published in: J. Math. Phys. **37** (1996) 3797–3814

Deriving the Standard Model from the Simplest two–point K–cycle

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Abstract

Basing on a differential algebra over the simplest two–point K–cycle and graded Lie algebras of homomorphisms of finite projective modules we derive the classical action of the standard model. This construction uses both the general framework of non–commutative geometry developed by A. Connes and ideas of the Mainz–Marseille approach to model building. We get a prediction of the Weinberg angle and constraints between the fermion masses and the masses of the W– and Higgs–bosons on tree–level, which differ from the relations obtained by Kastler and Schücker for the quaternionic K–cycle.

Keywords: standard model, non–commutative geometry
PACS: 12.10.Kt, 02.40.-k

I Introduction

This paper is the continuation of two earlier papers ([1], [2]) by R. Matthes, G. Rudolph and R. Wulkenhaar. In the first of them we presented an analysis of the structure of the differential algebra $\Lambda^*_{\mathcal{A}}$ canonically associated to the K–cycle $(\mathcal{A}, h, D)$ over the simplest two–point algebra $\mathcal{A} = C^\infty(X) \otimes (\mathbb{C} \oplus \mathbb{C})$. In the second one we constructed for a given finite projective right $\mathcal{A}$–module with Hermitian structure $\mathcal{E}$ the graded Lie algebra $\mathcal{H} = \mathrm{Hom}_{\mathcal{A}}(\mathcal{E}, \mathcal{E} \otimes_{\mathcal{A}} \Lambda^*_{\mathcal{A}})$ with natural derivation. We showed that a certain graded Lie subalgebra $\mathcal{H}_0$ of $\mathcal{H}$ provides a rigorous mathematical link between Connes' theory and the Mainz–Marseille model building scheme.

The K–cycle mentioned above together with a finite projective module $\mathcal{E} = e\mathcal{A}^2$ was used by Connes in [3] and [4] and by Connes and Lott in [5] to derive the

Salam–Weinberg model in a unified form. Within this scheme the above K–cycle cannot be used to obtain the full standard model. That is why Connes and Lott proposed a K–cycle over the algebra $C^\infty_{\mathbb{R}}(X) \otimes (\mathbb{C} \oplus \mathbb{H})$, see [5], [3] and [6]. A detailed exposition of these ideas was presented by Kastler in [7] and [8], see [9] for an earlier version, and by Kastler and Schücker in [10]. One obtains a prediction of the Weinberg angle and certain tree–level constraints between the masses of the fermions and the masses of the W–, Z–, and Higgs–bosons ([10], [11]), which we review at the end of this introduction.

Another way of obtaining the standard model by non–commutative geometry is the Mainz–Marseille model ([12], [13], [14]), which uses the graded Lie algebra $\Lambda^*(X) \otimes spl\,(2,1)$ of matrix–valued differential forms, see also [15]. We have shown in [2] that the assumptions of this approach are natural within Connes' framework. For the present paper the construction of the fermionic sector of the standard model given in [13] is of particular interest. This construction makes use of the theory of representations of the graded Lie algebra $spl\,(2,1)$ in a finite dimensional vector space [15].

Our strategy is the following: In section II we give a review of some results obtained in [1] and [2]. In section III we construct an isomorphism $\mathbf{i}$ of graded Lie algebras from $\mathcal{H}_0$, which we constructed in [2] within Connes' theory, onto its image, for which the fermionic sector in [13] serves as a guiding line. Next, in section IV, we give an embedding of a subspace of $\mathbf{i}(\mathcal{H}_0)$ into the algebra of bounded operators on the Hilbert space of fermions. This allows to construct the fermionic action (section V), the electroweak sector of the bosonic action of the standard model (section VI) and the chromodynamics sector (section VII). After a Wick rotation to Minkowski space and a reparameterization this action coincides with the classical action of the standard model, where some free parameters of the standard model are fixed, see section VIII.

Thus, our construction rests – as the construction of Connes, Lott and Kastler – upon a K–cycle. However, we take a K–cycle over the algebra $\mathcal{A} = C^\infty(X) \otimes (\mathbb{C} \oplus \mathbb{C})$, while the above authors start with a K–cycle over the much bigger algebras $C^\infty_{\mathbb{R}}(X) \otimes (\mathbb{C} \oplus \mathbb{H})$ and $C^\infty_{\mathbb{R}}(X) \otimes (\mathbb{C} \oplus \mathrm{M}_3\mathbb{C})$. To compensate this difference we use two finite projective $\mathcal{A}$–modules as an additional input, one module for the electroweak sector and the other one for the chromodynamics sector. In the construction of Connes, Lott and Kastler the module is identical with the algebra of their K–cycle, so that in their version the calculus of finite projective modules is not necessary. In both versions the differential of the differential algebras associated to the K–cycles are composed of the classical exterior differential and a matrix–differential, which in both versions contains fermionic mass parameters. In the construction of Connes, Lott and Kastler the just mentioned differential algebra is of primary importance for the physical model. In our version the differential algebra plays only an auxiliary role, namely for building the graded Lie algebra $\mathcal{H}_0$. Moreover, the Hilbert space of our K–cycle is also auxiliary, we have to add our physical Hilbert space by hand.

As it was shown in [2] one can derive from $\mathcal{H}_0$ the graded Lie algebra $\Lambda^*(X)\otimes$ *spl* $(2,1)$, which is the starting point for the construction of the standard model in the Mainz–Marseille approach. The essential difference between $\mathcal{H}_0$ and $\Lambda^*(X)\otimes$ *spl* $(2,1)$ is that $\mathcal{H}_0$ carries the fermionic mass parameters of the differential algebra associated to the K–cycle. It are these fermionic mass parameters which relate the masses of the fermions to the masses of the W–, Z–and Higgs–bosons in the model of Connes, Lott and Kastler, as well as in our model. Since the fermionic mass parameters are absent in $\Lambda^*(X)\otimes$ *spl* $(2,1)$, there is no relation between fermion masses and boson masses in the Mainz–Marseille model.

Both in our model and in the Mainz–Marseille approach one needs certain representations of the graded Lie algebras $\mathcal{H}_0$ respectively $\Lambda^*(X)\otimes$ *spl* $(2,1)$, which are generalizations of classical *spl* $(2,1)$–representations [15]. The representations of $\Lambda^*(X)\otimes$ *spl* $(2,1)$ generalize reducible indecomposable representations of *spl* $(2,1)$, which give the possibility to describe mixing between fermion generations [13]. For our model we take generalizations of the simplest irreducible representations of *spl* $(2,1)$, because the fermion generations are intrinsically contained in $\mathcal{H}_0$. With these representations there enters a big number of additional parameters due to not canonically determined free normalization constants of $sl(2,\mathbb{C})\oplus gl(1,\mathbb{C})$–subrepresentations of *spl* $(2,1)$. In the Mainz–Marseille construction these parameters are fitted to the fermion masses and the Kobayashi–Maskawa matrix. In our model there is a subtle interplay between these normalization parameters and the intrinsic fermionic mass parameters of the differential calculus. In the final Lagrangian there occur only such combinations of these parameters that for the simplest scalar product there is effectively only one additional free parameter in our model compared with the simplest version of the model of Connes, Lott and Kastler.

From our formulation of the standard model we get tree–level predictions for the Weinberg angle θ_W and the ratios m_W/m_t , m_H/m_W and g_3/g_2 . Here, m_t , m_H and m_W are the masses of the top–quark, the Higgs–boson and the W–boson and g_3, g_2 the coupling constants of the strong and weak interactions respectively. We list our predictions (Model III, with the parameters $x>0$ and $\vartheta_1,\vartheta_2,\vartheta_3,\vartheta_4$) in a table below and compare them with corresponding tree–level predictions of the following non–commutative geometrical formulations of the standard model:

Model I: The construction based on a K–cycle over the algebras $C^\infty_{\mathbb{R}}(X)\otimes(\mathbb{C}\oplus\mathbb{H})$ and $C^\infty_{\mathbb{R}}(X)\otimes(\mathbb{C}\oplus \mathrm{M}_3\mathbb{C})$ as presented by Kastler and Schücker in [10]. The parameters are $R>-1$, $\alpha,\beta>0$.

Model II: The Mainz–Marseille model as presented in [13] and [14], predictions for the adjoint representation of *spl* $(2,1)$ and a general scalar product. The parameters are $r_0, r_1, r_2 > 0$.

	Model I	Model II	Model III
$(\frac{m_t}{m_W})^2$	$R+4>3$	—	$\frac{6+2x}{4-(1-x\sin^2\vartheta_1)\cos^2\vartheta_2} > \frac{3}{2}$
$(\frac{m_H}{m_W})^2$	$11+3R-\frac{8+2R}{7+R} > 7$	$\frac{2r_0r_2}{r_1^2}$	$\frac{8(x+3)(5\cos^2\vartheta_3+3x\cos^2\vartheta_4\sin^4\vartheta_1)\cos^4\vartheta_2}{3(4-(1-x\sin^2\vartheta_1)\cos^2\vartheta_2)^2}$
$\sin^2\theta_W$	$\frac{8+2R}{15+3R+\frac{2}{3}\alpha+2\beta} < \frac{2}{3}$	$\frac{1}{4}$	$\frac{1}{4} < \frac{3}{4}\frac{3+x}{5+3x} < \frac{9}{20}$
$(\frac{g_3}{g_2})^2$	$\frac{R+4}{2\alpha}$	—	$\frac{3+x}{4} > \frac{3}{4}$

In Model I there is enough freedom to repoduce the experimental values for m_W/m_t, $\sin^2\theta_W$ and g_3/g_2. Then m_H is uniquely determined. Also in Model III it is possible to reproduce the experimental values for m_W/m_t, g_3/g_2 and $\sin^2\theta_W \approx 0.25$. However, then there is only an upper limit for m_H in our model, see section VIII. In Model II there is no relation between fermion and boson masses and between g_3 and g_2. Moreover, this model does not give a prediction for m_H. However, in contrast to the other two models one obtains an experimentally well–confirmed relation between the Cabibbo angle and the quark masses, see [16].

For the simplest scalar products given by $R=0$, $\alpha=2$, $\beta=\frac{3}{2}$ in Model I, $r_0=r_1=r_2$ in Model II and $x=1$ in Model III one gets

	Model I	Model II	Model III
m_t/m_W	2	—	1.41 ... 1.63
m_H/m_W	3.14	1.41	0 ... 2.43
$\sin^2\theta_W$	0.414	0.25	0.375
g_3/g_2	1	—	1

Now, m_H is fixed in Model II. In Model III there is still only an upper limit for m_H, and the relation between m_t and m_W does not fit the experimental value as well as the corresponding relation in Model I does.

II Review of Earlier Results

In this section we give a review of some results, which were obtained in [1] and [2] and which we need for what follows. For technical reasons let for the moment X be a compact four dimensional Riemannian spin manifold. We denote by $L^2(X,S)$ the Hilbert space of square integrable sections of the spinor bundle over X, by F a finite dimensional Hilbert space, by C the Clifford bundle of the cotangent space over X and by C^k the set of those sections of C, whose values at each point $x \in X$ belong to the subspace spanned by products of less than or equal k elements of

$T^*_x X$ of the same parity. The simplest two–point K–cycle $(\mathcal{A}, h, D)$ consists of the algebra $\mathcal{A} = C^\infty(X)\otimes(\mathbb{C}\oplus\mathbb{C})$ acting on the Hilbert space $h = L^2(X,S)\otimes(F\oplus F)$ and a generalized Dirac operator $D = (D^{c\ell}\otimes \mathrm{id}_{F\oplus F}) \oplus (\gamma^5 \otimes \mathcal{M})$. Here, $D^{c\ell}$ is the classical Dirac operator and γ^5 the grading operator on $L^2(X,S)$. Moreover, $\mathcal{M} = \left(\begin{smallmatrix} 0 & M \\ M^* & 0\end{smallmatrix}\right)$, where M is an endomorphism of F.

Now we are going to specify the parameters. We take F and M as

$$F = \mathbb{C}^3 \oplus \mathbb{C}^3 \ , \quad M = \begin{pmatrix} -\mathrm{m}_l & 0 \\ 0 & -\mathrm{m}_q \end{pmatrix} \ , \tag{1}$$

where m_l and m_q are real diagonal 3×3–matrices with non–negative entries. In principle, we could take for m_l and m_q arbitrary complex 3×3–matrices, which can be written as $\mathrm{u}_1\delta\mathrm{u}_2$, with $\mathrm{u}_1, \mathrm{u}_2$ unitary and δ diagonal and positive. However, the unitary matrices $\mathrm{u}_1, \mathrm{u}_2$ can be absorbed by unitary transformations of the physical fields.

