Correlation functions of a just renormalizable tensorial group field theory: The melonic approximation

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Abstract

The D-colored version of tensor models has been shown to admit a large N-limit expansion. The leading contributions result from so-called melonic graphs which are dual to the D-sphere. This is a note about the Schwinger-Dyson equations of the tensorial φ_5^4-model (with propagator $1/\mathbf{p}^2$) and their melonic approximation. We derive the master equations for two- and four-point correlation functions and discuss their solution.

1   Introduction

The construction of a consistent quantum theory of gravity is one of the biggest open problems of fundamental physics. There are several approaches to this challenging issue. Tensor models belong to the promising candidates to understand quantum gravity (QG) in dimension $D \geq 3$ [1]-[4]. Tensor models represent an attempt to generalize matrix models (see, for instance [5] and the review article [6]). Tensor models are, as it will be evident, connected with group field theory [8], which, interestingly, is a proposal for a second quantization of

loop quantum gravity [7]. In combination with group field theory, tensor models lead to another framework called tensorial group field theory (TGFT). TGFT is quantum field theory (QFT) over group manifolds. It can also be viewed as a new proposal for quantum field theories based on a Feynman path integral, which generates random graphs describing simplicial pseudo manifolds.

A few years ago, Razvan Gurău [9]-[15] achieved a breakthrough for this program by discovering the generalization of 't Hooft's $1/N$-expansion [6]-[16]. This allows to understand statistical physics properties such as continuum limits, phase transitions and critical exponents (see [17]-[27] for more details). Concerning the renormalizability of tensor models, by modifying the propagator using radiative corrections of the form $1/\mathbf{p}^2$ [28], it has been shown that several models have this property [29]-[37]. Investigating the UV behavior of these renormalizable models, it has been shown that several models are asymptotically free [32] and [38]-[40].

Recently, important progress was made in the case of independent identically distributed (iid) tensor models. The correlation functions are solved analytically in the large N-limit, in which the dominant graphs are called "*melons*" [19]. This model corresponds to dynamical triangulations in three and higher dimensions. The susceptibility exponent is computed and the model is reminiscent of certain models of branched polymers [25]. Despite all these aesthetic results, the critical behavior of the large-N limit of the renormalizable models (*the melonic approximation*) is not yet explored. The phase transitions must be computed explicitly. This glimpse needs to be taking into account for the future development of the renormalizable TGFT program.

This paper extends previous work on Schwinger-Dyson equations for matrix and tensor models. The original motivation for this method was the construction of the φ_4^4-model on noncommutative Moyal space. This model is perturbatively renormalizable [41]-[43] and asymptotically safe in the UV regime [44]-[45]. The key step of the asymptotic safety proof [45] was extended in [46] to obtain a closed equation for the two-point function of the model. This equation was reduced in [47] to a fixed point problem for which existence of a solution was proved. All higher correlation functions were expressed in terms of the fixed point solution. In [48] the fixed point problem was numerically studied. This gave evidence for phase transitions and for reflection positivity of the Schwinger two-point function.

The noncommutative φ_4^4-model solved in [47] can be viewed as the quartic cousin of the Kontsevich model which is relevant for two-dimensional quantum gravity. This leads immediately to the question to extend the techniques of [46, 47] to tensor models of rank $D \geq 3$. In [50] one of us addressed the closed equation for correlation functions of rank 3 and 4 just renormalizable TGFT. The two-point functions are given perturbatively using the iteration method. The main challenge in this new direction is to perform the combinatorics of Feynman graphs and to solve the nontrivial integral equations of the correlators. The nonperturbative study of all correlation functions need to be investigated carefully.

In this paper we push further this program. For this, we consider the just renormalizable tensor model of the form φ_5^4 whose dynamics is described by the propagator of the form $1/\mathbf{p}^2$. In the melonic approximation, the Schwinger-Dyson equations are given. The closed equations of the two-point and four-point functions are derived and the solution of the former is obtained.

The paper is organized as follows. In section 2, proceeding from the definition of the model and its symmetries, we give the Ward-Takahashi identities which result from these symmetries. In section 3 we find the melonic approximation of the Schwinger-Dyson equation. In 4 we investigate the closed equation for two- and four-point functions. Section 5 is devoted to the study of the closed equation of the four-point correlation functions. In section 6 we solve the equation obtained. In Section 7, we conclude by a summary of our work and list open questions.

