

Thus, $\mathfrak{e}(\theta_0+j^2)\equiv\mathfrak{e}(\theta)$ is those representative of θ, which is orthogonal to the ideal $\mathbb{J}^2\mathfrak{g}$ with respect to $\langle\ ,\ \rangle_{\hat{\mathcal{H}}^2\mathfrak{g}}$:

$$S_B=\mathrm{Tr}_\omega((\mathfrak{e}(\theta))^2\,|D|^{-\mathrm{d}})\;,\qquad\qquad \mathrm{Tr}_\omega(\mathfrak{e}(\theta)\,\mathbb{J}^2\mathfrak{g}\,|D|^{-\mathrm{d}})\equiv 0\;. \tag{2.67}$$

The representative $\mathfrak{e}(\theta)$ is unique, because $\mathrm{Tr}_\omega(\ .\ |D|^{-\mathrm{d}})$ is positive definite:[1]

$$\begin{aligned}\mathrm{Tr}_\omega((\mathfrak{e}(\theta)+j^2)^2\,|D|^{-\mathrm{d}})&=\mathrm{Tr}_\omega((\mathfrak{e}(\theta))^2\,|D|^{-\mathrm{d}})+\mathrm{Tr}_\omega((j^2)^2|D|^{-\mathrm{d}})\\&>\mathrm{Tr}_\omega((\mathfrak{e}(\theta))^2\,|D|^{-\mathrm{d}})\;,\qquad\forall j^2\neq 0\;.\end{aligned}$$

III. L–Cycles over Functions ⊗ Matrix Lie Algebra

A. A Class of L–Cycles Relevant to Physics

Let $(\mathfrak{a},\mathbb{C}^F,\mathcal{M},\hat{\pi},\hat{\Gamma})$ be an L–cycle over a matrix Lie algebra $\mathfrak{a}$. In particular, we have a representation $\hat{\pi}$ of $\mathfrak{a}$ in the Lie algebra $\mathrm{M}_F\mathbb{C}$ of endomorphisms of the Hilbert space $\mathbb{C}^F$. Moreover, the grading operator $\hat{\Gamma}$ anticommutes with the generalized Dirac operator $\mathcal{M}$ and commutes with $\hat{\pi}(\mathfrak{a})$. Both $\mathcal{M}$ and $\hat{\Gamma}$ belong to $\mathrm{M}_F\mathbb{C}$.

Let X be a compact even dimensional Riemannian spin manifold, $\dim(X)=N\geq 4$, and let $C^\infty(X)$ be the algebra of real–valued smooth functions on X. Since $C^\infty(X)$ is a commutative algebra, the tensor product

$$\mathfrak{g}:=C^\infty(X)\otimes\mathfrak{a} \tag{3.1a}$$

over $\mathbb{R}$ is in a natural way a Lie algebra, where the commutator is given by

$$[f_1\otimes a_1,f_2\otimes a_2]\equiv f_1f_2\otimes[a_1,a_2]\;,\quad f_1,f_2\in C^\infty(X)\;,\quad a_1,a_2\in\mathfrak{a}\;. \tag{3.1b}$$

We introduce the Hilbert space

$$h:=L^2(X,S)\otimes\mathbb{C}^F\;, \tag{3.2}$$

where $L^2(X,S)$ denotes the Hilbert space of square integrable sections of the spinor bundle over X. The representation $\hat{\pi} : \mathfrak{a} \to \mathrm{End}\,(\mathbb{C}^F)$ and the $C^\infty(X)$–module structure of $L^2(X,S)$ induce a natural representation π of $\mathfrak{g}$ in $\mathscr{B}(h)$:

$$\pi(f \otimes a)(s \otimes \varphi) := fs \otimes \hat{\pi}(a)\varphi \;, \tag{3.3}$$

for $f \in C^\infty(X)$, $a \in \mathfrak{a}$, $s \in L^2(X,S)$ and $\varphi \in \mathbb{C}^F$. We denote by $\boldsymbol{\gamma}$ the grading operator and by D the classical Dirac operator on the Hilbert space $L^2(X,S)$, see Sec. III.B for more details. Then we put

$$D := \mathsf{D} \otimes \mathbb{1}_F + \boldsymbol{\gamma} \otimes \mathcal{M} \;, \tag{3.4}$$

$$\Gamma := \boldsymbol{\gamma} \otimes \hat{\Gamma} \;. \tag{3.5}$$

The operator $[D, \pi(f \otimes a)]$ is bounded on h for all $f \otimes a \in \mathfrak{g}$. Moreover, D is selfadjoint on h, because D and $\boldsymbol{\gamma}$ are selfadjoint on $L^2(X,S)$ and $\mathcal{M}$ is symmetrical. Next, Γ commutes with $\pi(\mathfrak{g})$ and anticommutes with D. Finally, $(\mathrm{id}_h + D^2)^{-1}$ is compact:[13] The operator $(\mathrm{id}_h + D^2)^{-1}$ is a pseudo–differential operator of order -2 with compact support and has, therefore, an extension to a continuous operator from H_s to H_{s+2} on the Sobolev scale $\{H_s\}$. Due to Rellich's lemma, the embedding $e : H_t \hookrightarrow H_s$ is compact for $t > s$. Thus, $(\mathrm{id}_h + D^2)^{-1}$ considered as

$$e \circ (\mathrm{id}_h + D^2)^{-1} : H_s \to H_s$$

is compact, and $(\mathfrak{g}, h, D, \pi, \Gamma)$ forms an L–cycle.

Finally, we briefly sketch how the physical data specified in the Introduction fit into this scheme. First, one constructs a Euclidian version of the gauge field theory. Now, X is the one–point compactification of the Euclidian space–time manifold. The completion of the space of fermions $\boldsymbol{\psi}$ yields the Hilbert space h of the L–cycle. In some cases, it may be necessary to work with several copies of the fermions. Given the (Lie) group of local gauge transformations $\mathscr{G}$, we take $\mathfrak{g}$ as the Lie algebra of $\mathscr{G}$. The representation $\pi : \mathfrak{g} \to \mathscr{B}(h)$ is just the differential $\tilde{\pi}_*$ of the group representation $\tilde{\pi}$. The matrix $\mathcal{M}$ occurring in the generalized Dirac operator (3.4) contains the fermionic mass parameters and possibly contributions required by the desired symmetry breaking scheme. However, it is necessary that $\boldsymbol{\gamma} \otimes \mathcal{M}$ coincides with the fermionic mass matrix $\widetilde{\mathcal{M}}$ on chiral fermions. The grading operator Γ represents the chirality properties of the fermions. We have $\boldsymbol{\gamma} = \gamma^5$ in four dimensions. After the Wick rotation to Minkowski space we use Γ to impose a chirality condition on h.

B. Notations and Techniques

This subsection is devoted to definitions and techniques related to sections of the Clifford bundle. We denote by $\Gamma^\infty(C)$ the set of smooth sections of the Clifford bundle C over X and by $C^k \subset \Gamma^\infty(C)$ the set of those sections of C, whose values at each point $x \in X$ belong to the subspace spanned by products of less than or equal k elements of $T^*_x X$ of the same parity. In particular, we identify $C^\infty(X) \equiv C^0$.

We recall[14] that there is an isomorphism of vector spaces

$$c : \Lambda^*(\Gamma^\infty(T^*X)) \to \Gamma^\infty(C) \tag{3.6}$$

between $\Gamma^\infty(C)$ and the exterior differential algebra $\Lambda^*(\Gamma^\infty(T^*X))$ of antisymmetrized tensor products of the vector space of smooth sections of the cotangent bundle over X. In particular, the restriction to the first degree yields a vector space isomorphism $c : \Gamma^\infty(T^*X) \to C^1$. Therefore, elements $c^1 \in C^1$ have the form $c^1 = c(\boldsymbol{\omega}^1)$, for $\boldsymbol{\omega}^1 \in \Gamma^\infty(T^*X)$. We use the following sign convention for the defining relation of the Clifford action:

$$\tfrac{1}{2}(c(\boldsymbol{\omega}^1)c(\tilde{\boldsymbol{\omega}}^1) + c(\tilde{\boldsymbol{\omega}}^1)c(\boldsymbol{\omega}^1)) \equiv \tfrac{1}{2}\{c(\boldsymbol{\omega}^1), c(\tilde{\boldsymbol{\omega}}^1)\} = g^{-1}(\boldsymbol{\omega}^1, \tilde{\boldsymbol{\omega}}^1)1 \in C^0 , \tag{3.7}$$

where $g^{-1} : \Gamma^\infty(T^*X) \times \Gamma^\infty(T^*X) \to C^\infty(X)$ is the inverse of the metric $g : \Gamma^\infty(T_*X) \times \Gamma^\infty(T_*X) \to C^\infty(X)$.

Let us define the notion of the exterior product $\wedge$:

$$c^1_1 \wedge c^1_2 \wedge \dots \wedge c^1_n := \frac{1}{n!} \sum_{\pi \in P^n} (-1)^{\mathrm{sign}(\pi)} c^1_{\pi(1)} c^1_{\pi(2)} \dots c^1_{\pi(n)} , \quad c^1_i \in C^1 , \tag{3.8}$$

where the sum runs over all permutations of the numbers $1, \dots, n$ and the product on the r.h.s. is pointwise the product in the Clifford algebra. Observe that $\wedge$ is associative and that the antisymmetrization (3.8) yields zero for $n > N = \dim(X)$.

Definition 11. *$\Lambda^n \subset C^n$ is the vector subspace generated by elements of the form* (3.8)*, with $\Lambda^0 \equiv C^0$, $\Lambda^1 \equiv C^1$ and $\Lambda^n \equiv \{0\}$ for $n < 0$ and $n > \dim(X)$.*

We define the interior product $\lrcorner : \Lambda^1 \times \Lambda^n \to \Lambda^{n-1}$ by

$$c^1_0 \lrcorner (c^1_1 \wedge c^1_2 \wedge \dots \wedge c^1_n) := \sum_{j=1}^n (-1)^{j+1} \tfrac{1}{2} \{c^1_0, c^1_j\} (c^1_1 \wedge \overset{j}{\overset{\vee}{\dots}} \wedge c^1_n) , \tag{3.9a}$$

$$c^1_1 \wedge \overset{j}{\overset{\vee}{\dots}} \wedge c^1_n := c^1_1 \wedge c^1_2 \wedge \dots \wedge c^1_{j-1} \wedge c^1_{j+1} \wedge \dots \wedge c^1_n . \tag{3.9b}$$

The interior product (3.9a) is extended to $\lrcorner : \Lambda^k \times \Lambda^n \to \Lambda^{n-k}$ by

$$\begin{aligned}(\tilde{c}^1_1 \wedge \tilde{c}^1_2 \wedge \dots \wedge \tilde{c}^1_k) &\lrcorner (c^1_1 \wedge c^1_2 \wedge \dots \wedge c^1_n) \\ &:= \tilde{c}^1_1 \lrcorner (\dots \lrcorner (\tilde{c}^1_{k-1} \lrcorner (\tilde{c}^1_k \lrcorner (c^1_1 \wedge c^1_2 \wedge \dots \wedge c^1_n))) \dots) .\end{aligned} \tag{3.10}$$

Lemma 12. *For $c^1_i \in C^1$ we have*

$$\tfrac{1}{2}(c^1_0(c^1_1 \wedge \dots \wedge c^1_n) + (-1)^n (c^1_1 \wedge \dots \wedge c^1_n) c^1_0) = c^1_0 \wedge c^1_1 \wedge c^1_2 \wedge \dots \wedge c^1_n , \tag{3.11a}$$

$$\tfrac{1}{2}(c^1_0(c^1_1 \wedge \dots \wedge c^1_n) - (-1)^n (c^1_1 \wedge \dots \wedge c^1_n) c^1_0) = c^1_0 \lrcorner (c^1_1 \wedge c^1_2 \wedge \dots \wedge c^1_n) . \tag{3.11b}$$

Proof: The assertion is clear for orthogonal bases. □

Let $\{e^j\}_{j=1}^N$ be an arbitrary selfadjoint basis of $\Gamma^\infty(T^*X)$ and $\{e_j\}_{j=1}^N$ its dual basis of $\Gamma^\infty(T_*X)$. Duality of $\{e_j\}_{j=1}^N$ and $\{e^j\}_{j=1}^N$ is understood in the sense

$$e^j(e_i) \equiv \langle e^j, e_i\rangle = \delta_i^j \tag{3.12}$$

and selfadjointness means $c(e^j) = c(e^j)^*$. Let ∇_v be the Levi–Civita covariant derivative with respect to the vector field $v \in \Gamma^\infty(T_*X)$. Then we define the exterior differential $\mathbf{d} : \Lambda^k \to \Lambda^{k+1}$ on Λ^* by

$$\mathbf{d}c^k := \sum_{j=1}^N c(e^j) \wedge \nabla_{e_j}(c^k)\;, \quad c^k \in \Lambda^k\;. \tag{3.13}$$