One shows that the algebra $\pi(\Omega^*)$ obtained from an involutive representation π of the universal differential algebra Ω^* over $\mathcal{A}$ has the structure

$$\begin{aligned} \pi(\Omega^*) &= \bigoplus_{k=0}^{\infty} \pi(\Omega^k) \ , \\ \pi(\Omega^k) &= \begin{bmatrix} \bigoplus_{r=0}^{m} C^{k-2r}\otimes \mathbb{C}\, M_1^r & ; & \bigoplus_{r=0}^{m} C^{k-2r-1}\gamma^5\otimes\mathbb{C}\, M_2^r \\ \bigoplus_{r=0}^{m} C^{k-2r-1}\gamma^5\otimes\mathbb{C}\, M_3^r & ; & \bigoplus_{r=0}^{m} C^{k-2r}\otimes\mathbb{C}\, M_4^r \end{bmatrix} \ , \end{aligned} \tag{2}$$

where

$$M_1^r := (MM^*)^r,\ M_2^r := M(M^*M)^r,\ M_3^r := M^*(MM^*)^r,\ M_4^r := (M^*M)^r \ . \tag{3}$$

In particular, we can identify $\mathcal{A}$ with $\pi(\Omega^0)$. We denote by $(m+1)$ the number of linear independent elements $(MM^*)^r$. In our case M and M^* commute with each other and generically we have $m = 6$. We define $L^n := C^n/C^{n-2}$, $L^* := \bigoplus_{k=0}^4 L^k$, and put $L^n = \{0\}$ for $n<0$. There is a graded algebra $\Lambda^*_{\mathcal{A}}$ associated with $\pi(\Omega^*)$ defined as follows:

$$\Lambda^*_{\mathcal{A}} := \bigoplus_{k=0}^{\infty} \Lambda^k_{\mathcal{A}} \ , \quad \Lambda^k_{\mathcal{A}} \equiv \sigma_k\circ\pi(\Omega^k) := \begin{cases} \pi(\Omega^k)\,/\,\pi(\Omega^{k-2}) & \text{for } \ k\geq 2 \ , \\ \pi(\Omega^k) & \text{for } \ k = 0,1 \ , \end{cases} \tag{4}$$

with multiplication

$$\Lambda^k_{\mathcal{A}}\times\Lambda^n_{\mathcal{A}} \ni (\lambda^k, \tilde\lambda^n) \mapsto \lambda^k\bullet\tilde\lambda^n := \sigma_{k+n}(\tau^k\tilde\tau^n) \in \Lambda^{k+n}_{\mathcal{A}} \ , \tag{5}$$

where $\tau^k \in \pi(\Omega^k)$, $\tilde{\tau}^n \in \pi(\Omega^n)$, such that $\sigma_k(\tau^k) = \lambda^k$, $\sigma_l(\tilde{\tau}^n) = \tilde{\lambda}^n$. One can show that elements $\lambda^k \in \Lambda^k_{\mathcal{A}}$ have the form

$$\lambda^k = \begin{pmatrix} \sum_{r=0}^{m} \alpha_1^{k-2r} \otimes \mathbb{C}\, M_1^r & ; \sum_{r=0}^{m} \alpha_2^{k-2r-1}\gamma^5 \otimes \mathbb{C}\, M_2^r \\ \sum_{r=0}^{m} \alpha_3^{k-2r-1}\gamma^5 \otimes \mathbb{C}\, M_3^r & ; \sum_{r=0}^{m} \alpha_4^{k-2r} \otimes \mathbb{C}\, M_4^r \end{pmatrix} , \quad \alpha_q^n \in L^n . \qquad (6)$$

Let ι_k denote the isomorphism of L^k onto $\Lambda^k(X)$, where $\Lambda^k(X)$ is the space of complex valued k–forms on X. We define the following operations:

$$\begin{aligned} [\gamma^5 \otimes \mathcal{M}, \lambda^k]_g &:= (\gamma^5 \otimes \mathcal{M}) \bullet \lambda^k - (-1)^k \lambda^k \bullet (\gamma^5 \otimes \mathcal{M}) , \\ \mathbf{d}\alpha^k &:= \iota_{k+1}^{-1} \circ d \circ \iota_k(\alpha^k) , \qquad \mathbf{d}^* \;:=\; \gamma^5 \mathbf{d} \gamma^5 , \\ \mathbf{D}\lambda^k &:= pr_{k+1} \circ (\, (\mathbf{d} - \mathbf{d}^*) \otimes \mathrm{id}_F \otimes \mathbb{1}_{2\times 2}\,)(\lambda^k) , \end{aligned} \qquad (7)$$

for $\alpha^k \in L^k$, $\lambda^k \in \Lambda^k_{\mathcal{A}}$, where pr_{k+1} denotes the projection from $\Lambda^{k+1}_{\mathcal{A}} \oplus \Lambda^{k-1}_{\mathcal{A}}$ onto $\Lambda^{k+1}_{\mathcal{A}}$. One shows that

$$\hat{d} := \mathbf{D} - i[\gamma^5 \otimes \mathcal{M}\,,\,.\,]_g \qquad (8)$$

is a graded differential on $\Lambda^*_{\mathcal{A}}$, and one can write down explicit multiplication and differential rules for elements of $\Lambda^*_{\mathcal{A}}$, see [1].

In [2] we considered finite projective right $\mathcal{A}$–modules with Hermitian structure $\mathcal{E} = e\mathcal{A}^p$, where $e \in \mathrm{End}_{\mathcal{A}}(\mathcal{E})$, fulfilling $e = e^2 = e^*$. Let $\mathcal{H}^k := \mathrm{Hom}_{\mathcal{A}}(\mathcal{E}, \mathcal{E} \otimes_{\mathcal{A}} \Lambda^k_{\mathcal{A}})$ be the set of homomorphisms of the right $\mathcal{A}$–module $\mathcal{E}$ to the right $\mathcal{A}$–module $\mathcal{E} \otimes_{\mathcal{A}} \Lambda^k_{\mathcal{A}}$. Elements $\varrho^k \in \mathcal{H}^k$ can be identified with $p \times p$–matrices of elements of $\Lambda^k_{\mathcal{A}}$, fulfilling $e\varrho^k e = \varrho^k$:

$$\varrho^k = \begin{pmatrix} \varrho^k_{11} \dots \varrho^k_{1j} \dots \varrho^k_{1p} \\ \vdots \;\; \ddots \;\; \vdots \;\; \ddots \;\; \vdots \\ \varrho^k_{i1} \dots \varrho^k_{ij} \dots \varrho^k_{ip} \\ \vdots \;\; \ddots \;\; \vdots \;\; \ddots \;\; \vdots \\ \varrho^k_{p1} \dots \varrho^k_{pj} \dots \varrho^k_{pp} \end{pmatrix} , \quad \varrho^k_{ij} \in \Lambda^k_{\mathcal{A}} , \quad i,j = 1,\dots,p . \qquad (9)$$

The space $\mathcal{H} = \bigoplus_{k=0}^{\infty} \mathcal{H}^k$ is an algebra with multiplication $\bullet$ given by the tensor product of the multiplication $\bullet$ in $\Lambda^*_{\mathcal{A}}$ and multiplication of $p \times p$–matrices. On $\mathcal{H}$ we have a canonical derivation $\mathcal{D}$ given by

$$\mathcal{D}\varrho = e\hat{d}(\varrho)e , \quad \varrho \in \mathcal{H} , \qquad (10)$$

where $\hat{d}$ means the componentwise action on matrix elements belonging to $\Lambda^*_{\mathcal{A}}$. With respect to the graded commutator

$$[\, \varrho^k \,,\, \tilde{\varrho}^n \,]_g \;:=\; \varrho^k \bullet \tilde{\varrho}^n - (-1)^{kn} \tilde{\varrho}^n \bullet \varrho^k , \quad \varrho^k \in \mathcal{H}^k , \quad \tilde{\varrho}^n \in \mathcal{H}^n , \qquad (11)$$

$\mathcal{H}$ is a graded Lie algebra with graded derivation $\mathcal{D}$.

We define on $\Lambda^*_{\mathcal{A}}$, see (6), a linear map $T_\Lambda : \Lambda^*_{\mathcal{A}} \to L^*$ by

$$T_\Lambda\Big(\begin{pmatrix} \sum_{t=0}^m \alpha_1^{k-2t} \otimes M_1^t & ; & \sum_{t=0}^m \alpha_2^{k-2t-1}\gamma^5 \otimes M_2^t \\ \sum_{t=0}^m \alpha_3^{k-2t-1}\gamma^5 \otimes M_3^t & ; & \sum_{t=0}^m \alpha_4^{k-2t} \otimes M_4^t \end{pmatrix} \Big) := \sum_{t=0}^m (\alpha_1^{k-2t} - \alpha_4^{k-2t}) \;, \tag{12}$$

which can be interpreted as a generalized trace. We have proved in [2] that there exists a graded Lie subalgebra $\mathcal{H}_0$ of $\mathcal{H}$ defined by

$$\mathcal{H}_0 = \bigoplus_{k=0}^{\infty} \mathcal{H}_0^k \;, \quad \mathcal{H}_0^k = \{ \; \varrho^k \in \mathcal{H}^k \;, \quad \sum_{i=1}^p T_\Lambda\,(\varrho_{ii}^k) = 0 \; \} \tag{13}$$

(in the notation of (9)). Moreover, $\mathcal{D}$ is a graded derivation on $\mathcal{H}_0$.

Any connection ∇ on $\mathcal{E}$ has the form

$$\nabla = e\hat{d} + \rho \;, \quad \rho = -\rho^* \in \mathcal{H}^1 \;. \tag{14}$$

The situation found in physics suggests to consider special connections, namely such that $\rho \in \mathcal{H}_0^1$. This implies that the gauge group is restricted to

$$\mathcal{U} = \{ \; u \in \mathrm{Aut}_{\mathcal{A}}(\mathcal{E}) \;, \quad uu^* = u^*u = e \;, \quad u\mathbf{d}u^* \in \mathcal{H}_0^1 \; \} \;, \tag{15}$$

see [2], where $\mathbf{d}u^*$ means the action of the differential $\mathbf{d}$ on the L^*–component of $u^* \in \mathcal{U}$. Gauge transformations of the connection ∇ are given by $u\nabla u^*$.

III An Isomorphism of Graded Lie Algebras

Here we fix the module $\mathcal{E}$ by taking $p = 2$ and for the projector $e = \begin{pmatrix} e' & 0 \\ 0 & \mathbb{1}_{\mathcal{A}} \end{pmatrix}$, where $\mathbb{1}_{\mathcal{A}} = \begin{pmatrix} 1 \otimes \mathrm{id}_F & 0 \\ 0 & 1 \otimes \mathrm{id}_F \end{pmatrix} \in \mathcal{A}$ is the identity of $\mathcal{A}$ and $e' = \begin{pmatrix} 1 \otimes \mathrm{id}_F & 0 \\ 0 & 0 \end{pmatrix} \in \mathcal{A}$. Then from (6), (9), (12), (13) and the discussion in [2] we get that elements $\varrho^k \in \mathcal{H}_0^k$ have the form

$$\varrho^k = \sum_{r=0}^m \left(\begin{array}{cc|cc} \frac{1}{2}(\alpha_0^{k-2r} + \alpha_3^{k-2r}) \otimes M_1^r & 0 & \alpha_-^{k-2r} \otimes M_1^r & \alpha_4^{k-2r-1}\gamma^5 \otimes M_2^r \\ 0 & 0 & 0 & 0 \\ \hline \alpha_+^{k-2r} \otimes M_1^r & 0 & \frac{1}{2}(\alpha_0^{k-2r} - \alpha_3^{k-2r}) \otimes M_1^r & \alpha_5^{k-2r-1}\gamma^5 \otimes M_2^r \\ \alpha_6^{k-2r-1}\gamma^5 \otimes M_3^r & 0 & \alpha_7^{k-2r-1}\gamma^5 \otimes M_3^r & \alpha_0^{k-2r} \otimes M_4^r \end{array} \right), \tag{16}$$