The proof that $\mathbf{d}$ is indeed a graded differential uses the fact that the Levi–Civita connection has vanishing torsion, see (with different sign conventions) Ref. 14. There is a natural scalar product $\langle\;,\;\rangle_{\Lambda^*}$ on Λ^*:

$$\langle c^k, \tilde{c}^l\rangle_{\Lambda^*} := \int_X \mathrm{v}_g\; \mathrm{tr}_c({c^k}^*\tilde{c}^l)\;, \quad c^k \in \Lambda^k\;, \quad \tilde{c}^l \in \Lambda^l\;, \tag{3.14}$$

where $\mathrm{tr}_c : \Gamma^\infty(C) \to C^\infty(X)$ is pointwise the trace in the Clifford algebra and v_g the canonical volume form on X. The scalar product (3.14) vanishes for $k \neq l$. Via this scalar product we define the codifferential $\mathbf{d}^* : \Lambda^k \to \Lambda^{k-1}$ on Λ^* as the operator dual to the exterior differential $\mathbf{d}$:

$$\langle \mathbf{d}c^k, \tilde{c}^{k+1}\rangle_{\Lambda^*} =: \langle c^k, \mathbf{d}^*\tilde{c}^{k+1}\rangle_{\Lambda^*}\;, \quad \forall c^k \in \Lambda^k\;, \quad c^{k+1} \in \Lambda^{k+1}\;. \tag{3.15}$$

Lemma 13. *Within our conventions one has the representation*

$$\mathbf{d}^* c^k = -\sum_{j=1}^N c(e^j) \,\lrcorner\, \nabla_{e_j}(c^k)\;. \tag{3.16}$$

Proof: The proof is straightforward. One has to use Lemma 12, the invariance of the trace under cyclic permutations, the Leibniz rule for ∇_v and the identity $\nabla_v(\mathrm{v}_g) \equiv 0$ for the Levi–Civita connection. □

Note that – in contrast to what its name suggests – $\mathbf{d}^*$ is not a derivation. Using (3.16) one easily derives for $c_i^1 \in C^1 \equiv \Lambda^1$ the formula

$$\begin{aligned} &\mathbf{d}^*(c_1^1 \wedge c_2^1 \wedge \cdots \wedge c_n^1) \\ &\quad = \textstyle\sum_{k=1}^n \big(-(-1)^{k+1} \nabla_{g^{-1}(c^{-1}(c_k^1))}(c_1^1 \wedge \overset{k}{\overset{\vee}{\ldots}} \wedge c_n^1) + (-1)^{k+1}\mathbf{d}^*(c_k^1)(c_1^1 \wedge \overset{k}{\overset{\vee}{\ldots}} \wedge c_n^1)\big)\;, \end{aligned} \tag{3.17}$$

where g^{-1} is treated as an isomorphism from $\Gamma^\infty(T^*X)$ to $\Gamma^\infty(T_*X)$.

In terms of the above introduced selfadjoint bases $\{e^j\}_{j=1}^N$ of $\Gamma^\infty(T^*X)$ and $\{e_j\}_{j=1}^N$ of $\Gamma^\infty(T_*X)$, the classical Dirac operator is given by[14]

$$\mathsf{D} = \sum_{j=1}^N \mathrm{i}c(e^j)\nabla^S_{e_j} \;. \tag{3.18}$$

Here, ∇^S_v is the Clifford covariant derivative on $L^2(X,S)$ with respect to the vector field v. It has the property

$$[\nabla^S_v, c(\boldsymbol{\omega})] = c(\nabla_v\boldsymbol{\omega}) \equiv \nabla_v c(\boldsymbol{\omega}) \;, \tag{3.19}$$

for any differential form $\boldsymbol{\omega}$. With (3.13) this gives immediately

$$[\mathsf{D},f] = \textstyle\sum_{j=1}^N \mathrm{i}c(e^j)[\nabla^S_{e_j},f] \equiv \mathrm{i}\mathbf{d}f \equiv \mathrm{i}c(\mathsf{d}f) \;, \quad f\in C^\infty(X) \;, \tag{3.20}$$

where d is the usual exterior differential on the exterior differential algebra. The grading operator on $L^2(X,S)$ is $\boldsymbol{\gamma} = -\mathrm{i}^{N/2}c(\mathrm{v}_g)$, fulfilling

$$\begin{aligned}\mathsf{D}\boldsymbol{\gamma}+\boldsymbol{\gamma}\mathsf{D} &= \mathrm{i}^{-1+N/2}\textstyle\sum_{j=1}^N\big(c(e^j)[\nabla^S_{e_j},c(\mathrm{v}_g)] + (c(e^j)c(\mathrm{v}_g)+c(\mathrm{v}_g)c(e^j))\nabla^S_{e_j}\big)\\ &= \mathrm{i}^{-1+N/2}\textstyle\sum_{j=1}^N\big(c(e^j)c(\nabla_{e^j}(\mathrm{v}_g)) + 2c(e^j)\wedge c(\mathrm{v}_g)\nabla^S_{e_j}\big) \equiv 0 \;,\end{aligned} \tag{3.21}$$

because of the properties $\nabla_v(\mathrm{v}_g)\equiv 0$ and $c(e^j)\wedge c(\mathrm{v}_g)\in\Lambda^{N+1}\equiv 0$. Therefore, the Dirac operator D is an odd first order differential operator. One has $\boldsymbol{\gamma}^2 = (-1)^{N/2}c(\mathrm{v}_g)c(\mathrm{v}_g) = \det g^{-1}$. If we restrict ourselves to an orthogonal metric, which we do for the rest of this work, then we have $\boldsymbol{\gamma}^2 = 1$.

Next, using (3.13), (3.16) and Lemma 12 we have for $c^k\in\Lambda^k$

$$\begin{aligned}(-\mathrm{i}\mathsf{D})c^k-(-1)^kc^k(-\mathrm{i}\mathsf{D}) &= \textstyle\sum_{j=1}^N\big(c(e^j)[\nabla^S_{e_j},c^k] + (c(e^j)c^k-(-1)^kc^kc(e^j))\nabla^S_{e_j}\big)\\ &= \mathbf{d}c^k-\mathbf{d}^*c^k+2\textstyle\sum_{j=1}^N c(e^j)\,\lrcorner\, c^k\,\nabla^S_{e_j}\\ &= \mathbf{d}c^k-\mathbf{d}^*c^k+2\textstyle\sum_{i=1}^k(-1)^{i+1}c^1_1\wedge \overset{i}{\overset{\vee}{\ldots}}\wedge c^1_k\,\nabla^S_{g^{-1}(c^{-1}(c^1_i))} \;,\end{aligned} \tag{3.22}$$

if $c^k = c^1_1\wedge c^1_2\wedge\dots c^1_k$, $c^1_i\in\Lambda^1$. The last identity in (3.22) is due to

$$\begin{aligned}2\textstyle\sum_{j=1}^N c(e^j)\,\lrcorner\, c^k\,\nabla^S_{e^j} &= \textstyle\sum_{j=1}^N\sum_{i=1}^k(-1)^{i+1}\{c(e^j),c^1_i\}\,c^1_1\wedge \overset{i}{\overset{\vee}{\ldots}}\wedge c^1_k\,\nabla^S_{e^j}\\ &= 2\textstyle\sum_{j=1}^N\sum_{i=1}^k(-1)^{i+1}g^{-1}(e^j,c^{-1}(c^1_i))\,c^1_1\wedge \overset{i}{\overset{\vee}{\ldots}}\wedge c^1_k\,\nabla^S_{e_j}\\ &= 2\textstyle\sum_{i=1}^k(-1)^{i+1}c^1_1\wedge \overset{i}{\overset{\vee}{\ldots}}\wedge c^1_k\,\nabla^S_{g^{-1}(c^{-1}(c^1_i))} \;.\end{aligned}$$

In particular,

$$[\mathsf{D}^2,f] = \Delta f - 2\nabla^S_{\mathrm{grad}\,f} \;, \quad f\in C^\infty(X) \;, \tag{3.23}$$

where $\mathrm{grad}\,f := g^{-1}(\mathsf{d}f)$ is the vector field dual to $\mathsf{d}f$ and Δ the scalar Laplacian,

$$\Delta f \equiv \mathbf{d}^*\mathbf{d}f = -\textstyle\sum_{i,j=1}^N g^{-1}(e^i,e^j)(\nabla_{e_i}\nabla_{e_j}-\nabla_{\nabla_{e_i}e_j})(f) \;. \tag{3.24}$$

C. The Representation of $\Omega^*\mathfrak{g}$ on the Hilbert Space

For physical applications we are interested in the case that the matrix Lie algebra $\mathfrak{a}$ decomposes into

$$\mathfrak{a} = \mathfrak{a}' \oplus \mathfrak{a}'' \;. \tag{3.25}$$

Here, $\mathfrak{a}'$ is unitary and semisimple, i.e. a direct sum of simple unitary Lie algebras, and $\mathfrak{a}''$ is a direct sum of copies of the Abelian Lie algebra $\mathrm{u}(1)$, each of them represented in the form $\mathrm{u}(1)_{(i)} = \mathbb{R}\mathrm{b}_{(i)}$. In particular, direct sum means that elements of different direct sum subspaces always commute. For each copy of $\mathrm{u}(1)$, the representation $\hat{\pi}(\mathrm{b})$ shall have the following property: There exist $\lambda^z \in \mathbb{R}$ such that

$$[\hat{\pi}(\mathrm{b}), \mathcal{M}] = \textstyle\sum_{z\geq 2} \lambda^z \underbrace{[\hat{\pi}(\mathrm{b}), [\dots[\hat{\pi}(\mathrm{b}), [\hat{\pi}(\mathrm{b})}_{z}, \mathcal{M}]]\dots]] \;. \tag{3.26}$$

For simplicity, we restrict ourselves to the case $\mathfrak{a}'' = \mathrm{u}(1)$, where (3.26) is given by

$$\begin{aligned} [\hat{\pi}(\mathrm{b}), [\hat{\pi}(\mathrm{b}), [\hat{\pi}(\mathrm{b}), \mathcal{M}]]] &= [\hat{\pi}(\mathrm{b}), \mathcal{M}] \quad \text{or} \\ [\hat{\pi}(\mathrm{b}), [\hat{\pi}(\mathrm{b}), [\hat{\pi}(\mathrm{b}), \mathcal{M}]]] &= -[\hat{\pi}(\mathrm{b}), \mathcal{M}] \;. \end{aligned} \tag{3.27}$$

The extension to the general case is obvious.

Our goal is to construct the graded differential Lie algebra $\Omega^*_D\mathfrak{g}$ associated to the L–cycle $(\mathfrak{g}, h, D, \pi, \Gamma)$, see Sec. II.C. For this purpose we first have to construct the graded Lie algebra $\pi(\Omega^*\mathfrak{g})$ associated to this L–cycle. We denote by $\hat{\pi}(\Omega^*\mathfrak{a})$ the corresponding graded Lie algebra associated to the L–cycle $(\mathfrak{a}, \mathbb{C}^F, \mathcal{M}, \hat{\pi}, \hat{\Gamma})$. From (3.20) we get

$$[D, \pi(f\otimes a)] = \mathrm{i}\mathbf{d}f \otimes \hat{\pi}(a) + f\boldsymbol{\gamma} \otimes [\mathcal{M}, \hat{\pi}(a)] \;, \quad a \in \mathfrak{a} \;, \quad f \in C^\infty(X) \;, \tag{3.28}$$

where $\mathbf{d}$ is the exterior differential (3.13). Using that C^0 is an Abelian algebra, that elements of C^0 commute with elements of C^1 and that $\hat{\pi}$ is a representation we obtain for elements of $\pi(\Omega^1\mathfrak{g})$, see (2.25) and (2.30),

$$\begin{aligned} &\pi\big(\textstyle\sum_{\alpha,z\geq 0} [\iota(f^z_\alpha \otimes a^z_\alpha), [\dots[\iota(f^1_\alpha\otimes a^1_\alpha), \iota(d(f^0_\alpha \otimes a^0_\alpha))]\dots]]\big) \\ &\qquad = \textstyle\sum_{\alpha,z\geq 0} [\pi(f^z_\alpha \otimes a^z_\alpha), [\dots[\pi(f^1_\alpha \otimes a^1_\alpha), [-\mathrm{i}D, \pi(f^0_\alpha\otimes a^0_\alpha)]]\dots]] \\ &\qquad = \textstyle\sum_{\alpha,z\geq 0} f^z_\alpha \cdots f^1_\alpha \mathbf{d} f^0_\alpha \otimes \hat{\pi}([a^z_\alpha, [\dots[a^1_\alpha, a^0_\alpha]\dots]]) & \text{(3.29a)} \\ &\qquad + \textstyle\sum_{\alpha,z\geq 0} f^z_\alpha \cdots f^1_\alpha f^0_\alpha \boldsymbol{\gamma} \otimes \hat{\pi}([\iota(a^z_\alpha), [\dots[\iota(a^1_\alpha), \iota(da^0_\alpha)]\dots]]) \;. & \text{(3.29b)} \end{aligned}$$

Here we have $f^j_\alpha \in C^0$, $a^j_\alpha \in \mathfrak{a}$, and d denotes the universal differential on both the universal differential Lie algebras over $\mathfrak{g}$ and $\mathfrak{a}$; it is clear from the context on which of them. The same notational simplification was used for the factorization mappings ι. There are two different contributions in this formula, (3.29a) belongs to $C^1 \otimes \hat{\pi}(\Omega^0\mathfrak{a})$ and (3.29b) to $C^0\boldsymbol{\gamma} \otimes \hat{\pi}(\Omega^1\mathfrak{a})$. If it was possible to put all f^0_α equal to constants without changing the range of (3.29b) then the lines (3.29a) and (3.29b) would be independent. This is possible iff for all $f^0_0 \in C^\infty(X)$ and $a^0_0 \in \mathfrak{a}$ there exists a solution of the equation

$$f^0_0 \otimes \hat{\pi}(\iota(da^0_0)) = \textstyle\sum_{\alpha,z\geq 1} f^z_\alpha \cdots f^1_\alpha f^0_\alpha \boldsymbol{\gamma} \otimes \hat{\pi}([\iota(a^z_\alpha),[\ldots[\iota(a^1_\alpha),\iota(da^0_\alpha)]\ldots]]) \;.$$

But this is indeed the case, due to (3.26) for $a^0_0 \in \mathfrak{a}''$ and the fact that $\mathfrak{a}'$ is semisimple. Namely, for a semisimple Lie algebra $\mathfrak{a}'$ we have[12] $[\mathfrak{a}',\mathfrak{a}'] = \mathfrak{a}'$. This means that

$$\forall\, a' \in \mathfrak{a} \;\; \exists\, a'_\alpha, \tilde{a}'_\alpha \in \mathfrak{a}' \;\;:\;\; a' = \textstyle\sum_\alpha [a'_\alpha, \tilde{a}'_\alpha] \;. \tag{3.30}$$

Then, $\iota(da') = \sum_\alpha \big([\iota(a'_\alpha),\iota(d\tilde{a}'_\alpha)] - [\iota(\tilde{a}'_\alpha),\iota(da'_\alpha)]\big)$. Here we see the importance of the restrictions imposed to $\mathfrak{a}$, we will meet further examples in the sequel.

Now, from the definition (2.25) of $\Omega^1\mathfrak{a}$ there follows that (3.29b) can attain any element of $C^0\boldsymbol{\gamma} \otimes \hat{\pi}(\Omega^1\mathfrak{a})$. We split elements $a^j_\alpha \in \mathfrak{a}$ according to (3.25). Since commutators containing elements of the Abelian part vanish, there is a non–vanishing contribution of elements of $\mathfrak{a}''$ to (3.29a) only from the term $\mathbf{d}\tilde{f}^0_0 \otimes \hat{\pi}(a^0_0)$, for $a^0_0 \in \mathfrak{a}''$. Therefore, the coefficient of elements of $\hat{\pi}(\mathfrak{a}'')$ is the Clifford action of a total differential. We denote the space $\mathbf{d}C^0 \subset C^1$ by B^1 ("[co]boundary"). In the case of the semisimple Lie algebra $\mathfrak{a}'$ the line (3.29a) attains any element of $C^1 \otimes \hat{\pi}(\mathfrak{a}')$, due to (3.30). Thus, we get the final result

$$\pi(\Omega^1\mathfrak{g}) = (\Lambda^1 \otimes \hat{\pi}(\mathfrak{a}')) \oplus (B^1 \otimes \hat{\pi}(\mathfrak{a}'')) \oplus (\Lambda^0\boldsymbol{\gamma} \otimes \hat{\pi}(\Omega^1\mathfrak{a})) \;. \tag{3.31}$$

This means that elements $\tau^1 \in \pi(\Omega^1\mathfrak{g})$ are of the form

$$\tau^1 = \sum_\alpha \big(c^1_\alpha \otimes \hat{\pi}(a'_\alpha) + b^1_\alpha \otimes \hat{\pi}(a''_\alpha) + f_\alpha \boldsymbol{\gamma} \otimes \hat{\pi}(\omega^1_\alpha)\big) \;, \tag{3.32}$$

where $c^1_\alpha \in C^1$, $b^1_\alpha \in B^1$, $f_\alpha \in C^0$, $a'_\alpha \in \mathfrak{a}'$, $a''_\alpha \in \mathfrak{a}''$ and $\omega^1_\alpha \in \Omega^1\mathfrak{a}$.

Proposition 14.

$$\pi(\Omega^n\mathfrak{g}) = (\Lambda^n \otimes \hat{\pi}(\mathfrak{a}')) \oplus \Big(\bigoplus_{j=1}^n \Lambda^{n-j}\boldsymbol{\gamma}^j \otimes (\hat{\pi}(\Omega^j\mathfrak{a}) + \hat{\pi}(T^{j-2}_n\mathfrak{a}))\Big) \;, \tag{3.33}$$

for $n \geq 2$. Here, $\hat{\pi}(T^j_n\mathfrak{a})$ is zero for $j<0$, $n<j{+}2$ or $n>N{+}j{+}2$. For $j \geq 0$ and $j{+}2 \leq n \leq N{+}j{+}2$ it is recursively defined by

$$\begin{aligned}
&\hat{\pi}(T^0_2\mathfrak{a}) := \{\hat{\pi}(\mathfrak{a}),\hat{\pi}(\mathfrak{a})\}\;, \qquad \hat{\pi}(T^0_{N+2}\mathfrak{a}) := [\hat{\pi}(\mathfrak{a}),\{\hat{\pi}(\mathfrak{a}),\hat{\pi}(\mathfrak{a}')\}]\;, \\
&\hat{\pi}(T^0_n\mathfrak{a}) := \{\hat{\pi}(\mathfrak{a}),\hat{\pi}(\mathfrak{a}')\}\;, \qquad 3 \leq n \leq N{+}1\;,
\end{aligned} \tag{3.34a}$$

$$\begin{aligned}
\hat{\pi}(T^j_n\mathfrak{a}) &:= \{\hat{\pi}(\mathfrak{a}),\hat{\pi}(\Omega^j\mathfrak{a}) + \hat{\pi}(T^{j-2}_{j+1}\mathfrak{a})\} + [\hat{\pi}(\Omega^1\mathfrak{a}),\hat{\pi}(T^{j-1}_{j+1}\mathfrak{a})]_g\;, \\
&\qquad 2{+}j \leq n \leq N{+}j{+}1\;, \quad j>0\;, \\
\hat{\pi}(T^j_{N+j+2}\mathfrak{a}) &:= [\hat{\pi}(\mathfrak{a}),\hat{\pi}(T^j_{j+2}\mathfrak{a})] + [\hat{\pi}(\Omega^1\mathfrak{a}),\hat{\pi}(T^{j-1}_{N+j+1}\mathfrak{a})]_g\;, \quad j>0\;.
\end{aligned} \tag{3.34b}$$

Proof: The proposition is proved by induction. We need the following two identities:

$$\begin{aligned}(\tilde{c}^1\otimes \hat{\pi}(\tilde{a}))(c^{n-j}\boldsymbol{\gamma}^j \otimes A^j) - (-1)^n(c^{n-j}\boldsymbol{\gamma}^j \otimes A^j)(\tilde{c}^1 \otimes \hat{\pi}(\tilde{a}))\\ = \tfrac{1}{2}(\tilde{c}^1 c^{n-j} + (-1)^{n-j}c^{n-j}\tilde{c}^1)\boldsymbol{\gamma}^j \otimes (\hat{\pi}(\tilde{a})A^j - A^j\hat{\pi}(\tilde{a}))\\ +\tfrac{1}{2}(\tilde{c}^1 c^{n-j} - (-1)^{n-j}c^{n-j}\tilde{c}^1)\boldsymbol{\gamma}^j \otimes (\hat{\pi}(\tilde{a})A^j + A^j\hat{\pi}(\tilde{a}))\;,\end{aligned} \tag{3.35a}$$

$$\begin{aligned}(\tilde{f}\boldsymbol{\gamma} \otimes \hat{\pi}(\tilde{\omega}^1))(c^{n-j}\boldsymbol{\gamma}^j \otimes A^j) - (-1)^n(c^{n-j}\boldsymbol{\gamma}^j \otimes A^j)(\tilde{f}\boldsymbol{\gamma} \otimes \hat{\pi}(\tilde{\omega}^1))\\ = (-1)^{n-j}\tilde{f}c^{n-j}\boldsymbol{\gamma}^{j+1} \otimes (\hat{\pi}(\tilde{\omega}^1)A^j - (-1)^j A^j\hat{\pi}(\tilde{\omega}^1))\;,\end{aligned} \tag{3.35b}$$

for $\tilde{c}^1 \in \Lambda^1$, $c^n \in \Lambda^n$, $\tilde{f} \in \Lambda^0$, $\tilde{a} \in \mathfrak{a}$, $\tilde{\omega}^1 \in \Omega^1\mathfrak{a}$ and any $A^j \in \mathrm{M}_F\mathbb{C}$. We shall write (3.32) in the form

$$\tau^1 = \textstyle\sum_\alpha \big(c^1_\alpha \otimes \hat{\pi}(a_\alpha) + f_\alpha\boldsymbol{\gamma} \otimes \hat{\pi}(\omega_\alpha)\big)\;,$$

where $\sum_\alpha c^1_\alpha \otimes \hat{\pi}(a_\alpha) \equiv \sum_\alpha \big(c^{1\prime}_\alpha \otimes \hat{\pi}(a'_\alpha) + c^{1\prime\prime}_\alpha \otimes \hat{\pi}(a''_\alpha)\big)$.

Using (3.35a), (3.35b) and Lemma 12 we obtain from (2.26) the following form of elements $\tau^2 \in \pi(\Omega^2\mathfrak{g})$:

$$\begin{aligned}\tau^2 &= \sum\nolimits_\alpha (\tau^1_\alpha\tilde{\tau}^1_\alpha + \tilde{\tau}^1_\alpha\tau^1_\alpha)\\ &= \sum\nolimits_{\alpha,\beta,\gamma} \big(c^1_{\alpha\beta} \wedge \tilde{c}^1_{\alpha\gamma} \otimes [\hat{\pi}(a_{\alpha\beta}),\hat{\pi}(\tilde{a}_{\alpha\gamma})] + f_{\alpha\beta}\tilde{f}_{\alpha\gamma} \otimes [\hat{\pi}(\omega^1_{\alpha\beta}),\hat{\pi}(\tilde{\omega}^1_{\alpha\gamma})]_g\\ &\quad +\tilde{f}_{\alpha\gamma}c^1_{\alpha\beta}\boldsymbol{\gamma} \otimes [\hat{\pi}(a_{\alpha\beta}),\hat{\pi}(\tilde{\omega}^1_{\alpha\gamma})] + f_{\alpha\beta}\tilde{c}^1_{\alpha\gamma}\boldsymbol{\gamma} \otimes [\hat{\pi}(\tilde{a}_{\alpha\gamma}),\hat{\pi}(\omega^1_{\alpha\beta})]\big) + \kappa^0\;,\end{aligned} \tag{3.36a}$$

$$\kappa^0 = \sum\nolimits_{\alpha,\beta,\gamma} c^1_{\alpha\beta} \lrcorner\, \tilde{c}^1_{\alpha\gamma} \otimes \{\hat{\pi}(a_{\alpha\beta}),\hat{\pi}(\tilde{a}_{\alpha\gamma})\}\;. \tag{3.36b}$$

All five occurring different types of tensor products are independent. This is due to the fact that for non–vanishing $\tilde{c}^1 \in \Lambda^1$ and $c^n \in \Lambda^n$ the equality $\tilde{c}^1 \wedge c^n = 0$ implies $\tilde{c}^1 \lrcorner\, c^n \neq 0$ and $\tilde{c}^1 \lrcorner\, c^n = 0$ implies $\tilde{c}^1 \wedge c^n \neq 0$, see Lemma 12. First, κ^0 attains each element of $\Lambda^0 \otimes \{\hat{\pi}(\mathfrak{a}),\hat{\pi}(\mathfrak{a})\}$. Moreover, $\sum_\alpha f\tilde{f} \otimes [\hat{\pi}(\omega^1_\alpha),\hat{\pi}(\tilde{\omega}^1_\alpha)]_g$ gives an arbitrary element of $\Lambda^0 \otimes \hat{\pi}(\Omega^2\mathfrak{a})$ and each term in (3.36a) containing $\boldsymbol{\gamma}$ an arbitrary element of $\Lambda^1\boldsymbol{\gamma} \otimes \hat{\pi}(\Omega^1\mathfrak{a})$. The only not obvious elements are those of the form $[\mathcal{M},\hat{\pi}(a)]$. However, they can be represented by (3.27) for $a = a''$ and for $a = a'$ due to (3.30) by

$$[\mathcal{M},\hat{\pi}(\textstyle\sum_\alpha [a'_\alpha,\tilde{a}'_\alpha])] = \sum_\alpha([[\mathcal{M},\hat{\pi}(a'_\alpha)],\hat{\pi}(\tilde{a}'_\alpha)] + [\hat{\pi}(a'_\alpha),[\mathcal{M},\hat{\pi}(\tilde{a}'_\alpha)]])\;. \tag{3.37}$$

Finally, $\sum_{\alpha,\beta,\gamma} c^1_{\alpha\beta} \wedge \tilde{c}^1_{\alpha\gamma} \otimes [\hat{\pi}(a'_{\alpha\beta}),\hat{\pi}(\tilde{a}'_{\alpha\gamma})]$ represents an arbitrary element of $\Lambda^2 \otimes \hat{\pi}(\mathfrak{a}')$, because possible contributions from $\mathfrak{a}''$ are cancelled by the commutator. Collecting these results, we arrive at (3.33), for $n = 2$. For $n > 2$ one proceeds by induction, see Ref. 15. □

Thus, the computation of $\pi(\Omega^n\mathfrak{g})$ is reduced to an iterative multiplication of matrices only.