where $\alpha_f^n \in L^n$, $f = 0,+,-,3,4,5,6,7$. Due to (1) and (3) we have $M_s^r \in \mathrm{M}_3\mathbb{C} \otimes \mathrm{M}_2\mathbb{C}$, which means $\varrho^k \in L^* \otimes \mathrm{M}_4\mathbb{C} \otimes \mathrm{M}_3\mathbb{C} \otimes \mathrm{M}_2\mathbb{C}$. In (16) we considered ϱ^k as a 4×4–matrix with $L^* \otimes \mathrm{M}_3\mathbb{C} \otimes \mathrm{M}_2\mathbb{C}$–valued entries. Of course, ϱ^k can also be treated as a 2×2–matrix with $L^* \otimes \mathrm{M}_4\mathbb{C} \otimes \mathrm{M}_3\mathbb{C}$–valued entries. With regard to this view, ϱ^k has the form

$$\varrho^k = \begin{pmatrix} \varrho_l^k & 0 \\ 0 & \varrho_q^k \end{pmatrix} , \tag{17}$$

$$\varrho_l^k = \sum_{r=0}^{m} \left(\begin{array}{cc|cc} \frac{1}{2}(\alpha_0^{k-2r} + \alpha_3^{k-2r}) \otimes \mathrm{l}_1^r & 0 & \alpha_-^{k-2r} \otimes \mathrm{l}_1^r & \alpha_4^{k-2r-1}\gamma^5 \otimes \mathrm{l}_2^r \\ 0 & 0 & 0 & 0 \\ \hline \alpha_+^{k-2r} \otimes \mathrm{l}_1^r & 0 & \frac{1}{2}(\alpha_0^{k-2r} - \alpha_3^{k-2r}) \otimes \mathrm{l}_1^r & \alpha_5^{k-2r-1}\gamma^5 \otimes \mathrm{l}_2^r \\ \alpha_6^{k-2r-1}\gamma^5 \otimes \mathrm{l}_3^r & 0 & \alpha_7^{k-2r-1}\gamma^5 \otimes \mathrm{l}_3^r & \alpha_0^{k-2r} \otimes \mathrm{l}_4^r \end{array}\right),$$

$$\varrho_q^k = \sum_{r=0}^{m} \left(\begin{array}{cc|cc} \frac{1}{2}(\alpha_0^{k-2r} + \alpha_3^{k-2r}) \otimes \mathrm{q}_1^r & 0 & \alpha_-^{k-2r} \otimes \mathrm{q}_1^r & \alpha_4^{k-2r-1}\gamma^5 \otimes \mathrm{q}_2^r \\ 0 & 0 & 0 & 0 \\ \hline \alpha_+^{k-2r} \otimes \mathrm{q}_1^r & 0 & \frac{1}{2}(\alpha_0^{k-2r} - \alpha_3^{k-2r}) \otimes \mathrm{q}_1^r & \alpha_5^{k-2r-1}\gamma^5 \otimes \mathrm{q}_2^r \\ \alpha_6^{k-2r-1}\gamma^5 \otimes \mathrm{q}_3^r & 0 & \alpha_7^{k-2r-1}\gamma^5 \otimes \mathrm{q}_3^r & \alpha_0^{k-2r} \otimes \mathrm{q}_4^r \end{array}\right),$$

where

$$\begin{aligned} &\mathrm{l}_1^r = (\mathrm{m}_l\mathrm{m}_l^*)^r \ , \ \mathrm{l}_2^r = -\mathrm{m}_l(\mathrm{m}_l^*\mathrm{m}_l)^r \ , \ \mathrm{l}_3^r = -\mathrm{m}_l^*(\mathrm{m}_l\mathrm{m}_l^*)^r \ , \ \mathrm{l}_4^r = (\mathrm{m}_l^*\mathrm{m}_l)^r \ , \\ &\mathrm{q}_1^r = (\mathrm{m}_q\mathrm{m}_q^*)^r, \ \mathrm{q}_2^r = -\mathrm{m}_q(\mathrm{m}_q^*\mathrm{m}_q)^r, \ \mathrm{q}_3^r = -\mathrm{m}_q^*(\mathrm{m}_q\mathrm{m}_q^*)^r, \ \mathrm{q}_4^r = (\mathrm{m}_q^*\mathrm{m}_q)^r \ . \end{aligned} \tag{18}$$

We introduce an isomorphism $\mathbf{i}$ of graded Lie algebras, which generalizes certain representations ([15], [13]) of the graded Lie algebra *spl* $(2,1)$ to the graded Lie algebra $\mathcal{H}_0$:

$$\mathbf{i}(\varrho^k) = \begin{pmatrix} \mathbf{i}_l(\varrho_l^k) & 0 \\ 0 & \mathbf{i}_q(\varrho_q^k) \otimes \mathbb{1}_{3\times 3} \end{pmatrix} , \tag{19}$$

$$\mathbf{i}_l(\varrho_l^k) = \sum_{r=0}^{m} \begin{pmatrix} \frac{1}{2}(\alpha_0^{k-2r} + \alpha_3^{k-2r}) \otimes \mathrm{l}_1^r & \alpha_-^{k-2r} \otimes \mathrm{l}_1^r & \alpha_4^{k-2r-1}\gamma^5 \otimes \epsilon \mathrm{l}_2^r \\ \alpha_+^{k-2r} \otimes \mathrm{l}_1^r & \frac{1}{2}(\alpha_0^{k-2r} - \alpha_3^{k-2r}) \otimes \mathrm{l}_1^r & \alpha_5^{k-2r-1}\gamma^5 \otimes \epsilon \mathrm{l}_2^r \\ \alpha_6^{k-2r-1}\gamma^5 \otimes \epsilon^{-1}\mathrm{l}_3^r & \alpha_7^{k-2r-1}\gamma^5 \otimes \epsilon^{-1}\mathrm{l}_3^r & \alpha_0^{k-2r} \otimes \mathrm{l}_4^r \end{pmatrix}, \tag{20}$$

$$\mathbf{i}_q(\varrho_q^k) = \sum_{r=0}^{m} \begin{pmatrix} \begin{array}{c|c|c|c} \begin{matrix}(\frac{1}{2}\alpha_3^{k-2r} - \frac{1}{6}\alpha_0^{k-2r}) \\ \otimes \mathrm{q}_1^r\end{matrix} & \alpha_-^{k-2r} \otimes \mathrm{q}_1^r & \begin{matrix}\alpha_7^{k-2r-1}\gamma^5 \\ \otimes\beta\sqrt{\frac{2}{3}}\,\mathrm{q}_2^r\end{matrix} & \begin{matrix}\alpha_4^{k-2r-1}\gamma^5 \\ \otimes\sqrt{\frac{1}{3}}\,\mathrm{q}_2^r\chi\gamma\end{matrix} \\ \hline \alpha_+^{k-2r} \otimes \mathrm{q}_1^r & \begin{matrix}(-\frac{1}{2}\alpha_3^{k-2r} - \frac{1}{6}\alpha_0^{k-2r}) \\ \otimes \mathrm{q}_1^r\end{matrix} & \begin{matrix}-\alpha_6^{k-2r-1}\gamma^5 \\ \otimes\beta\sqrt{\frac{2}{3}}\,\mathrm{q}_2^r\end{matrix} & \begin{matrix}\alpha_5^{k-2r-1}\gamma^5 \\ \otimes\sqrt{\frac{1}{3}}\,\mathrm{q}_2^r\chi\gamma\end{matrix} \\ \hline \begin{matrix}-\alpha_5^{k-2r-1}\gamma^5 \\ \otimes\beta^{-1}\sqrt{\frac{2}{3}}\,\mathrm{q}_3^r\end{matrix} & \begin{matrix}\alpha_4^{k-2r-1}\gamma^5 \\ \otimes\beta^{-1}\sqrt{\frac{2}{3}}\,\mathrm{q}_3^r\end{matrix} & \begin{matrix}-\frac{2}{3}\alpha_0^{k-2r} \\ \otimes\mathrm{q}_4^r\end{matrix} & 0 \\ \hline \begin{matrix}\alpha_6^{k-2r-1}\gamma^5 \\ \otimes(\chi\gamma)^{-1}\sqrt{\frac{1}{3}}\,\mathrm{q}_3^r\end{matrix} & \begin{matrix}\alpha_7^{k-2r-1}\gamma^5 \\ \otimes(\chi\gamma)^{-1}\sqrt{\frac{1}{3}}\,\mathrm{q}_3^r\end{matrix} & 0 & \begin{matrix}\frac{1}{3}\alpha_0^{k-2r} \\ \otimes\chi^{-1}\mathrm{q}_4^r\chi\end{matrix} \end{array} \end{pmatrix} . \tag{21}$$

The isomorphism $\mathbf{i}$ fulfils $\mathbf{i}([\varrho^k, \tilde\varrho^n]_g) = \mathbf{i}(\varrho^k)\,\mathbf{i}(\tilde\varrho^n) - (-1)^{kn}\mathbf{i}(\tilde\varrho^n)\,\mathbf{i}(\varrho^k)$, for $\varrho^k \in \mathcal{H}_0^k$ and $\tilde\varrho^n \in \mathcal{H}_0^n$, where the multiplication $\mathbf{i}(\varrho^k)\,\mathbf{i}(\tilde\varrho^n)$ is the natural combination of the multiplication $\wedge$ in L^* and matrix multiplication.

In the above formulae ϵ and β are invertible diagonal 3×3–matrices which, therefore, commute with $\mathrm{m}_l, \mathrm{m}_l^*, \mathrm{m}_q, \mathrm{m}_q^*$. For the invertible 3×3–matrices γ and χ we have to demand $(\chi\gamma)^{-1}\mathrm{m}_q^*\mathrm{m}_q\chi\gamma = \chi^{-1}\mathrm{m}_q^*\mathrm{m}_q\chi$, which is achieved by taking $\chi\gamma\chi^{-1}$ diagonal. The matrix χ need not to be unitary. In principle, we could in an analogous way introduce matrices such as χ in the third row and column in (20) and (21), too. However, this can be reabsorbed by unitary transformations of the physical fields. Thus, the freedom in the choice of the isomorphism $\mathbf{i}$, modulo unitary transformations, introduced a lot of additional free parameters in (20) and (21). But, we shall see at the end of section V that all these parameters are (not uniquely) fixed by the physical model. The genuine free parameters are the six eigenvalues of M , see (1).

The motivation to consider just the isomorphism (20), (21) comes from our paper [2], where we constructed a partial homomorphism of $\mathcal{H}_0$ onto $\Lambda^*(X) \otimes$ *spl* $(2,1)$, and the paper [13], where similar looking representations of *spl* $(2,1)$ were used to write down a fermionic Lagrangian for the standard model.

IV Bounded Operators on the Hilbert Space of Fermions

Here we are going to associate to each $\mathbf{i}(\varrho^k) \in \mathbf{i}(\mathcal{H}_0^k)$, $k = 0,1,2$, a bounded operator on the Hilbert space

$$\begin{aligned} &H = H_l \oplus H_q \ , \\ &H_l = L^2(X,S)\otimes\mathbb{C}^3\otimes\mathbb{C}^3 \ , \quad H_q = L^2(X,S)\otimes(\mathbb{C}^4\oplus\mathbb{C}^4\oplus\mathbb{C}^4)\otimes\mathbb{C}^3 \ . \end{aligned} \tag{22}$$

First, because of $L^0 \equiv C^0$ and $L^1 \equiv C^1$ we can regard elements of $\mathbf{i}(\mathcal{H}_0^0)$ and $\mathbf{i}(\mathcal{H}_0^1)$ in a natural way as bounded operators on H , see (20) and (21), where the matrices l_s^r and q_s^r act on the last $\mathbb{C}^3$–components. Next, we associate to the local basis $[\gamma^\mu]\wedge[\gamma^\nu]$ in L^2 the operator $\mathbf{c}([\gamma^\mu]\wedge[\gamma^\nu]) := \gamma^\mu\cdot\gamma^\nu \in B(L^2(X,S))$, for $1\le\mu<\nu\le 4$. By linear extension we obtain a vector space isomorphism $\mathbf{c}$ of L^2 onto its image. The isomorphism $\mathbf{c}$ induces the vector space isomorphism $\tilde{\mathbf{c}} : \mathbf{i}(\mathcal{H}_0^2)\to\tilde{\mathbf{c}}\circ\mathbf{i}(\mathcal{H}_0^2)$, $\tilde{\mathbf{c}}\circ\mathbf{i}(\mathcal{H}_0^2)\subset B(H)$:

$$\tilde{\mathbf{c}}\circ\mathbf{i}(\varrho^2) := \begin{pmatrix} (\mathbf{c}\otimes\mathrm{id}_{\mathbb{C}^3\otimes\mathbb{C}^3})(\mathbf{i}_l(\varrho_l^2)) & 0 \\ 0 & \{(\mathbf{c}\otimes\mathrm{id}_{\mathbb{C}^4\otimes\mathbb{C}^3})(\mathbf{i}_q(\varrho_q^2))\}\otimes\mathbb{1}_{3\times3}\end{pmatrix} , \tag{23}$$

for $\varrho^2\in\mathcal{H}_0^2$. Next, we define a vector subspace $\tilde{\mathcal{T}}^2\subset B(H)$ as

$$\tilde{\mathcal{T}}^2 = \{\ \sum_\alpha(\mathbf{i}(\varrho_\alpha^1)\cdot\mathbf{i}(\tilde{\varrho}_\alpha^1)+\mathbf{i}(\tilde{\varrho}_\alpha^1)\cdot\mathbf{i}(\varrho_\alpha^1))\ ,\quad \varrho_\alpha^1,\tilde{\varrho}_\alpha^1\in\mathcal{H}_0^1\ ,\quad \text{finite sum}\ \}\ , \tag{24}$$

where $\cdot$ denotes the multiplication in $B(H)$, and put

$$\mathcal{T}^2 = \tilde{\mathbf{c}}\circ\mathbf{i}(\mathcal{H}_0^2)+\tilde{\mathcal{T}}^2\ . \tag{25}$$