D. Main Theorem

To derive the structure of $\Omega^*_D\mathfrak{g}$ we first define in analogy to (2.40)

$$\hat\sigma(\sum_{\alpha,z\geq 0}[\iota(a^z_\alpha),[\dots[\iota(a^1_\alpha),\iota(da^0_\alpha)]\dots]]) := \sum_{\alpha,z\geq 0}[\hat\pi(a^z_\alpha),[\dots[\hat\pi(a^1_\alpha),[\mathcal{M}^2,\hat\pi(a^0_\alpha)]]\dots]] \;, \tag{3.38}$$

for $a^i_\alpha\in\mathfrak{a}$. We extend $\hat\sigma$ to a linear map $\hat\sigma_\mathfrak{g}:\Omega^*\mathfrak{g}\to\Gamma^\infty(C)\otimes \mathrm{M}_F\mathbb{C}$ by

$$\begin{aligned}&\hat\sigma_\mathfrak{g}(\iota(f\otimes a)) := 0\;, \qquad \hat\sigma_\mathfrak{g}(\iota(d(f\otimes a))) := f\otimes\hat\sigma(\iota(da))\;,\\ &\hat\sigma_\mathfrak{g}([\omega^k,\tilde\omega^l]) := [\hat\sigma_\mathfrak{g}(\omega^k),\pi(\tilde\omega^l)]_g + (-1)^k[\pi(\omega^k),\hat\sigma_\mathfrak{g}(\tilde\omega^l)]_g\;,\end{aligned} \tag{3.39}$$

for $f\in C^\infty(X)\,,\; a\in\mathfrak{a}\,,\; \omega^k\in\Omega^k\mathfrak{g}$ and $\tilde\omega^l\in\Omega^l\mathfrak{g}$.

Theorem 15. *For* $\{\hat\pi(\mathfrak{a}''),\hat\pi(\mathfrak{a}'')\}\cap\hat\pi(\Omega^2\mathfrak{a})=0$ *we have*

$$\begin{aligned}\pi(\mathcal{J}^n\mathfrak{g}) = &\bigoplus_{j=2}^n \Lambda^{n-j}\boldsymbol{\gamma}^j\otimes(\hat\pi(\mathcal{J}^j\mathfrak{a})+\tilde K^{j-2}_n\mathfrak{a})\\ &+B^N\boldsymbol{\gamma}^n\otimes(\{\hat\pi(\mathfrak{a}),\hat\pi(\Omega^{n-N-2}\mathfrak{a})+\hat\pi(T^{n-N-4}_{n-2}\mathfrak{a})\}\cap\hat\pi(\Omega^{n-N}\mathfrak{a}))\;,\end{aligned} \tag{3.40}$$

where $B^N=\mathbf{d}\Lambda^{N-1}\,,\; \tilde K^0_n\mathfrak{a}\equiv\hat\pi(T^0_n\mathfrak{a})$ *and*

$$\begin{aligned}\tilde K^j_n\mathfrak{a} = &\{\hat\pi(\mathfrak{a}),\hat\pi(\Omega^j\mathfrak{a})+\tilde K^{j-2}_{n-1}\mathfrak{a}\}+[\hat\pi(\Omega^1\mathfrak{a}),\tilde K^{j-1}_{n-1}\mathfrak{a}]_g \qquad\qquad (3.41\text{a})\\ &+\hat\sigma(\hat\pi^{-1}(\hat\pi(T^{j-1}_{j+1}\mathfrak{a})\cap\hat\pi(\Omega^{j+1}\mathfrak{a})))\;,\quad 2{+}j\leq n\leq N{+}j{+}1\;,\quad j>0\;,\\ \tilde K^j_{N+j+2}\mathfrak{a} = &[\hat\pi(\mathfrak{a}),\tilde K^j_{N+j+1}\mathfrak{a}]+[\hat\pi(\Omega^1\mathfrak{a}),\tilde K^{j-1}_{N+j+1}\mathfrak{a}]_g \qquad\qquad (3.41\text{b})\\ &+\hat\sigma(\hat\pi^{-1}(\hat\pi(T^{j-1}_{N+j+1}\mathfrak{a})\cap\hat\pi(\Omega^{j+1}\mathfrak{a})))\;,\quad j>0\;.\end{aligned}$$

If $\{\hat\pi(\mathfrak{a}''),\hat\pi(\mathfrak{a}'')\}\cap\hat\pi(\Omega^2\mathfrak{a})\neq 0$ *then* $\pi(\mathcal{J}^3\mathfrak{g})$ *must be replaced by*

$$\pi(\mathcal{J}^3\mathfrak{g}) = \pi(\mathcal{J}^3\mathfrak{g})\restriction_{(3.40)} + B^1\otimes(\{\hat\pi(\mathfrak{a}''),\hat\pi(\mathfrak{a}'')\}\cap\hat\pi(\Omega^2\mathfrak{a}))\;.$$

Proof: The proof consists in deriving a formula for $\sigma(\omega^k)$ for a given $\omega^k\in\Omega^k\mathfrak{g}$. Taking $\omega^k\in\Omega^k\mathfrak{g}\cap\ker\pi$, we can derive the structure of $\pi(\mathcal{J}^{k+1}\mathfrak{g})$, see (2.40). We start with $k=1$ and proceed for higher degrees by induction.

Before, we provide a property of $\hat\pi(\Omega^1\mathfrak{a})$ which we need in the proof. We consider the splitting

$$\hat\omega^1 = d(\iota(a')+\iota(a''))+\textstyle\sum_{\alpha,z\geq 1}[\iota(a^z_\alpha),[\dots,[\iota(a^2_\alpha),[\iota(a^1_\alpha),\iota(da^0_\alpha)]]\dots]]\in\Omega^1\mathfrak{a}\;,$$

for $a'=\sum_\beta[a'_\beta,\tilde a'_\beta]\in\mathfrak{a}'$ and $a''\in\mathfrak{a}''$. Due to (3.27) and (3.30) we can replace $\omega^1_0:=\iota(d(a'+a''))$ by

$$\begin{aligned}\hat\omega^1_0 = &\pm\tfrac{5}{4}[\iota(\mathrm{b}),[\iota(\mathrm{b}),\iota(da'')]]-\tfrac{1}{4}[\iota(\mathrm{b}),[\iota(\mathrm{b}),[\iota(\mathrm{b}),[\iota(\mathrm{b}),\iota(da'')]]]]\\ &+\textstyle\sum_\beta\big([\iota(a'_\beta),\iota(d\tilde a'_\beta)]-[\iota(\tilde a'_\beta),\iota(da'_\beta)]\big)\;.\end{aligned}$$

Here, in the first term the plus sign (minus sign) stands if in (3.27) the equation with the plus sign (minus sign) is realized. Indeed, we have

$$\hat{\pi}(\hat{\omega}_0^1) \equiv \hat{\pi}(\omega_0^1) \;, \qquad\qquad \hat{\sigma}(\hat{\omega}_0^1) \equiv \hat{\sigma}(\omega_0^1) \;. \tag{3.42}$$

The first formula is due to (3.27) for a'' and due to the Jacobi identity for a'. The $\mathfrak{a}'$–part of the second formula in (3.42) follows immediately from the Jacobi identity. The proof for the $\mathfrak{a}''$–part consists of algebraic manipulations of (3.27), which are not difficult but rather lengthy so that they are not listed in this work. The importance of the identities (3.42) is that already elements of $\Omega^1\mathfrak{a}$, which do not contain terms labelled by $z=0$, are sufficient for the construction of $\hat{\pi}(\Omega^1\mathfrak{a})$ and $\hat{\sigma}(\Omega^1\mathfrak{a})$.

Using (3.29) we can represent elements $\omega^1 \in \Omega^1\mathfrak{g}$ as

$$\begin{aligned}
&\omega^1 = \textstyle\sum_{\alpha,z\geq 0}[\iota(f_\alpha^z\otimes a_\alpha^z),[\dots[\iota(f_\alpha^1\otimes a_\alpha^1),\iota(d(f_\alpha^0\otimes a_\alpha^0))]\dots]] \;, && (3.43\text{a})\\
&\Rightarrow \qquad \pi(\omega^1) = \textstyle\sum_{\alpha,z\geq 0}\big(\hat{c}_\alpha^{1,z}\otimes\hat{\pi}(\hat{a}_\alpha^z) + \hat{f}_\alpha^z\boldsymbol{\gamma}\otimes\hat{\pi}(\hat{\omega}_\alpha^{1,z})\big) \;, &&\\
&\hat{f}_\alpha^z = f_\alpha^z\cdots f_\alpha^1 f_\alpha^0 \in \Lambda^0 \;, \qquad \hat{c}_\alpha^{1,z} = f_\alpha^z\cdots f_\alpha^1\mathbf{d}f_\alpha^0 \in \Lambda^1 \;, && (3.43\text{b})\\
&\hat{a}_\alpha^z = [a_\alpha^z,[\dots[a_\alpha^1,a_\alpha^0]\dots]] \in \mathfrak{a} \;, \quad \hat{\omega}_\alpha^{1,z} = [\iota(a_\alpha^z),[\dots[\iota(a_\alpha^1),\iota(da_\alpha^0)]\dots]] \in \Omega^1\mathfrak{a} \;, &&
\end{aligned}$$

where $a_\alpha^i\in\mathfrak{a}$ and $f_\alpha^i\in\Lambda^0$. Applying the map σ to ω^1 in (3.43a) we get – using (3.23) and $D^2 \equiv \mathsf{D}^2\otimes\mathbb{1}_F + 1\otimes\mathcal{M}^2$, see (3.4) –

$$\begin{aligned}
&\sigma(\omega^1) = \textstyle\sum_{\alpha,z\geq 0}[f_\alpha^z\otimes\hat{\pi}(a_\alpha^z),[\dots[f_\alpha^1\otimes\hat{\pi}(a_\alpha^1),[D^2,f_\alpha^0\otimes\hat{\pi}(a_\alpha^0)]]\dots]] \equiv \sum_{j=0}^3 s_j \;, &&\\
&s_0 = \hat{\sigma}_{\mathfrak{g}}(\omega^1) = \textstyle\sum_{\alpha,z\geq 0} f_\alpha^z\cdots f_\alpha^1 f_\alpha^0\otimes[\hat{\pi}(a_\alpha^z),[\dots[\hat{\pi}(a_\alpha^1),[\mathcal{M}^2,\hat{\pi}(a_\alpha^0)]]\dots]] \;, && (3.44\text{a})\\
&s_1 = \textstyle\sum_{\alpha,z\geq 0} f_\alpha^z\cdots f_\alpha^1(\Delta f_\alpha^0)\otimes\hat{\pi}([a_\alpha^z,[\dots[a_\alpha^1,a_\alpha^0]\dots]]) \;, && (3.44\text{b})\\
&s_2 = -2\textstyle\sum_{\alpha,z\geq 0} f_\alpha^z\cdots f_\alpha^1\nabla^S_{\mathrm{grad}\,f_\alpha^0}\otimes\hat{\pi}([a_\alpha^z,[\dots[a_\alpha^1,a_\alpha^0]]\dots]]) \;, && (3.44\text{c})\\
&s_3 = 2\textstyle\sum_{\alpha,z\geq 1}\big(f_\alpha^z\cdots f_\alpha^2\nabla_{\mathrm{grad}\,f_\alpha^0}(f_\alpha^1)\otimes[\hat{\pi}(a_\alpha^z),[\dots[\hat{\pi}(a_\alpha^2),\hat{\pi}(a_\alpha^0)\hat{\pi}(a_\alpha^1)]\dots]] &&\\
&\quad + f_\alpha^z\cdots f_\alpha^3\nabla_{\mathrm{grad}\,f_\alpha^0}(f_\alpha^2)f_\alpha^1\otimes[\hat{\pi}(a_\alpha^z),[\dots[\hat{\pi}(a_\alpha^3),\hat{\pi}([a_\alpha^1,a_\alpha^0])\hat{\pi}(a_\alpha^2)]\dots]] &&\\
&\quad + \dots + \nabla_{\mathrm{grad}\,f_\alpha^0}(f_\alpha^z)f_\alpha^{z-1}\cdots f_\alpha^1\otimes\hat{\pi}([a_\alpha^{z-1},[\dots[a_\alpha^1,a_\alpha^0]\dots]])\hat{\pi}(a_\alpha^z)\big) \;. && (3.44\text{d})
\end{aligned}$$

From properties of covariant derivatives we find

$$f_\alpha^z\cdots f_\alpha^1\nabla^S_{\mathrm{grad}\,f_\alpha^0} = \nabla^S_{f_\alpha^z\cdots f_\alpha^1 g^{-1}(\mathsf{d}f_\alpha^0)} = \nabla^S_{g^{-1}(f_\alpha^z\cdots f_\alpha^1\mathsf{d}f_\alpha^0)} \;.$$

Next, using (3.13) and (3.16) one easily shows

$$f_\alpha^z\cdots f_\alpha^1(\Delta f_\alpha^0) = \mathbf{d}^*(f_\alpha^z\cdots f_\alpha^1\mathbf{d}f_\alpha^0) + \nabla_{\mathrm{grad}\,f_\alpha^0}(f_\alpha^z\cdots f_\alpha^1) \;. \tag{3.45}$$

Then, the sum of s_3 and the part of s_1 corresponding to the second term on the r.h.s. of (3.45) will be denoted by $\hat{s}(\omega^1)$:

$$\begin{aligned}
\hat{s}(\omega^1) &= s_3 + \textstyle\sum_{\alpha,z\geq 1}\nabla_{\mathrm{grad}\,f_\alpha^0}(f_\alpha^z\cdots f_\alpha^1)\otimes\hat{\pi}(\hat{a}_\alpha^z) \qquad\qquad (3.46)\\
&= \textstyle\sum_{\alpha,z\geq 1}\big(f_\alpha^z\cdots f_\alpha^2\nabla_{\mathrm{grad}\,f_\alpha^0}(f_\alpha^1)\otimes[\hat{\pi}(a_\alpha^z),[\dots[\hat{\pi}(a_\alpha^2),\{\hat{\pi}(a_\alpha^0),\hat{\pi}(a_\alpha^1)\}]\dots]]\\
&\quad + f_\alpha^z\cdots f_\alpha^3\nabla_{\mathrm{grad}\,f_\alpha^0}(f_\alpha^2)f_\alpha^1\otimes[\hat{\pi}(a_\alpha^z),[\dots[\hat{\pi}(a_\alpha^3),\{\hat{\pi}([a_\alpha^1,a_\alpha^0]),\hat{\pi}(a_\alpha^2)\}]\dots]]\\
&\quad + \dots + \nabla_{\mathrm{grad}\,f_\alpha^0}(f_\alpha^z)f_\alpha^{z-1}\cdots f_\alpha^1\otimes\{\hat{\pi}([a_\alpha^{z-1},[\dots[a_\alpha^1,a_\alpha^0]\dots]]),\hat{\pi}(a_\alpha^z)\}\big)\\
&\in \Lambda^0\otimes\{\hat{\pi}(\mathfrak{a}),\hat{\pi}(\mathfrak{a})\} \;.
\end{aligned}$$

Observe that the terms labelled by $z=0$ do not occur in (3.46). Collecting the results we find

$$\sigma(\omega^1) = \hat{s}(\omega^1) + \hat{\sigma}_{\mathfrak{g}}(\omega^1) + \sum_{\alpha,z\geq 0} \big(-2\nabla^S_{g^{-1}\circ c^{-1}(\hat{c}^{1,z}_\alpha)} \otimes \hat{\pi}(\hat{a}^z_\alpha) + \mathbf{d}^*(\hat{c}^{1,z}_\alpha) \otimes \hat{\pi}(\hat{a}^z_\alpha)\big) \;. \tag{3.47}$$

Next, we discuss the relation between $\pi(\omega^1)$ and $\sigma(\omega^1)$. It is clear that $\hat{s}(\omega^1) \in \Lambda^0 \otimes \{\hat{\pi}(\mathfrak{a}), \hat{\pi}(\mathfrak{a})\}$ and $\hat{\sigma}_{\mathfrak{g}}(\omega^1) \in \Lambda^0 \otimes \hat{\sigma}(\Omega^1\mathfrak{a})$, the question is to which amount they are determined by $\pi(\omega^1)$. To answer this question we first consider

$$\omega^1 = \textstyle\sum_\alpha \sum_{A=1}^3 [\iota(\tilde{\mathrm{f}}_{\alpha A} \otimes \tilde{a}_\alpha), \iota(d(\mathrm{f}_{\alpha A} \otimes a_\alpha))] \;, \quad a_\alpha, \tilde{a}_\alpha \in \mathfrak{a} \;, \tag{3.48}$$

where

$$\begin{aligned} &\tilde{\mathrm{f}}_{\alpha 1} = f_\alpha \;, && \tilde{\mathrm{f}}_{\alpha 2} = -\tfrac{1}{2} \;, && \tilde{\mathrm{f}}_{\alpha 3} = -\tfrac{1}{2}(f_\alpha)^2 \;, \\ &\mathrm{f}_{\alpha 1} = f_\alpha \tilde{f}_\alpha \;, && \mathrm{f}_{\alpha 2} = (f_\alpha)^2 \tilde{f}_\alpha \;, && \mathrm{f}_{\alpha 3} = \tilde{f}_\alpha \;, \end{aligned}$$

for $f_\alpha, \tilde{f}_\alpha \in C^\infty(X)$. These functions have the properties

$$\textstyle\sum_{A=1}^3 \tilde{\mathrm{f}}_{\alpha A} \mathrm{f}_{\alpha A} = 0 \;, \qquad \sum_{A=1}^3 \tilde{\mathrm{f}}_{\alpha A} \mathbf{d}(\mathrm{f}_{\alpha A}) = 0 \;, \qquad \sum_{A=1}^3 \mathbf{d}(\tilde{\mathrm{f}}_{\alpha A}) \mathrm{f}_{\alpha A} = 0 \;, \tag{3.49a}$$

$$\textstyle\sum_{A=1}^3 \nabla_{\mathrm{grad}(\tilde{\mathrm{f}}_{\alpha A})}(\mathrm{f}_{\alpha A}) = \tilde{f}_\alpha \nabla_{\mathrm{grad}(f_\alpha)}(f_\alpha) = \tilde{f}_\alpha g^{-1}(\mathsf{d} f_\alpha, \mathsf{d} f_\alpha) \;. \tag{3.49b}$$

Due to (3.49a) we have $\pi(\omega^1) = 0$ and $\hat{\sigma}_{\mathfrak{g}}(\omega^1) = 0$, but for (3.46) we get

$$\begin{aligned} \hat{s}(\omega^1) &\equiv \textstyle\sum_\alpha \sum_{A=1}^3 \nabla_{\mathrm{grad}\, \tilde{\mathrm{f}}_{\alpha A}}(\mathrm{f}_{\alpha A}) \otimes \{\hat{\pi}(\tilde{a}_\alpha), \hat{\pi}(a_\alpha)\} \\ &= \textstyle\sum_\alpha \tilde{f}_\alpha \nabla_{\mathrm{grad}\, f_\alpha}(f_\alpha) \otimes \{\hat{\pi}(\tilde{a}_\alpha), \hat{\pi}(a_\alpha)\} \;. \end{aligned}$$

Thus, $\hat{s}(\omega^1)$ is independent of $\pi(\omega^1)$. Since (3.49b) – for an appropriate choice of $f_\alpha, \tilde{f}_\alpha$ – attains each given function on X (using a partition of unity if necessary), $\hat{s}(\omega^1)$ attains each element of $\Lambda^0 \otimes \{\hat{\pi}(\mathfrak{a}), \hat{\pi}(\mathfrak{a})\} \equiv \Lambda^0 \otimes \hat{\pi}(T^0_2\mathfrak{a})$. Now we prove

Lemma 16. $\hat{\sigma}_{\mathfrak{g}}(\ker \pi \cap \Omega^1\mathfrak{g}) = \Lambda^0 \otimes \hat{\sigma}(\ker \hat{\pi} \cap \Omega^1\mathfrak{a}) \equiv \Lambda^0 \otimes \hat{\pi}(\mathcal{J}^2\mathfrak{a})$.

Proof of Lemma 16: We introduce a linear map $\hat{\pi}_{\mathfrak{g}} : \Omega^*\mathfrak{g} \to \mathscr{B}(h)$ by

$$\begin{aligned} &\hat{\pi}_{\mathfrak{g}}(\iota(f \otimes a)) := f \otimes \hat{\pi}(a) \;, \qquad \hat{\pi}_{\mathfrak{g}}(\iota(d(f \otimes a))) := f\boldsymbol{\gamma} \otimes [-\mathrm{i}\mathcal{M}, \hat{\pi}(a)] \;, \\ &\hat{\pi}_{\mathfrak{g}}([\omega, \tilde{\omega}]) := [\hat{\pi}_{\mathfrak{g}}(\omega), \hat{\pi}_{\mathfrak{g}}(\tilde{\omega})]_g \;, \end{aligned}$$

for $f \in C^\infty(X)$, $a \in \mathfrak{a}$, $\omega, \tilde{\omega} \in \Omega^*\mathfrak{g}$. For $\omega^1 \in \Omega^1\mathfrak{g}$ given by (3.43a) we have

$$\begin{aligned} \pi(\omega^1) &= \textstyle\sum_{\alpha,z\geq 0} \big(\hat{c}^{1,z}_\alpha \otimes \hat{\pi}(\hat{a}^z_\alpha) + \hat{f}^z_\alpha \boldsymbol{\gamma} \otimes \hat{\pi}(\hat{\omega}^{1,z}_\alpha)\big) \;, \\ \hat{\pi}_{\mathfrak{g}}(\omega^1) &= \textstyle\sum_{\alpha,z\geq 0} \hat{f}^z_\alpha \boldsymbol{\gamma} \otimes \hat{\pi}(\hat{\omega}^{1,z}_\alpha) \;, \\ \hat{\sigma}_{\mathfrak{g}}(\omega^1) &= \textstyle\sum_{\alpha,z\geq 0} \hat{f}^z_\alpha \otimes \hat{\sigma}(\hat{\omega}^{1,z}_\alpha) \;. \end{aligned} \tag{3.50}$$

For $\omega^1 \in \ker\pi$ we have $\sum_{\alpha,z\geq 0} \hat{c}_\alpha^{1,z} \otimes \hat{\pi}(\hat{a}_\alpha^z) = 0$ and $\sum_{\alpha,z\geq 0} \hat{f}_\alpha^z \boldsymbol{\gamma} \otimes \hat{\pi}(\hat{\omega}_\alpha^{1,z}) = 0$, because Λ^1 and Λ^0 are independent. But this means

$$(\ker\pi \cap \Omega^1\mathfrak{g}) \subset (\ker\hat{\pi}_\mathfrak{g} \cap \Omega^1\mathfrak{g}) \;\Rightarrow\; \hat{\sigma}_\mathfrak{g}(\ker\pi \cap \Omega^1\mathfrak{g}) \subset \hat{\sigma}_\mathfrak{g}(\ker\hat{\pi}_\mathfrak{g} \cap \Omega^1\mathfrak{g}) \;. \quad (3.51)$$

It is intuitively clear from (3.50) that

$$\hat{\sigma}_\mathfrak{g}(\ker\hat{\pi}_\mathfrak{g} \cap \Omega^1\mathfrak{g}) = \Lambda^0 \otimes \hat{\sigma}(\ker\hat{\pi} \cap \Omega^1\mathfrak{a}) \equiv \Lambda^0 \otimes \hat{\pi}(\mathcal{J}^2\mathfrak{a}) \;, \quad (3.52)$$

see (2.44). The justification for (3.52) gives the formalism of skew–tensor products, see Ref. 16 for the general scheme and Ref. 15 for the application to our case. Now, by virtue of (3.42) it suffices to take

$$\omega^1 = \textstyle\sum_\alpha \sum_{\beta,z\geq 1} [\iota(1\otimes a_{\alpha\beta}^z), [\dots, [\iota(1\otimes a_{\alpha\beta}^2), [\iota(f_\alpha \otimes a_{\alpha\beta}^1), \iota(d(1\otimes a_{\alpha\beta}^0))]]\dots]] \;,$$

with

$$\hat{\omega}_\alpha^1 := \textstyle\sum_{\beta,z\geq 1} [\iota(a_{\alpha\beta}^z), [\dots, [\iota(a_{\alpha\beta}^2), [\iota(a_{\alpha\beta}^1), \iota(da_{\alpha\beta}^0)]]\dots]] \in \ker\hat{\pi} \cap \Omega^1\mathfrak{a} \;, \quad \forall\alpha \,,$$

where $f_\alpha \in \Lambda^0$ and $a_{\alpha\beta}^i \in \mathfrak{a}$. It is obvious that $\pi(\omega^1) \equiv 0$ and that $\sigma(\omega^1) = \hat{\sigma}_\mathfrak{g}(\omega^1) = \sum_\alpha f_\alpha \otimes \hat{\sigma}(\hat{\omega}_\alpha^1)$ attains each element of $\Lambda^0 \otimes \hat{\pi}(\mathcal{J}^2\mathfrak{a})$. □

(Lemma 16)

We define a linear map ∇_Ω from $\pi(\Omega^*\mathfrak{g})$ to (unbounded) operators on h,

$$\begin{aligned} \nabla_\Omega(c^{n-j}\boldsymbol{\gamma}^j \otimes A^j) &:= \nabla^S_{c^{n-j}}\boldsymbol{\gamma}^j \otimes A^j \;, \quad n-j > 0 \;, \\ \nabla_\Omega(f\boldsymbol{\gamma}^n \otimes A^n) &:= 0 \;, \quad f \in C^\infty(X) \;, \end{aligned} \quad (3.53)$$

where $c^{n-j} \in \Lambda^{n-j}$ and $A^j \in \mathrm{M}_F\mathbb{C}$. Here and in the sequel a covariant derivative with respect to elements of Λ^n is understood in the sense

$$\nabla_{c_1^1\wedge c_2^1\wedge\dots\wedge c_n^1} := \textstyle\sum_{l=1}^k (-1)^{l+1} c_1^1 \wedge \overset{l}{\overset{\vee}{\dots}} \wedge c_n^1 \nabla_{g^{-1}\circ c^{-1}(c_l^1)} \;, \quad c_i^1 \in \Lambda^1 \;, \quad (3.54)$$

where $c^{-1} : \Lambda^1 \to \Gamma^\infty(T^*X)$ and $g^{-1} : \Gamma^\infty(T^*X) \to \Gamma^\infty(T_*X)$ are isomorphisms.