One easily convinces oneself that another characterization of this space is

$$\begin{aligned}\mathcal{T}^2 &= \tilde{\mathbf{c}}\circ\mathbf{i}(\mathcal{H}_0^2)\oplus\mathbf{i}(\mathcal{H}_0^0)\oplus\Delta\ ,\\ \Delta &= \{\ \mathbf{i}(\mathrm{diag}\,(\alpha_3^0,0,-\alpha_3^0,0)\otimes\mathrm{id}_F)\cdot\mathbf{i}(\mathrm{diag}\,(\tilde{\alpha}_3^0,0,-\tilde{\alpha}_3^0,0)\otimes\mathrm{id}_F)\\ &\qquad +\mathbf{i}(\mathrm{diag}\,(\alpha_0^0,0,\alpha_0^0,2\alpha_0^0)\otimes\mathrm{id}_F)\cdot\mathbf{i}(\mathrm{diag}\,(\tilde{\alpha}_0^0,0,\tilde{\alpha}_0^0,2\tilde{\alpha}_0^0)\otimes\mathrm{id}_F)\ ,\\ &\qquad \alpha_3^0,\tilde{\alpha}_3^0,\alpha_0^0,\tilde{\alpha}_0^0\in C^0\ \}\ .\end{aligned} \tag{26}$$

Let $\mathbf{p}_2$ be the projection of $\mathcal{T}^2=\tilde{\mathbf{c}}\circ\mathbf{i}(\mathcal{H}_0^2)\oplus\mathbf{i}(\mathcal{H}_0^0)\oplus\Delta$ onto its first component $\tilde{\mathbf{c}}\circ\mathbf{i}(\mathcal{H}_0^2)$. Then we have

$$\tilde{\mathbf{c}}^{-1}\circ\mathbf{p}_2(\mathbf{i}(\varrho^1)\cdot\mathbf{i}(\tilde{\varrho}^1)+\mathbf{i}(\tilde{\varrho}^1)\cdot\mathbf{i}(\varrho^1)) = \mathbf{i}([\varrho^1,\tilde{\varrho}^1]_g)\ , \tag{27}$$

which is in some sense an analogy to (5).

On $B(H)$ there is a natural scalar product given by the Dixmier trace Tr_ω :

$$< b,\tilde{b} >_{B(H)} = \mathrm{Tr}_\omega(b^*\tilde{b}|D|^{-4})\ ,\quad b,\tilde{b}\in B(H)\ , \tag{28}$$

where D is the generalized Dirac operator of the K–cycle and * denotes the involution in $B(H)$. In the case considered here the Dixmier trace can be expressed by a combination of the usual trace over the matrix structure including the trace in the Clifford algebra and integration over the manifold X :

$$< b,\tilde{b} >_{B(H)} = \frac{1}{32\pi^2}\int_X v_g\,\mathrm{tr}\,(b^*\,\tilde{b})\ , \tag{29}$$

where v_g is the canonical volume form on X and the factor $\frac{1}{32\pi^2}$ is taken from [6]. By restriction of the scalar product (28) to $\mathcal{T}^2\subset B(H)$ we get a natural scalar product on $\mathcal{T}^2$. However, since elements of $\mathcal{T}^2$ are diagonal with respect to the splitting $H = H_l \oplus H_q$ in (22), we can take as a scalar product $<\,,\,>_{\mathcal{T}^2}$ on $\mathcal{T}^2$ a convex linear combination of the partial traces in $B(H_l)$ and $B(H_q)$, in the same way as in [11]:

$$< b, \tilde b >_{\mathcal{T}^2} = < zb, \tilde b >_{B(H)} \;, \quad z = \begin{pmatrix} x \mathbb{1}_{9\times 9} & 0 \\ 0 & \mathbb{1}_{36\times 36} \end{pmatrix} \;, \tag{30}$$

for $b, \tilde b \in \mathcal{T}^2$ and $0 < x < \infty$.

The direct sum decomposition $\mathcal{T}^2 = \tilde{\mathbf{c}} \circ \mathbf{i}(\mathcal{H}^2_0) \oplus \ker \mathbf{p}_2$ is not an orthogonal decomposition with respect to $<\,,\,>_{\mathcal{T}^2}$. Let $\mathbf{s}_2$ be the orthogonal projection from $\mathcal{T}^2$ onto the orthogonal complement of the subspace $\ker \mathbf{p}_2$ (after a completion with respect to the scalar product). Then we define the canonical embedding $\mathbf{e} : \mathcal{H}^2_0 \to B(H)$ by

$$\mathbf{e}(\varrho^2) = \mathbf{s}_2 \circ \tilde{\mathbf{c}} \circ \mathbf{i}(\varrho^2) \;, \quad \varrho^2 \in \mathcal{H}^2 \;. \tag{31}$$

In the same way as in [1] one can show that $\mathbf{e}(\varrho^2)$ is given by (20) and (21) if we replace

$$\alpha^2_f \mapsto \mathbf{c}(\alpha^2_f) \;, \quad f = 0, +, -, 3 \;, \tag{32}$$

$$\mathrm{l}^2_s \mapsto \tilde{\mathrm{l}}^2_s := \mathrm{l}^2_s - \frac{\mathrm{tr}\ \mathrm{l}^2_s}{3} \mathbb{1}_{3\times 3} \;, \quad \mathrm{q}^2_s \mapsto \tilde{\mathrm{q}}^2_s := \mathrm{q}^2_s - \frac{\mathrm{tr}\ \mathrm{q}^2_s}{3} \mathbb{1}_{3\times 3} \;, \quad s = 1, 4 \;. \tag{33}$$

Thus, the operators on H associated to $\mathbf{i}(\varrho^k) \in \mathbf{i}(\mathcal{H}^k_0)$, $k = 0, 1, 2$, are $\mathbf{i}(\varrho^k)$ themselves for $k = 0, 1$ and $\mathbf{e}(\varrho^k)$ for $k = 2$. Of course, this construction can be extended to $\mathbf{i}(\mathcal{H}^k_0)$, $k > 2$, but we do not need this for model building.

V The Fermionic Action

The connection form ρ is a skew–adjoint element of $\mathcal{H}^1_0$, see (14). Therefore, we have

$$\rho = \begin{pmatrix} (\frac{1}{2}A^3 + \frac{1}{2}A^0) \otimes \mathrm{id}_F & 0 & A^- \otimes \mathrm{id}_F & -i\Phi^1\gamma^5 \otimes M \\ 0 & 0 & 0 & 0 \\ A^+ \otimes \mathrm{id}_F & 0 & (-\frac{1}{2}A^3 + \frac{1}{2}A^0) \otimes \mathrm{id}_F & -i\Phi^2\gamma^5 \otimes M \\ -i\overline{\Phi}^1\gamma^5 \otimes M^* & 0 & -i\overline{\Phi}^2\gamma^5 \otimes M^* & A^0 \otimes \mathrm{id}_F \end{pmatrix} \;, \tag{34}$$

where $A^0 = -(A^0)^*$, $A^3 = -(A^3)^*$, $A^+ = -(A^-)^* \in L^1$ and Φ^1, $\Phi^2 \in L^0$. Applying the isomorphism $\mathbf{i}$ and abbreviating $\mathbb{1} = \mathbb{1}_{3\times 3}$, we get from (18), (20) and (21)

$$\mathbf{i}_l(\rho_l) = \begin{pmatrix} (\frac{1}{2}A^3 + \frac{1}{2}A^0) \otimes \mathbb{1} & A^- \otimes \mathbb{1} & i\Phi^1\gamma^5 \otimes \epsilon \mathrm{m}_l \\ A^+ \otimes \mathbb{1} & (-\frac{1}{2}A^3 + \frac{1}{2}A^0) \otimes \mathbb{1} & i\Phi^2\gamma^5 \otimes \epsilon \mathrm{m}_l \\ i\overline{\Phi}^1\gamma^5 \otimes \epsilon^{-1}\mathrm{m}^*_l & i\overline{\Phi}^2\gamma^5 \otimes \epsilon^{-1}\mathrm{m}^*_l & A^0 \otimes \mathbb{1} \end{pmatrix} , \tag{35}$$

$$\mathbf{i}_q(\rho_q)=\left(\begin{array}{c|c|c|c} (\frac{1}{2}A^3-\frac{1}{6}A^0)\otimes\mathbb{1} & A^-\otimes\mathbb{1} & \begin{array}{c} i\overline{\Phi}^2\gamma^5 \\ \otimes\sqrt{\frac{2}{3}}\beta\mathrm{m}_q \end{array} & \begin{array}{c} i\Phi^1\gamma^5 \\ \otimes\sqrt{\frac{1}{3}}\mathrm{m}_q\chi\gamma \end{array} \\ \hline A^+\otimes\mathbb{1} & (-\frac{1}{2}A^3-\frac{1}{6}A^0)\otimes\mathbb{1} & \begin{array}{c} -i\overline{\Phi}^1\gamma^5 \\ \otimes\sqrt{\frac{2}{3}}\beta\mathrm{m}_q \end{array} & \begin{array}{c} i\Phi^2\gamma^5 \\ \otimes\sqrt{\frac{1}{3}}\mathrm{m}_q\chi\gamma \end{array} \\ \hline \begin{array}{c} -i\Phi^2\gamma^5 \\ \otimes\sqrt{\frac{2}{3}}\beta^{-1}\mathrm{m}_q^* \end{array} & \begin{array}{c} i\Phi^1\gamma^5 \\ \otimes\sqrt{\frac{2}{3}}\beta^{-1}\mathrm{m}_q^* \end{array} & -\frac{2}{3}A^0\otimes\mathbb{1} & 0 \\ \hline \begin{array}{c} i\overline{\Phi}^1\gamma^5 \\ \otimes\sqrt{\frac{1}{3}}(\chi\gamma)^{-1}\mathrm{m}_q^* \end{array} & \begin{array}{c} i\overline{\Phi}^2\gamma^5 \\ \otimes\sqrt{\frac{1}{3}}(\chi\gamma)^{-1}\mathrm{m}_q^* \end{array} & 0 & \frac{1}{3}A^0\otimes\mathbb{1} \end{array}\right). \tag{36}$$