Now we can express (3.47) in terms of $\pi(\omega^1)$. For given $\tau^1 \in \pi(\Omega^1\mathfrak{g})$ let $\pi^{-1}(\tau^1) \in \Omega^1\mathfrak{g}$ be an arbitrary but fixed representative and $\omega^1 \in \Omega^1\mathfrak{g}$ be any representative. Then, the set $\{\sigma(\omega^1)\}$ of all elements $\sigma(\omega^1)$ fulfilling the just introduced conditions is

$$\{\sigma(\omega^1)\} = \Lambda^0 \otimes (\hat{\pi}(T_2^0\mathfrak{a}) + \hat{\pi}(\mathcal{J}^2\mathfrak{a})) + \hat{\sigma}_\mathfrak{g}(\pi^{-1}(\tau^1)) - 2\nabla_\Omega(\tau^1) + \mathbf{d}^*\tau^1 \;. \quad (3.55)$$

Putting $\tau^1 = 0$, i.e. $\omega^1 \in \ker\pi \cap \Omega^1\mathfrak{g}$, we obtain immediately the assertion of the theorem for $n = 2$.

Formula (3.55) is the starting point for the construction of $\sigma(\Omega^n\mathfrak{g})$, $n \geq 2$, out of (2.42). The result is:

Lemma 17. *For given $\tau^n \in \pi(\Omega^n\mathfrak{g})$ let $\pi^{-1}(\tau^n) \in \Omega^n\mathfrak{g}$ be an arbitrary but fixed representative and $\omega^n \in \Omega^n\mathfrak{g}$ be any representative. Then we have for $n=2$*

$$\begin{aligned}\{\sigma(\omega^2)\} &= \Lambda^1 \otimes (\hat\pi(T_3^0\mathfrak{a}) + \hat\pi(\mathcal{J}^2\mathfrak{a})) + \Lambda^0\boldsymbol{\gamma} \otimes (\tilde K_3^1\mathfrak{a} + \hat\pi(\mathcal{J}^3\mathfrak{a})) \\ &\quad + \hat\sigma_{\mathfrak{g}}(\pi^{-1}(\tau^2)) - 2\nabla_\Omega(\tau^2) + \mathbf{d}^*\tau^2 - \mathbf{d}\big(\tau^2 \restriction_{\Lambda^0\otimes\{\hat\pi(\mathfrak{a}''),\hat\pi(\mathfrak{a}'')\}}\big)\end{aligned} \tag{3.56}$$

and for $n \geq 3$

$$\begin{aligned}\{\sigma(\omega^n)\} &= \hat\sigma_{\mathfrak{g}}(\pi^{-1}(\tau^n)) - 2\nabla_\Omega(\tau^n) + \mathbf{d}^*\tau^n + \sum_{j=2}^{n+1} \Lambda^{n+1-j}\boldsymbol{\gamma}^j \otimes (\tilde K_{n+1}^{j-2}\mathfrak{a} + \hat\pi(\mathcal{J}^j\mathfrak{a})) \\ &\quad - \mathbf{d}\big(\tau^n \restriction_{\Lambda^{N-1}\boldsymbol{\gamma}^{n+1}\otimes\{\hat\pi(\mathfrak{a}),\hat\pi(\Omega^{n-N-1}\mathfrak{a})+\hat\pi(T_{n-1}^{n-N-3}\mathfrak{a})\}}\big)\ .\end{aligned} \tag{3.57}$$

Remarks on the proof of Lemma 17: The Lemma is proved by induction exploiting formula (2.42). The proof is very technical and too long to display in this work. For the details see Ref. 15. It is clear that the proof of Lemma 17 finishes the proof of Theorem 15. Here, for $n=2$, one has to take into account that for $\{\hat\pi(\mathfrak{a}''),\hat\pi(\mathfrak{a}'')\} \cap \hat\pi(\Omega^2\mathfrak{a}) = 0$ and $\tau^2 = 0$ we have $\mathbf{d}\big(\tau^2 \restriction_{\Lambda^0\otimes\{\hat\pi(\mathfrak{a}''),\hat\pi(\mathfrak{a}'')\}}\big) = 0$. If $\{\hat\pi(\mathfrak{a}''),\hat\pi(\mathfrak{a}'')\} \cap \hat\pi(\Omega^2\mathfrak{a}) \neq 0$ then a non–vanishing $\Lambda^0 \otimes \{\hat\pi(\mathfrak{a}''),\hat\pi(\mathfrak{a}'')\}$–part of $\tau^2 = 0$ can be compensated by $\Lambda^0 \otimes \hat\pi(\Omega^2\mathfrak{a})$, giving the contribution $B^1 \otimes (\{\hat\pi(\mathfrak{a}''),\hat\pi(\mathfrak{a}'')\} \cap \hat\pi(\Omega^2\mathfrak{a}))$ to the ideal $\pi(\mathcal{J}^3\mathfrak{g})$. The same argumentation yields the boundary terms in the second line of (3.40). □

E. The Structure of $\Omega_D^*\mathfrak{g}$, Commutator and Differential

As an immediate consequence of Theorem 15 we find

Corollary 18. *If $\{\hat\pi(\mathfrak{a}''),\hat\pi(\mathfrak{a}'')\} \cap \hat\pi(\Omega^2\mathfrak{a}) = 0$ we have for $n \geq 2$*

$$\begin{aligned}\Omega_D^n\mathfrak{g} &= \big(\Lambda^n \otimes \hat\pi(\mathfrak{a}')\big) \oplus \big(\Lambda^{n-1}\boldsymbol{\gamma} \otimes \hat\pi(\Omega^1\mathfrak{a})\big) \oplus \\ &\quad \oplus \bigoplus_{j=2}^{n} \big(\Lambda^{n-j}\boldsymbol{\gamma}^j \otimes \big((\hat\pi(\Omega^j\mathfrak{a}) + \hat\pi(T_n^{j-2}\mathfrak{a})) \mod (\hat\pi(\mathcal{J}^j\mathfrak{a}) + \tilde K_n^{j-2}\mathfrak{a}\big)\big) \\ &\quad \mod \delta_{n-N}^j B^N\boldsymbol{\gamma}^n \otimes (\{\hat\pi(\mathfrak{a}),\hat\pi(\Omega^{n-N-2}\mathfrak{a}) + \hat\pi(T_{n-1}^{n-N-4}\mathfrak{a})\} \cap \hat\pi(\Omega^{n-N}\mathfrak{a}))\big)\ .\end{aligned} \tag{3.58}$$

If $\{\hat\pi(\mathfrak{a}''),\hat\pi(\mathfrak{a}'')\} \cap \hat\pi(\Omega^2\mathfrak{a}) \neq 0$ then $\Omega_D^3\mathfrak{g}$ must be replaced by

$$\Omega_D^3\mathfrak{g} = \Omega_D^3\mathfrak{g} \restriction_{(3.58)} \mod B^1 \otimes (\{\hat\pi(\mathfrak{a}''),\hat\pi(\mathfrak{a}'')\} \cap \hat\pi(\Omega^2\mathfrak{a}))\ .$$ □

Therefore, the construction of $\Omega_D^n\mathfrak{g}$ is reduced to the problem of finding the factor space $(\hat\pi(\Omega^j\mathfrak{a}) + \hat\pi(T_n^{j-2}\mathfrak{a}))\,/\,(\hat\pi(\mathcal{J}^j\mathfrak{a}) + \tilde K_n^{j-2}\mathfrak{a})$. Here, only the matrix Lie algebra $\mathfrak{a}$ plays a rôle. The influence of the Λ^*–part to $\Omega_D^n\mathfrak{g}$ is almost trivial.

Next, we derive explicit formulae for the commutator and the differential of elements of $\Omega_D^*\mathfrak{g}$. For the sake of an easier notation we restrict ourselves to the case $\{\hat\pi(\mathfrak{a}''),\hat\pi(\mathfrak{a}'')\} \cap \hat\pi(\Omega^2\mathfrak{a}) = 0$ and $(\{\hat\pi(\mathfrak{a}),\hat\pi(\Omega^{n-N-2}\mathfrak{a}) + \hat\pi(T_{n-1}^{n-N-4}\mathfrak{a})\} \cap$

$\hat{\pi}(\Omega^{n-N}\mathfrak{a})) = 0$. If these conditions are not fulfilled then there are obvious modifications to $\Omega^3_D\mathfrak{g}$ and $\Omega^n_D\mathfrak{g}$, $n \geq N+2$, see Corollary 18.

Due to Corollary 18 and (3.32) we represent elements $\varrho^n \in \Omega^n_D\mathfrak{g}$ as

$$\varrho^n = \sum_\alpha \sum_{j=0}^n c_\alpha^{n-j}\boldsymbol{\gamma}^j \otimes (\hat{\pi}(\omega_\alpha^j) + \tilde{\mathcal{J}}^j_n\mathfrak{a}) , \tag{3.59a}$$

$$\tilde{\mathcal{J}}^j_n\mathfrak{a} := \hat{\pi}(\mathcal{J}^j\mathfrak{a}) + \tilde{K}^{j-2}_n\mathfrak{a} , \qquad \tilde{\mathcal{J}}^0_n\mathfrak{a} \equiv 0 , \qquad \tilde{\mathcal{J}}^1_n\mathfrak{a} \equiv 0 , \tag{3.59b}$$

$$\begin{array}{lll} n \geq 2 : & c_\alpha^{n-j} \in \Lambda^{n-j} , \quad \hat{\pi}(\omega_\alpha^0) \in \hat{\pi}(\mathfrak{a}') , \quad \hat{\pi}(\omega_\alpha^j) \in \hat{\pi}(\Omega^j\mathfrak{a}) \text{ for } j > 0 , \\ n = 1 : & c_\alpha^1 \in \Lambda^1 \text{ if } \hat{\pi}(\omega_\alpha^0) \in \hat{\pi}(\mathfrak{a}') , \quad c_\alpha^1 \in B^1 \text{ if } \hat{\pi}(\omega_\alpha^0) \in \hat{\pi}(\mathfrak{a}'') , \\ & c_\alpha^0 \in \Lambda^0 , \quad \hat{\pi}(\omega_\alpha^1) \in \hat{\pi}(\Omega^1\mathfrak{a}) , \\ n = 0 : & c_\alpha^0 \in \Lambda^0 , \quad \hat{\pi}(\omega_\alpha^0) \in \hat{\pi}(\mathfrak{a}) . \end{array} \tag{3.59c}$$

The formula for the graded commutator of elements of $\Omega^*_D\mathfrak{g}$ is very simple,

$$\begin{aligned} &\big[\sum_\alpha \sum_{i=0}^k c_\alpha^{k-i}\boldsymbol{\gamma}^i \otimes (\hat{\pi}(\omega_\alpha^i) + \tilde{\mathcal{J}}^i_k\mathfrak{a}), \sum_\beta \sum_{j=0}^l \tilde{c}_\beta^{l-j}\boldsymbol{\gamma}^j \otimes (\hat{\pi}(\tilde{\omega}_\beta^j) + \tilde{\mathcal{J}}^j_l\mathfrak{a})\big]_g \\ &\quad = \sum_{\alpha,\beta} \sum_{i=0}^k \sum_{j=0}^l (-1)^{i(l-j)}\, c_\alpha^{k-i} \wedge \tilde{c}_\beta^{l-j}\boldsymbol{\gamma}^{i+j} \otimes ([\hat{\pi}(\omega_\alpha^i), \hat{\pi}(\tilde{\omega}_\alpha^j)]_g + \tilde{\mathcal{J}}^{i+j}_{k+l}\mathfrak{a}) , \end{aligned} \tag{3.60}$$

because if the product between c_α^{k-i} and $\tilde{c}_\beta^{l-j}$ is not completely antisymmetrized then we get a combination of graded anticommutators of elements of $\hat{\pi}(\Omega^*\mathfrak{a})$ in the second component of the tensor product, which contributes to the ideal $\pi(\mathcal{J}^*\mathfrak{g})$. Thus, the graded commutator of elements of $\Omega^*_D\mathfrak{g}$ is given by the combination of the exterior product of the Λ^*–parts and the graded commutator of the $\hat{\pi}(\Omega^*\mathfrak{a})$–parts modulo $\pi(\mathcal{J}^*\mathfrak{g})$, where a graded sign due to the exchange with $\boldsymbol{\gamma}$ must be added.