The connection ∇ can be extended to an operator $\nabla:\mathcal{E}\otimes_{\mathcal{A}}h\to\mathcal{E}\otimes_{\mathcal{A}}h$,

$$\begin{aligned}\nabla(\xi\otimes_{\mathcal{A}}\psi)&=(i\nabla\xi)\otimes_{\mathcal{A}}\psi+\xi\otimes_{\mathcal{A}}D\psi=e\{((D\otimes\mathbb{1}_{2\times2})\xi-\xi D)+i\rho\xi\}\otimes_{\mathcal{A}}\psi+\xi\otimes_{\mathcal{A}}D\psi\\ &=(e(D\otimes\mathbb{1}_{2\times2})e+i\rho)(\xi\otimes_{\mathcal{A}}\psi)=(D^{cl}+\mu+i\rho)(\xi\otimes_{\mathcal{A}}\psi)\ ,\\ \mu&=e(\gamma^5\otimes\mathcal{M}\otimes\mathbb{1}_{2\times2})e\ ,\end{aligned} \tag{37}$$

for $\xi\in\mathcal{E}$, $\psi\in h$, see [5]. We have used $\hat{d}a=-i[D,a]$ for $a\in\mathcal{A}$, see [1], and $L^n\equiv C^n$ for $n=0,1$. We have $\mu\in\mathcal{H}_0^1$, hence we can apply the isomorphism $\mathbf{i}$, and we find

$$\mathbf{i}_l(\mu_l)=\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\gamma^5\otimes\epsilon\mathrm{m}_l \\ 0 & -\gamma^5\otimes\epsilon^{-1}\mathrm{m}_l^* & 0 \end{pmatrix}, \tag{38}$$

$$\mathbf{i}_q(\mu_q)=\begin{pmatrix} 0 & 0 & -\gamma^5\otimes\sqrt{\frac{2}{3}}\beta\mathrm{m}_q & 0 \\ 0 & 0 & 0 & -\gamma^5\otimes\sqrt{\frac{1}{3}}\mathrm{m}_q\chi\gamma \\ \gamma^5\otimes\sqrt{\frac{2}{3}}\frac{1}{\beta}\mathrm{m}_q^* & 0 & 0 & 0 \\ 0 & -\gamma^5\otimes\sqrt{\frac{1}{3}}(\chi\gamma)^{-1}\mathrm{m}_q^* & 0 & 0 \end{pmatrix}.$$

The situation found in nature demands to use a pseudo–Riemannian manifold X_M instead of the Riemannian manifold X. We are interested in the case that X_M is the Minkowski space. We convert the results obtained so far for the Euclidian manifold X by a Wick rotation to Minkowski space. If we denote by $L^2(X_M,S)$ the space of square integrable sections of the spinor bundle over X_M then instead of H, see (22), we have to take the space

$$H_M=\{L^2(X_M,S)\otimes\mathbb{C}^3\oplus\mathbb{C}^3\}\oplus\{L^2(X_M,S)\otimes(\mathbb{C}^4\oplus\mathbb{C}^4\oplus\mathbb{C}^4)\otimes\mathbb{C}^3\}\ . \tag{39}$$

On H_M we have the invariant product

$$<\Psi,\tilde{\Psi}>_{H_M}:=\int_{X_M}v_M\ \Psi^*\gamma^0\tilde{\Psi}\ ,\quad \Psi,\tilde{\Psi}\in H_M\ , \tag{40}$$

where v_M is the canonical volume form on X_M. Due to (37) the natural fermionic action is

$$S_F = \tfrac{1}{2} < \Psi, (D^{c\ell} + \mathbf{i}(\mu + i\rho_M))\Psi >_{H_M} + h.c.\ , \quad \Psi \in H_M\ , \tag{41}$$

where ρ_M is the connection form ρ rotated to Minkowski space and $h.c.$ denotes the Hermitian conjugate of the preceding term. We take

$$\Psi = (\Psi_l, \Psi_q)^T\ , \quad \Psi_l = (\nu_L, e_L, e_R)^T\ , \quad \Psi_q = (u_L, d_L, u_R, d_R)^T\ , \tag{42}$$

where $e_L, \nu_L, e_R \in L^2(X_M, S) \otimes \mathbb{C}^3$ and $u_L, d_L, u_R, d_R \in L^2(X, S) \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$, with

$$(\gamma^5 \otimes \mathrm{id}) f_L = -f_L\ , \quad (\gamma^5 \otimes \mathrm{id}) f_R = f_R\ , \quad f = e, \nu, u, d\ . \tag{43}$$

We obtain, denoting the Wick rotated fields $A^{0,+,-,3}$ and $\Phi^{1,2}$ by the same symbols,

$$\begin{aligned}
S_F &= \textstyle\int_{X_M} v_M\ (\mathscr{L}_l + \mathscr{L}'_l + \mathscr{L}_q + \mathscr{L}'_q)\ ,\\
\mathscr{L}_l &= \begin{pmatrix}\nu_L^*\, ;\, e_L^*\end{pmatrix} \Big\{\gamma^0 \begin{pmatrix} D^{c\ell} + i(\frac{1}{2}A^3 + \frac{1}{2}A^0) & iA^- \\ iA^+ & D^{c\ell} + i(-\frac{1}{2}A^3 + \frac{1}{2}A^0)\end{pmatrix} \otimes \mathbb{1}\Big\} \begin{pmatrix}\nu_L \\ e_L\end{pmatrix}\\
&\quad + e_R^* \{\gamma^0 (D^{c\ell} + iA^0) \otimes \mathbb{1}\} e_R\ ,\\
\mathscr{L}'_l &= -\{e_R^* \Big\{\gamma^0 \begin{pmatrix}\overline{\Phi}^1\, ;\, \overline{\Phi}^2 + 1\end{pmatrix} \otimes \tfrac{1}{2}(\epsilon^* - \epsilon^{-1})\mathrm{m}_l^*\Big\} \begin{pmatrix}\nu_L \\ e_L\end{pmatrix} + h.c.\}\ ,\\
\mathscr{L}_q &= \begin{pmatrix}u_L^*\, ;\, d_L^*\end{pmatrix} \Big\{\gamma^0 \begin{pmatrix} D^{c\ell} + i(\frac{1}{2}A^3 - \frac{1}{6}A^0) & iA^- \\ iA^+ & D^{c\ell} + i(-\frac{1}{2}A^3 - \frac{1}{6}A^0)\end{pmatrix} \otimes \mathbb{1}_c \otimes \mathbb{1}\Big\} \begin{pmatrix}u_L \\ d_L\end{pmatrix}\\
&\quad + u_R^* \{\gamma^0 (D^{c\ell} - \tfrac{2}{3} iA^0) \otimes \mathbb{1}_c \otimes \mathbb{1}\} u_R + d_R^* \{\gamma^0 (D^{c\ell} + \tfrac{1}{3} iA^0) \otimes \mathbb{1}_c \otimes \mathbb{1}\} d_R\ ,\\
\mathscr{L}'_q &= -\{u_R^* \Big\{\gamma^0 \begin{pmatrix}\Phi^2 + 1\, ;\, -\Phi^1\end{pmatrix} \otimes \mathbb{1}_c \otimes \tfrac{1}{2}\sqrt{\tfrac{2}{3}}(\beta^* + \beta^{-1})\mathrm{m}_q^*\Big\} \begin{pmatrix}u_L \\ d_L\end{pmatrix} + h.c.\}\\
&\quad -\{d_R^* \Big\{\gamma^0 \begin{pmatrix}\overline{\Phi}^1\, ;\, \overline{\Phi}^2 + 1\end{pmatrix} \otimes \mathbb{1}_c \otimes \tfrac{1}{2}\sqrt{\tfrac{1}{3}}((\chi\gamma)^* - (\chi\gamma)^{-1})\mathrm{m}_q^*\Big\} \begin{pmatrix}u_L \\ d_L\end{pmatrix} + h.c.\}\ ,
\end{aligned} \tag{44}$$

where $\mathbb{1}_c = \mathbb{1}_{3\times3}$ is the identity acting on the colour space of the quarks, see section VII. Comparing this with the classical fermionic action [17], we read off the mass matrices of the fermions

$$\mathrm{m}_e = \tfrac{1}{2}(\epsilon^* - \epsilon^{-1})\mathrm{m}_l^*\ , \quad \mathrm{m}_u = \tfrac{1}{2}\sqrt{\tfrac{2}{3}}(\beta^* + \beta^{-1})\mathrm{m}_q^*\ , \quad \mathrm{m}'_d = \tfrac{1}{2}\sqrt{\tfrac{1}{3}}((\chi\gamma)^* - (\chi\gamma)^{-1})\mathrm{m}_q^*\ , \tag{45}$$

where $e \equiv (e, \mu, \tau)^T$, $u \equiv (u, c, t)^T$, $d \equiv (d, s, b)^T$. The matrices $\mathrm{m}_e, \mathrm{m}_u$ are diagonal. We can assume that their diagonal matrix elements are positive, otherwise this is achieved by unitary transformations $e_R \mapsto \mathrm{u}_e e_R$, $u_R \mapsto \mathrm{u}_u u_R$ with diagonal unitary matrices $\mathrm{u}_e, \mathrm{u}_u$. However, the matrix m'_d is – in general – an arbitrary non–diagonal 3×3–matrix. There exist unitary matrices $\mathrm{u}_1, \mathrm{u}_2$, so that $\mathrm{m}'_d = \mathrm{u}_1 \mathrm{m}_d \mathrm{u}_2^*$ for a diagonal matrix m_d with positive diagonal matrix elements. The matrix u_1 can be absorbed by means of a unitary transformation

$d_R \mapsto \mathrm{u}_1 d_R$. But the matrix u_2^* cannot be absorbed by a unitary transformation $\binom{u_L}{d_L} \mapsto \left(\begin{smallmatrix} \mathrm{u}_2 & 0 \\ 0 & \mathrm{u}_2 \end{smallmatrix}\right)\binom{u_L}{d_L}$, because this would make the matrix m_u non–diagonal. All we can do are transformations $\binom{u_L}{d_L} \mapsto \left(\begin{smallmatrix} \delta_1 & 0 \\ 0 & \delta_1 \end{smallmatrix}\right)\binom{u_L}{d_L}$, $u_R \mapsto \delta_1 u_R$ and $d_R \mapsto \delta_2 d_R$ by diagonal unitary matrices δ_1, δ_2, so that m_u remains invariant and $\mathrm{u}_2^* \mapsto \delta_1^* \mathrm{u}_2^* \delta_2 =: V$. The matrix V is the famous Kobayashi–Maskawa matrix, which for an appropriate choice of δ_1, δ_2 can be brought into the standard form parametrized by three rotation angles and one phase. Thus, we can rewrite $\mathscr{L}_l'$ and $\mathscr{L}_q'$ as

$$
\begin{aligned}
\mathscr{L}_l' &= -\{e_R^* \Big\{\gamma^0 \left(\overline{\Phi}^1 \,;\, \overline{\Phi}^2 + 1\right) \otimes \mathrm{m}_e\Big\} \begin{pmatrix} \nu_L \\ e_L \end{pmatrix} + h.c.\} \;, \\
\mathscr{L}_q' &= -\{u_R^* \Big\{\gamma^0 \left(\Phi^2 + 1 \,;\, -\Phi^1\right) \otimes \mathrm{m}_u \otimes \mathbb{1}_c\Big\} \begin{pmatrix} u_L \\ d_L \end{pmatrix} + h.c.\} \\
&\quad -\{d_R^* \Big\{\gamma^0 \left(\overline{\Phi}^1 \,;\, \overline{\Phi}^2 + 1\right) \otimes \mathrm{m}_d V \otimes \mathbb{1}_c\Big\} \begin{pmatrix} u_L \\ d_L \end{pmatrix} + h.c.\} \;.
\end{aligned}
\tag{46}
$$

Now we count the free parameters on which the physical masses in our model depend. We can regard (45) as equations for computing the fermion masses and the Kobayashi–Maskawa matrix for given 3×3–matrices $\beta, \gamma, \epsilon, \chi, \mathrm{m}_l, \mathrm{m}_q$. But if we take the point of view that the fermion masses and the Kobayashi–Maskawa matrix (13 parameters) are given by experiment and if we keep the diagonal positive matrices m_l and m_q as additional free parameters then (45), with $\mathrm{m}_d' = \mathrm{m}_d V$, are equations to determine $\beta, \gamma, \epsilon, \chi$. This gives a system of quadratic equations for the matrix elements of $\beta, \gamma, \epsilon, \chi$. We take any of the (possibly complex) solutions of this system to fix the matrices $\beta, \gamma, \epsilon, \chi$. Then, there remain only the six free parameters in the diagonal of m_l and m_q.

We remark that the presented construction of the fermionic action of the standard model yields immediately the correct hypercharges of the fermions – namely the coefficients in front of $-\frac{1}{2}A^0$ in (35) and (36). This is possible, because we use the isomorphism $\mathbf{i}$ of graded Lie algebras, which allows embeddings into the space of bounded operators on the fermionic Hilbert space different from the fundamental embedding. For matrix algebras there exist – besides the trivial representation – only the fundamental representation and, hence, only the fundamental embedding. Therefore, in the derivation of standard model elaborated by Kastler [7] one must additionally consider the chromodynamics algebra and impose a generalized Poincaré duality condition ([3], [7]) in order to obtain the correct hypercharges.

VI The Bosonic Action

We recall [2] that the curvature θ of the connection ∇ is given by

$$
\theta = \mathcal{D}\rho + \tfrac{1}{2}[\rho, \rho]_g + \theta_0 = \mathbf{D}\rho - i[\mu, \rho]_g + \tfrac{1}{2}[\rho, \rho]_g + \theta_0 \;, \tag{47}
$$

where

$$\theta_0 = e\hat{d}(e) \bullet \hat{d}(e)e = \mathrm{diag}\,(MM^*, 0, 0, 0)\ . \tag{48}$$

The problem is that $\theta_0 \notin \mathcal{H}_0$, so that it is not possible to apply the isomorphism $\mathbf{i}$. We propose to replace here the isomorphism $\mathbf{i}$ by a linear mapping $\mathbf{i}'$ defined as follows. Looking at (16) we see that θ_0 is given by putting $\alpha_3^0 = \alpha_0^0 = 1$ and all other $\alpha_f^{0,1,2} = 0$, and then projecting away the last row and column. Thus, to the element $\tilde{\theta}_0 \in \mathcal{H}_0^2$ given by $\alpha_3^0 = \alpha_0^0 = 1$ we apply the isomorphism $\mathbf{i}$, giving $\mathbf{i}_l(\tilde{\theta}_{0,l}) = \mathrm{diag}\,(\mathrm{l}_1^1, 0, \mathrm{l}_4^1)$ and $\mathbf{i}_q(\tilde{\theta}_{0,q}) = \mathrm{diag}\,(\frac{1}{3}\mathrm{q}_1^1, -\frac{2}{3}\mathrm{q}_1^1, -\frac{2}{3}\mathrm{q}_4^1, \frac{1}{3}\mathrm{q}_4^1)$, see (20) and (21). Now we apply a reasonable projection:

$$\mathbf{i}_l'(\theta_{0,l}) := \mathrm{diag}\,(\mathrm{l}_1^1, 0, 0)\ , \quad \mathbf{i}_q'(\theta_{0,q}) := \mathrm{diag}\,(\tfrac{1}{3}\mathrm{q}_1^1, -\tfrac{2}{3}\mathrm{q}_1^1, -\tfrac{2}{3}\mathrm{q}_4^1, 0)\ . \tag{49}$$

The choice of $\mathbf{i}_l'(\theta_{0,l})$ is plausible, but I have no deeper explanation for $\mathbf{i}_q'(\theta_{0,q})$, except that we need this below. We recall that $\mathrm{l}_1^1 = \mathrm{l}_4^1 = \mathrm{m}_l\mathrm{m}_l^* \equiv |\mathrm{m}_l|^2$ and $\mathrm{q}_1^1 = \mathrm{q}_4^1 = \mathrm{m}_q\mathrm{m}_q^* \equiv |\mathrm{m}_q|^2$. Then, using (35), (36) and (38), the transported curvature is given by ($\mathrm{f} \in \{l, q\}$)

$$\mathbf{i}_\mathrm{f}(\theta_\mathrm{f}) = \mathbf{D}(\mathbf{i}_\mathrm{f}(\rho_\mathrm{f})) - i[\mathbf{i}_\mathrm{f}(\mu_\mathrm{f}), \mathbf{i}_\mathrm{f}(\rho_\mathrm{f})]_g + \tfrac{1}{2}[\mathbf{i}_\mathrm{f}(\rho_\mathrm{f}), \mathbf{i}_\mathrm{f}(\rho_\mathrm{f})]_g + \mathbf{i}_\mathrm{f}'(\theta_{0,\mathrm{f}})\ , \tag{50}$$

$$\mathbf{i}_l(\theta_l) = \left(\begin{array}{c|c|c}
\begin{array}{c}(\frac{1}{2}\mathbf{d}(A^3+A^0) + A^-\wedge A^+) \\ \otimes\mathbb{1} - (|\mathbf{\Phi}^1|^2 - 1)\otimes|\mathrm{m}_l|^2\end{array} &
\begin{array}{c}(\mathbf{d}A^- + A^3\wedge A^-)\otimes\mathbb{1} \\ -\mathbf{\Phi}^1(\overline{\mathbf{\Phi}}^2 + 1)\otimes|\mathrm{m}_l|^2\end{array} &
\begin{array}{c}i(\mathbf{d}\mathbf{\Phi}^1 + \frac{1}{2}(A^3 - A^0)\mathbf{\Phi}^1 \\ +A^-(\mathbf{\Phi}^2+1))\gamma^5\otimes\epsilon\mathrm{m}_l\end{array} \\ \hline
\begin{array}{c}(\mathbf{d}A^+ + A^+\wedge A^3)\otimes\mathbb{1} \\ -(\mathbf{\Phi}^2+1)\overline{\mathbf{\Phi}}^1\otimes|\mathrm{m}_l|^2\end{array} &
\begin{array}{c}(\frac{1}{2}\mathbf{d}(A^0 - A^3) + A^+\wedge A^-) \\ \otimes\mathbb{1} - (|\mathbf{\Phi}^2+1|^2 - 1)\otimes|\mathrm{m}_l|^2\end{array} &
\begin{array}{c}i(-\frac{1}{2}(A^3+A^0)(\mathbf{\Phi}^2+1) \\ +\mathbf{d}\mathbf{\Phi}^2 + A^+\mathbf{\Phi}^1)\gamma^5\otimes\epsilon\mathrm{m}_l\end{array} \\ \hline
\begin{array}{c}i(\mathbf{d}\overline{\mathbf{\Phi}}^1 - \frac{1}{2}\overline{\mathbf{\Phi}}^1(A^3 - A^0) \\ -(\overline{\mathbf{\Phi}}^2+1)A^+)\gamma^5\otimes\epsilon^{-1}\mathrm{m}_l^*\end{array} &
\begin{array}{c}i(\frac{1}{2}(\overline{\mathbf{\Phi}}^2+1)(A^3+A^0) \\ +\mathbf{d}\overline{\mathbf{\Phi}}^2 - \overline{\mathbf{\Phi}}^1 A^-)\gamma^5\otimes\epsilon^{-1}\mathrm{m}_l^*\end{array} &
\begin{array}{c}\mathbf{d}A^0\otimes\mathbb{1} - (|\mathbf{\Phi}^1|^2 + \\ |\mathbf{\Phi}^2+1|^2 - 1)\otimes|\mathrm{m}_l|^2\end{array}
\end{array}\right),$$

$$\mathbf{i}_q(\theta_q) = \left(\begin{array}{c|c|c|c}
\begin{array}{c}(\mathbf{d}(\frac{1}{2}A^3 - \frac{1}{6}A^0) \\ +A^-\wedge A^+)\otimes\mathbb{1} \\ +(\frac{2}{3}|\mathbf{\Phi}^2+1|^2 - \frac{1}{3} \\ -\frac{1}{3}|\mathbf{\Phi}^1|^2)\otimes|\mathrm{m}_q|^2\end{array} &
\begin{array}{c}(\mathbf{d}A^- + A^3\wedge A^-)\otimes\mathbb{1} \\ -\mathbf{\Phi}^1(\overline{\mathbf{\Phi}}^2+1) \\ \otimes|\mathrm{m}_q|^2\end{array} &
\begin{array}{c}i(\mathbf{d}\overline{\mathbf{\Phi}}^2 - \overline{\mathbf{\Phi}}^1 A^- \\ +\frac{1}{2}(\overline{\mathbf{\Phi}}^2+1)\times \\ (A^3+A^0))\gamma^5 \\ \otimes\sqrt{\frac{2}{3}}\,\beta\mathrm{m}_q\end{array} &
\begin{array}{c}i(\mathbf{d}\mathbf{\Phi}^1 + \\ \frac{1}{2}(A^3 - A^0)\mathbf{\Phi}^1 + \\ +A^-(\mathbf{\Phi}^2+1))\gamma^5 \\ \otimes\sqrt{\frac{1}{3}}\,\mathrm{m}_q\chi\gamma\end{array} \\ \hline
\begin{array}{c}(\mathbf{d}A^+ + A^+\wedge A^3) \\ \otimes\mathbb{1} \\ -(\mathbf{\Phi}^2+1)\overline{\mathbf{\Phi}}^1\otimes|\mathrm{m}_q|^2\end{array} &
\begin{array}{c}(\mathbf{d}(-\frac{1}{2}A^3 - \frac{1}{6}A^0) \\ +A^+\wedge A^-)\otimes\mathbb{1} \\ -(\frac{1}{3}|\mathbf{\Phi}^2+1|^2 + \frac{1}{3} \\ -\frac{2}{3}|\mathbf{\Phi}^1|^2)\otimes|\mathrm{m}_q|^2\end{array} &
\begin{array}{c}-i(\mathbf{d}\overline{\mathbf{\Phi}}^1 - \\ (\overline{\mathbf{\Phi}}^2+1)A^+ - \\ \frac{1}{2}\overline{\mathbf{\Phi}}^1(A^3 - A^0))\gamma^5 \\ \otimes\sqrt{\frac{2}{3}}\,\beta\mathrm{m}_q\end{array} &
\begin{array}{c}i(\mathbf{d}\mathbf{\Phi}^2 + A^+\mathbf{\Phi}^1 \\ -\frac{1}{2}(A^3+A^0)\times \\ (\mathbf{\Phi}^2+1))\gamma^5 \\ \otimes\sqrt{\frac{1}{3}}\,\mathrm{m}_q\chi\gamma\end{array} \\ \hline
\begin{array}{c}-i(\mathbf{d}\mathbf{\Phi}^2 + A^+\mathbf{\Phi}^1 - \\ \frac{1}{2}(A^3+A^0)\times \\ (\mathbf{\Phi}^2+1))\gamma^5 \\ \otimes\sqrt{\frac{2}{3}}\,\beta^{-1}\mathrm{m}_q^*\end{array} &
\begin{array}{c}i(\mathbf{d}\mathbf{\Phi}^1 + \\ A^-(\mathbf{\Phi}^2+1) + \\ \frac{1}{2}(A^3 - A^0)\mathbf{\Phi}^1)\gamma^5 \\ \otimes\sqrt{\frac{2}{3}}\,\beta^{-1}\mathrm{m}_q^*\end{array} &
\begin{array}{c}-\frac{2}{3}\mathbf{d}A^0\otimes\mathbb{1} \\ +\frac{2}{3}(|\mathbf{\Phi}^1|^2 \\ +|\mathbf{\Phi}^2+1|^2 - 2) \\ \otimes|\mathrm{m}_q|^2\end{array} &
0 \\ \hline
\begin{array}{c}i(\mathbf{d}\overline{\mathbf{\Phi}}^1 - \\ (\overline{\mathbf{\Phi}}^2+1)A^+ - \\ \frac{1}{2}\overline{\mathbf{\Phi}}^1(A^3 - A^0))\gamma^5 \\ \otimes\sqrt{\frac{1}{3}}\,(\chi\gamma)^{-1}\mathrm{m}_q^*\end{array} &
\begin{array}{c}i(\mathbf{d}\overline{\mathbf{\Phi}}^2 - \overline{\mathbf{\Phi}}^1 A^- \\ +\frac{1}{2}(\overline{\mathbf{\Phi}}^2+1)\times \\ (A^3+A^0))\gamma^5 \\ \otimes\sqrt{\frac{1}{3}}\,(\chi\gamma)^{-1}\mathrm{m}_q^*\end{array} &
0 &
\begin{array}{c}\frac{1}{3}\mathbf{d}A^0\otimes\mathbb{1} \\ -\frac{1}{3}(|\mathbf{\Phi}^1|^2 \\ +|\mathbf{\Phi}^2+1|^2 - 1) \\ \otimes\chi^{-1}|\mathrm{m}_q|^2\chi\end{array}
\end{array}\right).$$