Due to (3.22) and (3.54) we have for $c^k \in \Lambda^k$

$$(-\mathrm{iD})c^k - (-1)^k c^k(-\mathrm{iD}) = \mathbf{d}c^k - \mathbf{d}^*c^k + 2\nabla^S_{c^k} . \tag{3.61}$$

We apply Proposition 6 and Lemma 17 to (3.59a), using (3.53) and (3.39) and introducing $\tau^n := \sum_\alpha \sum_{j=0}^n c_\alpha^{n-j}\boldsymbol{\gamma}^j \otimes \hat{\pi}(\omega_\alpha^j) \in \pi(\Omega^n\mathfrak{g})$. This gives

$$\begin{aligned} d\varrho^n &\equiv \pi(d\pi^{-1}(\tau^n)) + \pi(\mathcal{J}^{n+1}\mathfrak{g}) \\ &= \textstyle\sum_\alpha \sum_{j=0}^n \big(((-\mathrm{iD})c_\alpha^{n-j} - (-1)^{n-j}c_\alpha^{n-j}(-\mathrm{iD}))\boldsymbol{\gamma}^j \otimes \hat{\pi}(\omega_\alpha^j) \\ &\quad + (-1)^{n-j}c_\alpha^{n-j}\boldsymbol{\gamma}^{j+1} \otimes ((-\mathrm{i}\mathcal{M})\hat{\pi}(\omega_\alpha^j) - (-1)^j\hat{\pi}(\omega_\alpha^j)(-\mathrm{i}\mathcal{M}))\big) \\ &\quad + \mathbf{d}^*\tau^n - 2\nabla_\Omega(\tau^n) + \hat{\sigma}_\mathfrak{g}(\pi^{-1}(\tau^n)) + \pi(\mathcal{J}^{n+1}\mathfrak{g}) \\ &= \textstyle\sum_\alpha \sum_{j=0}^n \big(\mathbf{d}c_\alpha^{n-j}\boldsymbol{\gamma}^j \otimes (\hat{\pi}(\omega_\alpha^j) + \tilde{\mathcal{J}}^j_{n+1}\mathfrak{a}) \\ &\quad + c_\alpha^{n-j}\boldsymbol{\gamma}^{j+1} \otimes ((-1)^{n-j}[-\mathrm{i}\mathcal{M}, \hat{\pi}(\omega_\alpha^j)]_g + \hat{\sigma}(\omega_\alpha^j) + \tilde{\mathcal{J}}^{j+1}_{n+1}\mathfrak{a})\big) . \end{aligned} \tag{3.62}$$

Let us say some words on the terms in (3.56) and (3.57) containing total differentials. In general, for

$$\tau^k := c^{k-j}\boldsymbol{\gamma}^j \otimes \hat{\pi}(\kappa_k^{j-2}) \in \Lambda^{k-j}\boldsymbol{\gamma}^j \otimes \hat{\pi}(T_k^{j-2}\mathfrak{a}) \subset \pi(\mathcal{J}^k\mathfrak{g})$$

we have $\mathbf{d}\tau^k \in \pi(\mathcal{J}^{k+1}\mathfrak{g})$. This is no longer true for $k = 2$ and $\hat{\pi}(\hat{\kappa}_2^0) = \{\hat{\pi}(a''), \hat{\pi}(\tilde{a}'')\}$, with $a'', \tilde{a}'' \in \mathfrak{a}''$. However, in this case the differential $\mathbf{d}\tau^2$ is eliminated by the counterterm $-\mathbf{d}\big(\tau^2 \restriction_{\Lambda^0 \otimes \{\hat{\pi}(\mathfrak{a}''), \hat{\pi}(\mathfrak{a}'')\}}\big)$ in (3.56). An analogous property holds for $k - j = N - 1$, where the terms $\mathbf{d}\tau^k$ are cancelled by the differentials in (3.57). Therefore, in the following formula for the differentiation rule on $\Omega^*_D\mathfrak{g}$ one must omit these boundary terms. Then we obtain a simple formula:

$$\begin{aligned} d\varrho^n = \big(&(\mathbf{d} \otimes \mathbb{1}_F)(\tau^n) + [\boldsymbol{\gamma} \otimes -\mathrm{i}\mathcal{M}, \tau^n]_g \\ &+ (1 \otimes \hat{\sigma} \circ \hat{\pi}^{-1}) \circ \tau^n \circ (\boldsymbol{\gamma} \otimes \mathbb{1}_F)\big) \mod \pi(\mathcal{J}^{n+1}\mathfrak{g}) , \end{aligned} \tag{3.63}$$

where $\tau^n \in \pi(\Omega^n\mathfrak{g})$ is an arbitrary representative of $\varrho^n \in \Omega^n_D\mathfrak{g}$. Here, the differential $\mathbf{d}$ ignores the grading operator $\boldsymbol{\gamma}$, i.e. $\mathbf{d}(c^k\boldsymbol{\gamma}) := (\mathbf{d}c^k)\boldsymbol{\gamma}$. The non–trivial part in this formula is to find the spaces $\tilde{\mathcal{J}}^j_{n+1}\mathfrak{a}$ constituting the ideal $\pi(\mathcal{J}^{n+1}\mathfrak{g})$. The differential $\mathbf{d}\tau^n$, the graded commutator with $\boldsymbol{\gamma}\otimes -\mathrm{i}\mathcal{M}$ and even the computation of $(1 \otimes \hat{\sigma} \circ \hat{\pi}^{-1})(\tau^n)$ are not difficult for a concrete example.

F. Local Connections

In the case under consideration, an L–cycle over the tensor product of the algebra of functions and a matrix Lie algebra, there exists the notion of locality. Our goal is to define a multiplication

$$\begin{aligned} &\tilde{\wedge} : \Lambda^k \times \Omega^n_D\mathfrak{g} \to \Omega^{k+n}_D\mathfrak{g} , \\ &\tilde{c}^k \tilde{\wedge} \Big(\sum_\alpha \sum_{j=0}^n c_\alpha^{n-j}\boldsymbol{\gamma}^j \otimes (\hat{\pi}(\omega^j_\alpha) + \tilde{\mathcal{J}}^j_n\mathfrak{a})\Big) := \sum_\alpha \sum_{j=0}^n (\tilde{c}^k \wedge c_\alpha^{n-j})\boldsymbol{\gamma}^j \otimes (\hat{\pi}(\omega^j_\alpha) + \tilde{\mathcal{J}}^j_{k+n}\mathfrak{a}) , \end{aligned} \tag{3.64}$$

see (3.59). However, we clearly have problems to do this on the whole differential Lie algebra $\Omega^*_D\mathfrak{g}$ due to the existence of the boundary spaces $\Lambda^0 \otimes \hat{\pi}(\mathfrak{a}'')$ in $\Omega^0_D\mathfrak{g} \equiv \pi(\mathfrak{g})$ and $B^1 \otimes \hat{\pi}(\mathfrak{a}'')$ in $\Omega^1_D\mathfrak{g} \equiv \pi(\Omega^1\mathfrak{g})$. These boundary spaces in general do not yield elements of $\Omega^*_D\mathfrak{g}$ when we multiply them by elements of Λ^*. Moreover, there are problems if the boundary terms $\delta^j_{n-N} B^N\boldsymbol{\gamma}^n \otimes (\{\hat{\pi}(\mathfrak{a}), \hat{\pi}(\Omega^{n-N-2}\mathfrak{a}) + \hat{\pi}(T^{n-N-4}_{n-1}\mathfrak{a})\} \cap \hat{\pi}(\Omega^{n-N}\mathfrak{a}))$ and $B^1 \otimes (\{\hat{\pi}(\mathfrak{a}''), \hat{\pi}(\mathfrak{a}'')\} \cap \hat{\pi}(\Omega^2\mathfrak{a}))$ in Corollary 18 are present. Therefore, formula (3.64) is understood to hold on subspaces of $\Omega^*_D\mathfrak{g}$, where no collision with boundary terms occurs. Then, the multiplication (3.64) is associative,

$$(c^k \wedge \tilde{c}^l)\tilde{\wedge}\varrho^n = c^k \tilde{\wedge}(\tilde{c}^l \tilde{\wedge} \varrho^n) , \tag{3.65}$$

for $\tilde{c}^k \in \Lambda^k$, $\tilde{\tilde{c}}^l \in \Lambda^l$ and $\varrho^n \in \Omega^n_D\mathfrak{g}$ (different from boundary spaces). In particular, $\Omega^n_D\mathfrak{g}$ carries a natural $C^\infty(X)$–module structure, where we omit the multiplication symbol $\tilde{\wedge}$ for simplicity:

$$f\Big(\sum_\alpha \sum_{j=0}^n c_\alpha^{n-j}\boldsymbol{\gamma}^j \otimes (\hat{\pi}(\omega^j_\alpha) + \tilde{\mathcal{J}}^j_n\mathfrak{a})\Big) := \sum_\alpha \sum_{j=0}^n (f c_\alpha^{n-j})\boldsymbol{\gamma}^j \otimes (\hat{\pi}(\omega^j_\alpha) + \tilde{\mathcal{J}}^j_n\mathfrak{a}) , \tag{3.66}$$

for $f \in C^\infty(X)$. Moreover, the Hilbert space $h = L^2(X,S) \otimes \mathbb{C}^F$ carries a natural $\Gamma^\infty(C)$–module structure induced by the $\Gamma^\infty(C)$–module structure of $L^2(X,S)$:

$$s^c(\textstyle\sum_\alpha s_\alpha \otimes \varphi_\alpha) := \sum_\alpha s^c s_\alpha \otimes \varphi_\alpha \;, \tag{3.67}$$

for $s^c \in \Gamma^\infty(C)$, $s_\alpha \in L^2(X,S)$ and $\varphi_\alpha \in \mathbb{C}^F$. The structures just introduced enable us to restrict the set of connections according to Definition 7 to the subset of local connections relevant for physical applications.

Definition 19. *A connection* (∇, ∇_h) *is called local connection iff for all* $f \in C^\infty(X)$, $\boldsymbol{\psi} \in h$ *and* $\varrho^n \in \Omega^n_D\mathfrak{g}$ *different from boundary spaces one has*

$$\nabla_h(f\boldsymbol{\psi}) = f\nabla_h(\boldsymbol{\psi}) + \mathbf{d}f(\boldsymbol{\psi}) \;, \tag{3.68a}$$
$$\nabla(f\varrho^n) = f\nabla(\varrho^n) + (\mathbf{d}f)\hat{\wedge}\varrho^n \;. \tag{3.68b}$$

The group of local gauge transformations is the group

$$\begin{aligned}\mathcal{U}_0(\mathfrak{g}) := \big\{\, & u \in \mathcal{U}(\mathfrak{g}) \subset \mathscr{B}(h) \;,\; fu\boldsymbol{\psi} = uf\boldsymbol{\psi} \;,\; \forall f \in C^\infty(X) \;,\; \forall \boldsymbol{\psi} \in h \;, \\ & (\mathrm{Ad}_u \nabla \mathrm{Ad}_{u^*} \;,\; u\nabla_h u^*) \;\; \textit{is a local connection if } (\nabla, \nabla_h) \textit{ is} \,\big\} \;.\end{aligned} \tag{3.68c}$$

We recall that a connection has the form $(\nabla = d + [\tilde\rho,\;.\;]_g \,,\, \nabla_h = -\mathrm{i}D + \rho)$, where $\rho \in \mathcal{H}^1\mathfrak{g}$ and $\tilde\rho := \rho + \tilde{\mathbb{c}}^1\mathfrak{a} \in \hat{\mathcal{H}}^1\mathfrak{g}$, see Proposition 8. The insertion into Definition 19 yields

$$\rho \circ f = f \circ \rho \;, \quad \forall f \in C^\infty(X) \;. \tag{3.69}$$

Therefore, $\rho \in \Gamma(C) \otimes \mathrm{M}_F\mathbb{C}$. Since $\rho \in \mathcal{H}^1\mathfrak{g}$, there can only occur classical smooth differential forms up to first degree in the $\Gamma(C)$–component of ρ. This means that

$$\begin{aligned}&\rho \in (\Lambda^1 \otimes \mathbb{r}^0\mathfrak{a}) \oplus (\Lambda^0\boldsymbol{\gamma} \otimes \mathbb{r}^1\mathfrak{a}) \;, \\ &\quad \mathbb{r}^0\mathfrak{a} = -(\mathbb{r}^0\mathfrak{a})^* = \hat\Gamma(\mathbb{r}^0\mathfrak{a})\hat\Gamma \subset \mathrm{M}_F\mathbb{C} \;, \qquad \mathbb{r}^1\mathfrak{a} = -(\mathbb{r}^1\mathfrak{a})^* = -\hat\Gamma(\mathbb{r}^1\mathfrak{a})\hat\Gamma \subset \mathrm{M}_F\mathbb{C} \;.\end{aligned} \tag{3.70}$$

If we compute graded commutators with $\pi(\Omega^*\mathfrak{g})$ we get

$$\begin{aligned}&[\mathbb{r}^0\mathfrak{a}, \hat\pi(\mathfrak{a})] \subset \hat\pi(\mathfrak{a}') \;, && [\mathbb{r}^0\mathfrak{a}, \hat\pi(\Omega^1\mathfrak{a})] \subset \hat\pi(\Omega^1\mathfrak{a}) \;, \\ &\{\mathbb{r}^0\mathfrak{a}, \hat\pi(\mathfrak{a})\} \subset \{\hat\pi(\mathfrak{a}), \hat\pi(\mathfrak{a})\} + \hat\pi(\Omega^2\mathfrak{a}) \;, && \{\mathbb{r}^0\mathfrak{a}, \hat\pi(\Omega^1\mathfrak{a})\} \subset \{\hat\pi(\mathfrak{a}), \hat\pi(\Omega^1\mathfrak{a})\} + \hat\pi(\Omega^3\mathfrak{a}) \;, \\ &[\mathbb{r}^1\mathfrak{a}, \hat\pi(\mathfrak{a})] \subset \hat\pi(\Omega^1\mathfrak{a}) \;, && \{\mathbb{r}^1\mathfrak{a}, \hat\pi(\Omega^1\mathfrak{a})\} \subset \hat\pi(\Omega^2\mathfrak{a}) + \{\hat\pi(\mathfrak{a}), \hat\pi(\mathfrak{a})\} \;.\end{aligned} \tag{3.71}$$

Moreover, one has to check that $[\rho, \pi(\mathcal{J}^n\mathfrak{g})]_g \subset \pi(\mathcal{J}^{n+1}\mathfrak{g})$. The same analysis for the group of local gauge transformations (3.68c) yields

$$\begin{aligned}\mathcal{U}_0(\mathfrak{g}) &= \exp(\Lambda^0 \otimes \mathbb{u}_0(\mathfrak{a})) \;, \qquad \text{where} \\ \mathbb{u}_0(\mathfrak{a}) &= \{\; u_0 \in \mathbb{r}^0\mathfrak{a} \;, \quad \hat\sigma \circ \hat\pi^{-1}(u_0) \subset \mathbb{c}^1\mathfrak{a} \;\} \;,\end{aligned} \tag{3.72}$$

see (2.49) and (2.55) for the notation.