Now we take the embedding $\mathbf{e}$ of θ into $B(H)$, which means to perform the replacements (32) and (33) in the above matrices, see section IV. We introduce the abbreviations $|\mathrm{m}|^2 := \mathrm{m}\mathrm{m}^*$, $|\mathrm{m}|^4 := (\mathrm{m}\mathrm{m}^*)^2$ and $|\mathrm{m}|^{-2} := (\mathrm{m}\mathrm{m}^*)^{-1}$, for a 3×3–matrix m. Thus, we get the Euclidean bosonic action – for an appropriate choice of the constants – as

$$
\begin{aligned}
S_B &= \frac{8\pi^2}{3(x+3)g_2^2}\mathrm{Tr}_\omega\,\{(z\mathbf{e}(\theta))^*\mathbf{e}(\theta)\} = \frac{1}{12(x+3)g_2^2}\int_X v_g\,\mathrm{tr}\,\{z(\mathbf{e}(\theta))^*\mathbf{e}(\theta)\} \\
&= \int_X v_g\,(\mathscr{L}_2+\mathscr{L}_1+\mathscr{L}_0)\,, \qquad (51)\\
\mathscr{L}_2 &= -\frac{1}{2g_2^2}\,\delta^{\mu\kappa}\delta^{\nu\lambda}\,\{\,\mathrm{tr}\,(F_{\mu\nu}F_{\kappa\lambda}) + \tfrac{9x+11}{6x+18}F'_{\mu\nu}F'_{\kappa\lambda}\,\}\,,\\
F_{\mu\nu} &= \begin{pmatrix} \frac12\partial_{[\mu}A^3_{\nu]} + A^-_{[\mu}A^+_{\nu]} & \partial_{[\mu}A^-_{\nu]} + A^3_{[\mu}A^-_{\nu]} \\ \partial_{[\mu}A^+_{\nu]} + A^+_{[\mu}A^3_{\nu]} & -\frac12\partial_{[\mu}A^3_{\nu]} - A^-_{[\mu}A^+_{\nu]} \end{pmatrix}\,, \quad F'_{\mu\nu} = \partial_{[\mu}A^0_{\nu]}\,,\\
\mathscr{L}_1 &= \frac{1}{3(x+3)g_2^2}\delta^{\mu\nu}\,(\,\overline{(D_\mu\Phi^1)}D_\nu\Phi^1 + \overline{(D_\mu(\Phi^2+1)}D_\nu(\Phi^2+1)\,)\times\\
&\quad \mathrm{tr}\,\{x(|\epsilon|^2+|\epsilon|^{-2})|\mathrm{m}_l|^2 + (2|\beta|^2+2|\beta|^{-2}+|\chi\gamma|^2+|\chi\gamma|^{-2})|\mathrm{m}_q|^2\}\,,\\
&\quad \begin{pmatrix} D_\mu\Phi^1 \\ D_\mu(\Phi^2+1)\end{pmatrix} = \begin{pmatrix}\partial_\mu\Phi^1\\ \partial_\mu\Phi^2\end{pmatrix} + \begin{pmatrix} \frac12(A^3_\mu - A^0_\mu) & A^-_\mu \\ A^+_\mu & -\frac12(A^3_\mu + A^0_\mu)\end{pmatrix}\begin{pmatrix}\Phi^1\\ \Phi^2+1\end{pmatrix}\,,\\
\mathscr{L}_0 &= \frac{2}{3(x+3)g_2^2}(|\Phi^1|^2+|\Phi^2+1|^2-1)^2\,\mathrm{tr}\,(x|\tilde{\mathrm{m}}_l|^4 + \tfrac53|\tilde{\mathrm{m}}_q|^4)\\
&\quad + \frac{1}{(x+3)g_2^2}\mathrm{tr}\,(\tfrac13 x|\tilde{\mathrm{m}}_l|^4 + |\tilde{\mathrm{m}}_q|^4)\,,
\end{aligned}
$$

with $X_{[\mu}Y_{\nu]} \equiv X_\mu Y_\nu - X_\nu Y_\mu$. We have used $\mathrm{tr}_C\,(1) = 4$, $\mathrm{tr}_C\,(\gamma^{\mu'}\cdot\gamma^{\nu'}) = 4\,\delta^{\mu'\nu'}$ and $\mathrm{tr}_C\,(\gamma^\mu\cdot\gamma^\nu\cdot\gamma^\kappa\cdot\gamma^\lambda) = 4(\delta^{\nu\kappa}\delta^{\mu\lambda} - \delta^{\mu\kappa}\delta^{\nu\lambda})$ for $\mu\neq\nu$, $\kappa\neq\lambda$, where tr_C denotes the trace in the space of sections of the Clifford bundle C. We remark that the part $\mathscr{L}_0$ of the Lagrangian would vanish if there was only one generation of fermions, because in this case a formula corresponding to (33) would give zero for $\tilde{m}_l$ and $\tilde{m}_q$. But manifestly there are three fermionic generations in nature. Next, we perform some reparameterizations. We put

$$
\begin{aligned}
&A^3_\mu = ig_2W^3_\mu\,,\ \ A^-_\mu = \frac{i}{2}g_2(W^1_\mu - iW^2_\mu)\,,\ \ A^+_\mu = \frac{i}{2}g_2(W^1_\mu + iW^2_\mu)\,,\\
&A^0_\mu = -\sqrt{\tfrac{3x+9}{9x+11}}\,ig_2W^0_\mu\,, \qquad (52)\\
&\Phi^{1,2} = \sqrt{\tfrac{3(x+3)g_2^2}{2\,\mathrm{tr}\,\{x(|\epsilon|^2+|\epsilon|^{-2})|\mathrm{m}_l|^2+(2|\beta|^2+2|\beta|^{-2}+|\chi\gamma|^2+|\chi\gamma|^{-2})|\mathrm{m}_q|^2\}}}\,\phi_{1,2}\,.
\end{aligned}
$$

This gives ($I.T.\equiv$ interaction terms, $C.C.\equiv$ cosmological constant)

$$
\mathscr{L}_2 = \frac14\delta^{\mu\kappa}\delta^{\nu\lambda}\{\sum_{a=1}^3 F^a_{\mu\nu}F^a_{\kappa\lambda} + F^0_{\mu\nu}F^a_{\kappa\lambda}\}\,, \qquad (53)
$$

$$\mathscr{L}_1 = \frac{1}{2}\delta^{\mu\nu}\{\sum_{j=1}^{2}(\partial_\mu\phi_j)(\partial_\nu\phi_j) + m_W^2(W_\mu^1 W_\nu^1 + W_\mu^2 W_\nu^2) + m_Z^2 Z_\mu Z_\nu\} + I.T.\ , \quad (54)$$

$$\mathscr{L}_0 = \frac{1}{2}m_H^2(\mathrm{Re}\,\phi_2)^2 + I.T. + C.C.\ , \quad (55)$$

where

$$F^a_{\mu\nu} = \partial_{[\mu}W^a_{\nu]} - g_2\sum\nolimits_{b,c=1}^{3}\varepsilon^a{}_{bc}W^b_\mu W^c_\nu\ , \quad F^0_{\mu\nu} = \partial_{[\mu}W^0_{\nu]}\ ,$$
$$Z = \cos\theta_W\ W^3 - \sin\theta_W\ W^0\ ,$$
$$P = \sin\theta_W\ W^3 + \cos\theta_W\ W^0\ ,$$
$$\sin^2\theta_W = \tfrac{3}{4}\tfrac{3+x}{5+3x}\ , \quad (56)$$
$$m_W = \sqrt{\tfrac{1}{6(x+3)}\mathrm{tr}\,\{x(|\epsilon|^2+|\epsilon|^{-2})|\mathrm{m}_l|^2+(2|\beta|^2+2|\beta|^{-2}+|\chi\gamma|^2+|\chi\gamma|^{-2})|\mathrm{m}_q|^2\}}\ , \quad (57)$$
$$m_Z = m_W\,/\cos\theta_W\ ,$$
$$m_H = \tfrac{2}{3\,m_W}\sqrt{\tfrac{1}{x+3}\mathrm{tr}\,(3x|\tilde{\mathrm{m}}_l|^4 + 5|\tilde{\mathrm{m}}_q|^4)}\ . \quad (58)$$

We can transform this action to Minkowski space by a Wick rotation of the physical fields, replacing $\delta^{\mu\nu} \mapsto -g^{\mu\nu}$ and introducing a global minus sign in the action. Then this action coincides with the electroweak sector of the classical bosonic action of the standard model [17], where the Weinberg angle θ_W and the masses m_W of the W–boson and m_H of the Higgs–boson are fixed, see (56), (57) and (58). Inserting the reparameterizations (52) into (44), we get precisely the electroweak sector of the fermionic action of the standard model [17]. According to [2], the gauge group associated to the module $\mathcal{E}$ discussed in the beginning of section III is isomorphic to $C^\infty_{\mathbb{R}}(X_M)\otimes(SU(2)\times U(1))$, where $C^\infty_{\mathbb{R}}(X_M)$ denotes the algebra of real smooth functions on the Minkowski space. The action is by construction invariant under gauge transformations.

VII The Chromodynamics Sector

The chromodynamics sector can be obtained from the module $\mathcal{E}_c = e_c\mathcal{A}^3$, with $e_c = \mathrm{diag}\,(e', e', e')$. The analysis of this case [2] shows that

$$\mathcal{H}^k_{0,c} = \bigoplus_{t=0}^{m} L^{k-2r}\otimes sl(3,\mathbb{C})\otimes M_1^r\ , \quad (59)$$

after omitting the rows and columns consisting of zeros only. Thus, elements $\varrho^k \in \mathcal{H}^k_{0,c}$ are of the form $\varrho^k = \sum_{r=0}^m G^{k-2r}\otimes M_1^r$, where $G^{k-2r}\in L^{k-2r}\otimes sl(3,\mathbb{C})$. We split $\mathcal{H}^k_{0,c}$ into its l_1^r– and q_1^r–part, we take the trivial representation $\mathbf{i}_{c,l}(\varrho^k_l)\equiv 0$ of the l_1^r–part and the fundamental representation (tensorized with $\mathbb{1}_{4\times4}$) of the

q_1^r–part:

$$\mathbf{i}_{c,q}(\varrho_q^k) = \sum_{r=0}^{m} \mathrm{diag}\,(G^{k-2r}, G^{k-2r}, G^{k-2r}, G^{k-2r}) \otimes \mathrm{q}_1^r \ . \tag{60}$$

Obviously, elements $\mathbf{i}_c(\varrho^k)$ are bounded operators on H for $k = 0, 1$. In analogy to the procedure in section IV we construct the space $\mathcal{T}_c^2 \subset B(H)$, which turns out to be

$$\mathcal{T}_c^2 = \tilde{\mathbf{c}}_c \circ \mathbf{i}_c(\mathcal{H}_{0,c}^2) + \mathbf{i}_c(C^0 \otimes gl(3,\mathbb{C}) \otimes \mathrm{id}_F) \ , \tag{61}$$

where $\tilde{\mathbf{c}}_c$ is defined in analogy to (23) and where we extended $\mathbf{i}_c\,|_{\mathcal{H}_{0,c}^0}$ naturally to $C^0 \otimes gl(3,\mathbb{C}) \otimes \mathrm{id}_F$.

From (59) we find for the chromodynamics connection form $\rho_c = G \otimes \mathrm{id}_F$, $G = -G^* \in L^1 \otimes su(3)$. After rotating to Minkowski space ($G \mapsto G_M$) we obtain the following contribution to the fermionic action:

$$\mathscr{L}_s = \Big(u_L^* \,;\, d_L^* \,;\, u_R^* \,;\, d_R^* \Big) \Big\{ \gamma^0 \begin{pmatrix} iG_M & 0 & 0 & 0 \\ 0 & iG_M & 0 & 0 \\ 0 & 0 & iG_M & 0 \\ 0 & 0 & 0 & iG_M \end{pmatrix} \otimes \mathbb{1} \Big\} \begin{pmatrix} u_L \\ d_L \\ u_R \\ d_R \end{pmatrix} \ . \tag{62}$$

The curvature θ_c of the chromodynamics connection ∇_c is given by

$$\theta = \mathbf{d}\rho_c + \frac{1}{2}[\rho_c, \rho_c]_g + \theta_{0,c} \ , \quad \theta_{0,c} = \mathbb{1}_c \otimes M_1^1 \ . \tag{63}$$

These equations follow from the discussion in [2]. We put in local bases

$$G = \frac{i}{2} g_3 G_\mu^a \gamma^\mu \otimes \lambda_a \ , \tag{64}$$

where λ_a , $a = 1, \dots, 8$, are the Gell–Mann matrices, fulfilling $[\lambda_b, \lambda_c] = \sum_{a=1}^8 2i f^a_{\ bc} \lambda_a$, $f^a_{\ bc} \in \mathbb{R}$, and $\mathrm{tr}\,(\lambda_a \lambda_b) = 2\,\delta_{ab}$. Denoting by $\mathbf{e}_c(\theta_c)$ the embedding of θ_c into $B(H)$, we obtain for the Euclidian bosonic action of the chromodynamics sector

$$\begin{aligned} S_B^c &= \frac{8\pi^2}{3(3+x)g_2^2}\, \mathrm{Tr}_\omega \left\{ z(\mathbf{e}_c(\theta_c))^* \mathbf{e}_c(\theta_c) \right\} = \int_X v_g\, \mathscr{L}_c \ , \\ \mathscr{L}_c &= \frac{1}{4} \sum_{a=1}^{8} \delta^{\mu\kappa} \delta^{\nu\lambda} F^a_{c,\mu\nu} F^a_{c,\kappa\lambda} + \frac{1}{g_3^2}\, \mathrm{tr}\,(|\tilde{\mathrm{m}}_q|^4) \ , \\ F^a_{c,\mu\nu} &= \partial_{[\mu} G^a_{\nu]} - g_3 \sum_{b,c=1}^{8} f^a_{\ bc} G^b_\mu G^c_\nu \ , \end{aligned} \tag{65}$$

$$(g_3/g_2)^2 = (3+x)/4 \ . \tag{66}$$

Rotation to Minkowski space ($G \mapsto G_M$, $\delta^{\mu\nu} \to -g^{\mu\nu}$, $v_g \mapsto v_M$ and global minus sign) transforms the first term in $\mathscr{L}_c$ into the bosonic action of chromodynamics

[17], the second term in $\mathscr{L}_c$ contributes to the "cosmological constant". The relation (66) between the coupling constants g_3 and g_2 of the strong and weak interactions respectively should not be taken too seriously, because it can easily be changed by a different normalization of the chromodynamics action. But if we take everywhere the simplest scalar product given by $x = 1$ then we get $g_3 = g_2$ from this model (just as in [11] and [10] for the simplest scalar product). According to [2], the gauge group associated to the module $\mathcal{E}_c$ is isomorphic to $C^\infty_{\mathbb{R}}(X_M) \otimes SU(3)$.