From (3.68b) one easily finds for the curvature of a local connection $\nabla^2 f = f\nabla^2$, for $f \in C^\infty(X)$. Thus,

$$\begin{aligned} f\theta f = \theta f &= f(\pi \circ d \circ \pi^{-1}(\rho) + \tfrac{1}{2}[\rho,\rho]_g + \pi(\mathcal{J}^2\mathfrak{g}) + \tilde{\mathfrak{c}}^2\mathfrak{a}) \\ &= (\pi \circ d \circ \pi^{-1}(\rho) + \tfrac{1}{2}[\rho,\rho]_g + \pi(\mathcal{J}^2\mathfrak{g}) + \tilde{\mathfrak{c}}^2\mathfrak{a})f \;. \end{aligned} \tag{3.73}$$

Here, $\pi \circ d \circ \pi^{-1}(\rho) + \pi(\mathcal{J}^2\mathfrak{g}) + \tilde{\mathfrak{c}}^2\mathfrak{a}$ is understood in the sense (2.48b). Hence, we must search for the subspace of $\tilde{\mathfrak{c}}^2\mathfrak{a}$ commuting with functions. This space has the structure

$$\tilde{\mathfrak{c}}^2\mathfrak{a} = (\Lambda^2 \otimes \mathfrak{c}^0\mathfrak{a}) \oplus (\Lambda^1\boldsymbol{\gamma} \otimes \mathfrak{c}^1\mathfrak{a}) \oplus (\Lambda^0 \otimes \mathfrak{c}^2\mathfrak{a}) \;, \quad \mathfrak{c}^i\mathfrak{a} \subset \mathrm{M}_F\mathbb{C} \;, \tag{3.74}$$

because possible Λ^*–contributions of higher degree are already orthogonal to any representative of θ, see (2.67). The spaces $\mathfrak{c}^i\mathfrak{a}$ have elementwise the following involution and $\mathbb{Z}_2$–grading properties:

$$\begin{aligned} &\mathfrak{c}^0\mathfrak{a} = -(\mathfrak{c}^0\mathfrak{a})^* = \hat\Gamma(\mathfrak{c}^0\mathfrak{a})\hat\Gamma \;, \qquad \mathfrak{c}^1\mathfrak{a} = -(\mathfrak{c}^1\mathfrak{a})^* = -\hat\Gamma(\mathfrak{c}^1\mathfrak{a})\hat\Gamma \;, \\ &\mathfrak{c}^2\mathfrak{a} = \;\;(\mathfrak{c}^2\mathfrak{a})^* = \hat\Gamma(\mathfrak{c}^2\mathfrak{a})\hat\Gamma \;. \end{aligned} \tag{3.75}$$

From (2.46) one finds after a decomposition into Λ^*–components the equations

$$\begin{aligned} \mathfrak{c}^0\mathfrak{a}\cdot\hat\pi(\mathfrak{a}') &= 0 \;, & \mathfrak{c}^0\mathfrak{a}\cdot\hat\pi(\Omega^1\mathfrak{a}) &= 0 \;, \\ \mathfrak{c}^1\mathfrak{a}\cdot\hat\pi(\mathfrak{a}') &= 0 \;, & \mathfrak{c}^1\mathfrak{a}\cdot\hat\pi(\Omega^1\mathfrak{a}) &= 0 \;, \\ [\mathfrak{c}^2\mathfrak{a},\hat\pi(\mathfrak{a}')] &= 0 \;, & [\mathfrak{c}^2\mathfrak{a},\hat\pi(\Omega^1\mathfrak{a})] &= 0 \;. \end{aligned} \tag{3.76a}$$

The restriction to $\hat\pi(\mathfrak{a}')$ is due to possible problems with the boundary spaces. Due to (3.73) it is convenient to define

$$\mathfrak{j}^0\mathfrak{a} := \mathfrak{c}^0\mathfrak{a} \;, \qquad \mathfrak{j}^1\mathfrak{a} := \mathfrak{c}^1\mathfrak{a} \;, \qquad \mathfrak{j}^2\mathfrak{a} := \mathfrak{c}^2\mathfrak{a} + \hat\pi(\mathcal{J}^2\mathfrak{a}) + \{\hat\pi(\mathfrak{a}),\hat\pi(\mathfrak{a})\} \;. \tag{3.76b}$$

We recall that the commutator and the differential in the curvature $\theta = d\rho + \frac{1}{2}[\rho,\rho]_g + \mathbb{J}^2\mathfrak{g}$ are indirectly defined via the graded Jacobi identity and the graded Leibniz rule (2.48b). The commutator and differential in $\pi(\Omega^*\mathfrak{g}) \mod \pi(\mathcal{J}^*\mathfrak{g})$ are given by (3.60) and (3.63). It is obvious that these formulae extend to local elements of $\hat{\mathcal{H}}^*\mathfrak{g}$. Only the map $\hat\sigma \circ \hat\pi^{-1}$ has to be extended to $\mathfrak{r}^*\mathfrak{a}$ via the graded Leibniz rule:

$$\begin{aligned} &[\hat\sigma \circ \pi^{-1}(\eta^k) + \hat\pi(\mathcal{J}^{k+1}\mathfrak{a}), \hat\pi(\omega^l) + \hat\pi(\mathcal{J}^l\mathfrak{a})]_g \\ &\qquad := \hat\sigma \circ \pi^{-1}([\eta^k,\hat\pi(\omega^l)]_g) - (-1)^k[\eta^k,\hat\sigma(\omega^l)]_g + \hat\pi(\mathcal{J}^{k+l+1}\mathfrak{a}) \;, \end{aligned} \tag{3.77}$$

for $\eta^k \in \mathfrak{r}^k\mathfrak{a}$ and $\omega^l \in \Omega^l\mathfrak{a}$. Then we find for the curvature

$$\begin{aligned} \theta = \big(&(\mathbf{d}\otimes \mathbb{1}_F)(\rho) + \{\boldsymbol{\gamma}\otimes -\mathrm{i}\mathcal{M},\rho\} + \tfrac{1}{2}\{\rho,\rho\} \\ &+ (1\otimes\hat\sigma\circ\pi^{-1})\circ\rho\circ(\boldsymbol{\gamma}\otimes\mathbb{1}_F)\big) \mod \mathbb{J}^2\mathfrak{g} \;, \end{aligned} \tag{3.78}$$

where we recall that $\mathbb{J}^2\mathfrak{g} = \Lambda^0 \otimes (\hat\pi(\mathcal{J}^2\mathfrak{a}) + \{\hat\pi(\mathfrak{a}),\hat\pi(\mathfrak{a})\}) + \tilde{\mathfrak{c}}^2\mathfrak{a}$.

In our case – $h = L^2(X,S) \otimes \mathbb{C}^F$ – we have $\mathscr{B}(h) = \mathscr{B}(L^2(X,S)) \otimes \mathrm{M}_F\mathbb{C}$. Then, the parameter d in (2.64) is equal to the dimension N of the manifold X, see Ref. 1. Moreover, the trace theorem[1] of Alain Connes says that in this case we have

$$\mathrm{Tr}_\omega((s^c \otimes \mathrm{m})\,|D|^{-N}) = \frac{1}{(\frac{N}{2})!(4\pi)^{\frac{N}{2}}} \int_X \mathrm{v}_g\ \mathrm{tr}_c(s^c)\ \mathrm{tr}(\mathrm{m})\ , \qquad (3.79)$$

where we recall that v_g denotes the canonical volume form on X, tr_c denotes the trace in the Clifford algebra $\mathrm{Cliff}_{\mathbb{C}}(\mathbb{R}^N)$, normalized by $\mathrm{tr}_c(1) = 2^{N/2}$, and $\mathrm{tr}(\mathrm{m})$ is the matrix–trace of $\mathrm{m} \in \mathrm{M}_F\mathbb{C}$. We use the trace theorem (3.79) for the construction of $\mathfrak{e}(\theta)$, see (2.67). For the curvature θ of a local connection we have according to the above considerations a decomposition

$$\theta = \textstyle\sum_\alpha \big(c^2_\alpha \otimes (\tau^0_\alpha + \mathbb{j}^0\mathfrak{a}) + c^1_\alpha \boldsymbol{\gamma} \otimes (\tau^1_\alpha + \mathbb{j}^1\mathfrak{a}) + c^0_\alpha \otimes (\tau^2_\alpha + \mathbb{j}^2\mathfrak{a})\big)\ , \qquad (3.80)$$

where $c^i \in \Lambda^i$ and $\tau^i \in \mathrm{M}_F\mathbb{C}$. Since $\Lambda^* = \bigoplus_{k=0}^N \Lambda^k$ is an orthogonal decomposition with respect to the scalar product (3.14) given by tr_c, we see that (2.67) is equivalent to finding for $i \in \{0,1,2\}$ and each α the elements $j^i_\alpha \in \mathbb{j}^i\mathfrak{a}$ satisfying

$$\mathrm{tr}(\tilde{j}^{\,i}\,(\tau^i_\alpha + j^i_\alpha)) = 0\ , \qquad \text{for all}\ \ \tilde{j}^{\,i} \in \mathbb{j}^i\mathfrak{a}\ . \qquad (3.81a)$$

These equations must be solved for the concrete L–cycle $(\mathfrak{a}, \mathbb{C}^F, \mathcal{M}, \hat{\pi}, \hat{\Gamma})$ and the concrete element τ^i_α, giving in the notation of (3.80)

$$\mathfrak{e}(\theta) = \textstyle\sum_\alpha \big(c^2_\alpha \otimes (\tau^0_\alpha + j^0_\alpha) + c^1_\alpha \boldsymbol{\gamma} \otimes (\tau^1_\alpha + j^1_\alpha) + c^0_\alpha \otimes (\tau^2_\alpha + j^2_\alpha)\big)\ . \qquad (3.81b)$$

Now, formula (2.65a) for the bosonic action takes the form (up to a constant)

$$S_B(\nabla) = \int_X \mathrm{v}_g\ \mathrm{tr}_c(\mathfrak{e}(\theta)^2)\ . \qquad (3.82a)$$

Here, tr_c contains both the traces in $\mathrm{Cliff}_{\mathbb{C}}(\mathbb{R}^N)$ and $\mathrm{M}_F\mathbb{C}$. For the fermionic action we obtain

$$S_F(\boldsymbol{\psi}, \nabla) = \langle \boldsymbol{\psi}, (D + \mathrm{i}\rho)\boldsymbol{\psi}\rangle_h = \int_X \mathrm{v}_g\ \boldsymbol{\psi}^*(D + \mathrm{i}\rho)\boldsymbol{\psi}\ . \qquad (3.82b)$$

This finishes our prescription towards gauge field theories. Let us recall what the essential steps are. One starts to select the L–cycle from the physical data or assumptions. We have learned that the matrix part of the L–cycle contains the essential information. Hence, we must construct the spaces $\hat{\pi}(\Omega^n\mathfrak{a})$ and the ideal $\hat{\pi}(\mathcal{J}^n\mathfrak{a})$ up to second (in some cases up to third) order. This is necessary to compute the spaces $\mathbb{r}^0\mathfrak{a}, \mathbb{r}^1\mathfrak{a}$ and $\mathbb{j}^0\mathfrak{a}, \mathbb{j}^1\mathfrak{a}, \mathbb{j}^2\mathfrak{a}$ constituting the connection form ρ and the ideal $\mathbb{J}^2\mathfrak{g}$. Then we have to compute the curvature θ of the connection and to select its representative $\mathfrak{e}(\theta)$ orthogonal to $\mathbb{J}^2\mathfrak{g}$. Finally, we write down the bosonic and fermionic actions. This scheme can be applied to a large class of physical models. Among them are the $\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)$–standard model[17] and the $\mathrm{SU}(5) \times \mathrm{U}(1)$–Grand Unification model.[18] The SU(5)–GUT can be obtained as a special case of the latter.

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