VIII Remarks on Mass Relations

In this section we are going to discuss the mass relations (57) and (58). This will be done only for $\mathrm{tr}\,(|\mathrm{m}_l|^2) \leq \mathrm{tr}\,(|\mathrm{m}_q|^2)$, because lepton masses are small compared with the quark masses of the same generation so that the case $\mathrm{tr}\,(|\mathrm{m}_l|^2) \leq \mathrm{tr}\,(|\mathrm{m}_q|^2)$ is more natural than the case $\mathrm{tr}\,(|\mathrm{m}_l|^2) > \mathrm{tr}\,(|\mathrm{m}_q|^2)$, see (45). Hence, we put

$$\mathrm{tr}\,(|\mathrm{m}_l|^2) = \sin^2\vartheta_1\,\mathrm{tr}\,(|\mathrm{m}_q|^2)\ .$$

From the second equation of (45) we obtain $\mathrm{tr}\,(|m_u|^2) \geq \frac{2}{3}\mathrm{tr}\,(|\mathrm{m}_q|^2)$ so that we put

$$\mathrm{tr}\,(|\mathrm{m}_q|^2) = \tfrac{3}{2}\cos^2\vartheta_2\,\mathrm{tr}\,(|m_u|^2) = \tfrac{3}{2}\,m_t^2\cos^2\vartheta_2\ .$$

Here and in the sequel we neglect the other fermion masses against the mass m_t of the top–quark. Then we get for (57)

$$\begin{aligned} m_W &= \sqrt{\tfrac{2}{3+x}\mathrm{tr}\,\{x(\tfrac{1}{3}|m_e|^2 + \tfrac{1}{6}|\mathrm{m}_l|^2) + (|m_u|^2 + |m_d|^2 - \tfrac{1}{6}|\mathrm{m}_q|^2)\}} \\ &= \sqrt{\tfrac{2}{3+x}(1 - \tfrac{1}{4}(1 - x\sin^2\vartheta_1)\cos^2\vartheta_2)}\,m_t < \sqrt{\tfrac{2}{3}}\,m_t\ . \end{aligned} \tag{67}$$

For $\mathrm{m} \in \mathrm{M}_3\mathbb{C}$ we have $\mathrm{tr}\,(|\tilde{\mathrm{m}}|^4) \equiv \mathrm{tr}\,\{(|\mathrm{m}|^2 - \frac{1}{3}(\mathrm{tr}\,(|\mathrm{m}|^2))\mathbb{1})^2\} \leq \frac{2}{3}(\mathrm{tr}\,(|\mathrm{m}|^2))^2$, see (33). Therefore, we put

$$\begin{aligned} \mathrm{tr}\,(|\tilde{\mathrm{m}}_q|^4) &= \tfrac{2}{3}\cos^2\vartheta_3(\mathrm{tr}\,(|\mathrm{m}_q|^2))^2 = \tfrac{3}{2}m_t^4\cos^2\vartheta_3\cos^4\vartheta_2\ , \\ \mathrm{tr}\,(|\tilde{\mathrm{m}}_l|^4) &= \tfrac{2}{3}\cos^2\vartheta_4(\mathrm{tr}\,(|\mathrm{m}_l|^2))^2 = \tfrac{3}{2}m_t^4\cos^2\vartheta_4\cos^4\vartheta_2\sin^4\vartheta_1\ . \end{aligned}$$

Now we find for (58)

$$m_H = \sqrt{\frac{20\cos^2\vartheta_3 + 12x\cos^2\vartheta_4\sin^4\vartheta_1}{12 - 3(1 - x\sin^2\vartheta_1)\cos^2\vartheta_2}}\,m_t\cos^2\vartheta_2 < 2\,m_t\ . \tag{68}$$

The experimental values $m_W = 80\,\mathrm{GeV}$ and $m_t = 174\,\mathrm{GeV}$ can be reproduced in the case $x\sin^2\vartheta_1 \ll 1$ by $x = 6.5\ldots9.6$, depending on ϑ_2, see (67). Then, by choosing ϑ_2 and ϑ_3, we can adjust m_H to any value smaller than $\sqrt{\frac{20}{9}}\, m_t = 259\,\mathrm{GeV}$, see (68). Moreover, we find for the Weinberg angle $\sin^2\theta_W = 0.27\ldots 0.29$, see (56), and for the ratio of the coupling constants of the strong and weak interactions $(g_3/g_2)^2 = 2.3\ldots 3.9$, see (66). In the case $x\sin^2\vartheta_1 \gg 4$ we can reproduce the m_W/m_t ratio for $\sin^2\vartheta \approx 1$ and $\cos^2\vartheta_2 \approx 1$. By choosing ϑ_4 we can adjust m_H to any value smaller than $2\,m_t = 346\,\mathrm{GeV}$. Moreover, in this case we have $\sin^2\theta_W = 0.25$ and $g_3 \gg g_2$. Hence, we have enough freedom to bring the tree–level predictions (67), (68), (56) and (66) of our model into agreement with experimental data. However, these "predictions" have only a heuristic value, because they do not survive the classical quantization procedure. But there seems to be only a weak scale dependence [18].

The simplest scalar product is given by $x = 1$. In this case we get

$$\begin{aligned} m_W &= \sqrt{\tfrac{1}{2}(1-\tfrac{1}{4}\cos^2\vartheta_1\cos^2\vartheta_2)}\, m_t\ , \quad \sqrt{\tfrac{1}{2}}\, m_t < m_W < \sqrt{\tfrac{3}{8}}\, m_t\ , \\ m_H &= \sqrt{\tfrac{20\cos^2\vartheta_3+12\cos^2\vartheta_4\sin^4\vartheta_1}{12-3\cos^2\vartheta_1\cos^2\vartheta_2}}\, m_t\cos^2\vartheta_2 < \sqrt{\tfrac{8}{3}}\, m_t\ , \\ \sin^2\theta_W &= \tfrac{3}{8}\ , \qquad g_3 = g_2\ . \end{aligned} \tag{69}$$

These mass relations differ from the relations $m_t = 2\,m_W$ and $m_H = 3.14\,m_W$ obtained by Kastler and Schücker in [11] and [10] for the quaternionic K–cycle together with the simplest scalar product. While the m_W/m_t ratio is approximately stable, we get for the m_H/m_t ratio only an upper limit. Thus, we see that in comparison to the simplest model by Connes, Lott and Kastler we get within our simplest model effectively one additional parameter determining the m_H/m_t ratio.

This means: Although we introduced a plenty of free parameters during the construction, these parameters occur in the final Lagrangians in the case of the simplest scalar product only in such combinations that we end up with nine parameters for the fermion masses, four parameters of the Kobayashi–Maskawa matrix, one undetermined coupling constant and one additional parameter, which determines the ratio between the masses of the Higgs–boson and the top–quark. For this last parameter we have only an upper limit. Therefore, our model is less predictive than the model by Connes, Lott and Kastler.

To summerize, the purpose of this paper was to present a different construction of the classical action of the standard model, which uses the simplest possible (non–classical) algebra and puts the complexity into the module. In non–commutative geometry a K–cycle replaces the notion of a manifold and a finite projective module over the algebra of this K–cycle the notion of a vector bundle over this manifold. Therefore, our approach (a K–cycle together with two associated modules – one for the electroweak sector and the other one for the chromodynamics sector) corresponds to a classical manifold with two associated fibre bundles over that manifold – a principal fibre bundle with structure group $SU(2)\times U(1)$ for the electroweak sector and a principal fibre bundle with structure group $SU(3)$ for the chromodynamics sector.

Acknowledgement

The author is grateful to G. Rudolph for helpful comments and discussions. This work was supported by the Deutsche Forschungsgemeinschaft.

References

[1] R. Matthes, G. Rudolph und R. Wulkenhaar, *On the structure of a differential algebra used by Connes and Lott*, Rep. Math. Phys. **38** (1996) 45–66.

[2] R. Matthes, G. Rudolph und R. Wulkenhaar, *On a certain construction of graded Lie algebras with derivation*, J. Geom. Phys. **20** (1996) 107–141.

[3] A. Connes, *Non commutative geometry*, Academic Press, New York 1994.

[4] A. Connes, *Essay on Physics and Non-commutative Geometry*, preprint IHES/M/89/69

[5] A. Connes and J. Lott, *The Metric Aspect of Noncommutative Geometry*, Proceedings of 1991 Cargèse Summer Conference, ed. by J. Fröhlich et al. (Plenum, New York 1992)

[6] J. C. Várilly and J. M. Gracia–Bondía, *Connes' noncommutative differential geometry and the Standard Model*, J. Geom. Phys. **12** (1993) 223–301.

[7] D. Kastler, *A Detailed Account of Alain Connes' Version of the Standard Model in Non–Commutative Differential Geometry III. State of the Art*, Rev. Math. Phys. **8** (1996) 103–166.

[8] D. Kastler, *State of the Art of Alain Connes' Version of the Standard Model of Elementary Particles in Non–commutative Differential Geometry*, preprint Marseille–Luminy CPT-92/P.2814 (1992).

[9] D. Kastler, *A Detailed Account of Alain Connes' Version of the Standard Model in Non–Commutative Geometry I and II*, Rev. Math. Phys. **5** (1993) 477–532.

[10] D. Kastler and T. Schücker, *The Standard Model à la Connes–Lott*, preprint `hep-th/9412185`.

[11] D. Kastler and T. Schücker, *Remarks on Alain Connes' Approach to the Standard Model in Non–Commutative Geometry*, Theor. Math. Phys. **92** (1993) 1075–1080.

[12] R. Coquereaux, G. Esposito–Farèse and G. Vaillant, *Higgs Fields as Yang–Mills Fields and Discrete Symmetries*, Nucl. Phys. B **353** (1991) 689–706.

[13] R. Coquereaux, G. Esposito–Farèse and F. Scheck, *Noncommutative Geometry and Graded Algebras in Electroweak Interactions*, Int. J. Mod. Phys. A **7** (1992) 6555–6593.

[14] R. Coquereaux, R. Häußling, N. A. Papadopoulos and F. Scheck, *Generalized Gauge Transformations and Hidden Symmetry in the Standard Model*, Int. J. Mod. Phys. A **7** (1992) 2809–2824.

[15] M. Scheunert, W. Nahm and V. Rittenberg, *Irreducible representations of the* $osp(2,1)$ *and* $spl(2,1)$ *graded Lie algebras*, J. Math. Phys. **18** (1977) 155–162.

[16] R. Häußling and F. Scheck, *Quark Mass Matrices and Generation Mixing in the Standard Model with Non–commutative Geometry*, Phys. Lett. B **336** (1994) 477–486.

[17] M. Guidry, *Gauge Field Theories*, Wiley, New York 1991.

[18] E. Álvarez, J. M. Gracia–Bondía and C. P. Martín, *A renormalization group analysis of the NCG constraints* $m_{\rm top} = 2\, m_W$, $m_{\rm Higgs} = 3.14\, m_W$, Phys. Lett. B **329** (1994) 259–262.