2 The Model

The model we will be mainly considering here is a tensorial φ^4-theory on $U(1)^5$, i.e. a field $\varphi : U(1)^5 \to \mathbb{C}$ and the following action, whose measure is the product of Haar measures for each $U(1)$-factor:

$$
\begin{aligned}
S[\bar{\varphi}, \varphi] \;\; = \;\; & \int_{U(1)^5} d\mathbf{g}\, \bar{\varphi}(\mathbf{g})(-\Delta + m^2)\varphi(\mathbf{g}) \\
& + \frac{\lambda}{2} \sum_{c=1}^{5} \int_{U(1)^{20}} d\mathbf{g}\, d\mathbf{g}'\, d\mathbf{h}\, d\mathbf{h}'\; \bar{\varphi}(\mathbf{g})\varphi(\mathbf{g}')\bar{\varphi}(\mathbf{h})\varphi(\mathbf{h}')K_c(\mathbf{g}, \mathbf{g}', \mathbf{h}, \mathbf{h}') \,.
\end{aligned}
\tag{1}
$$

Here, $\Delta = \sum_{\ell=1}^{5}\Delta_\ell$ and Δ_ℓ is the Laplace-Beltrami operator on $U(1)$ acting on colour-ℓ indices [34], bold variables stand for 5-dimensional variables ($\mathbf{g} = (g_1, \ldots, g_5)$), and K_c identifies group variables according to a vertex of colour $c \in \{1, 2, \ldots, 5\}$. Figure 1 shows the vertex of colour 1.

(a) A vertex of colour 1
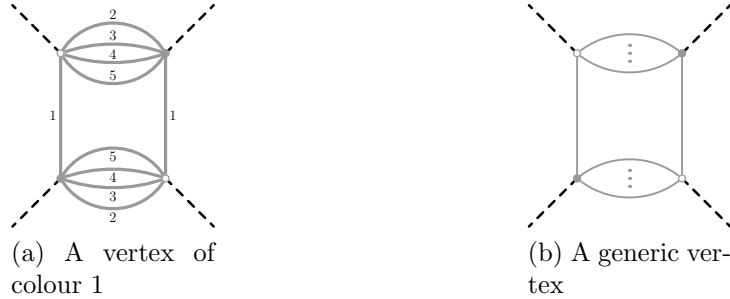
(b) A generic vertex

Figure 1: Vertices

The statistical physics description of the model is encoded in the partition function:

$$
\mathcal{Z}[\bar{J}, J] = \int D\varphi D\bar{\varphi}\, e^{-S[\bar{\varphi}, \varphi] + \langle \bar{J}, \varphi \rangle + \langle \bar{\varphi}, J \rangle} = e^{W[\bar{J}, J]},
\tag{2}
$$

where $\bar{J}, J : U(1)^5 \to \mathbb{C}$ represent the sources, and for fields $\psi_1, \psi_2 : U(1)^5 \to \mathbb{C}$,

$$
\langle \psi_1, \psi_2 \rangle := \int_{U(1)^5} d\mathbf{g}\, \psi_1(\mathbf{g})\psi_2(\mathbf{g}),
$$

3

and $W[\bar{J}, J]$ is the generating functional for the connected Green's functions. Then the N-point Green functions take the form

$$G_N(\mathbf{g}_1, \cdots, \mathbf{g}_{2N}) = \left. \frac{\partial \mathcal{Z}[\bar{J}, J]}{\partial J_1 \partial \bar{J}_1 \cdots \partial J_N \partial \bar{J}_N} \right|_{J_1, \ldots, J_N = \bar{J}_1, \ldots, J_N = 0}. \tag{3}$$

being J_i shorthand for $J(\mathbf{g}_i)$, for each $i = 1, \ldots, N$.

The correlation functions can be computed perturbatively by expanding the interaction part of the action (1):

$$G_N(\mathbf{g}_1, \cdots, \mathbf{g}_{2N}) \sim \sum_{n=0}^{\infty} \frac{(-\lambda)^n}{2^n n!} \int d\mu_C \, \bar{\varphi}(\mathbf{g}_1) \cdots \varphi(\mathbf{g}_{2N}) \tag{4}$$
$$\times \left[\sum_{c=1}^{5} \int_{\mathrm{U}(1)^{20}} d\mathbf{g} \, d\mathbf{g}' \, d\mathbf{h} \, d\mathbf{h}' \, \bar{\varphi}(\mathbf{g})\varphi(\mathbf{g}')\bar{\varphi}(\mathbf{h})\varphi(\mathbf{h}')K_c(\mathbf{g}, \mathbf{g}', \mathbf{h}, \mathbf{h}') \right]^n$$

where $d\mu_C$ is the Gaussian measure with covariance C (the propagator), i.e.

$$\int d\mu_C(\bar{\varphi}, \varphi) \, \bar{\varphi}(\mathbf{g})\varphi(\mathbf{g}') = C(\mathbf{g}, \mathbf{g}'), \quad \int d\mu_C(\bar{\varphi}, \varphi) \, \varphi(\mathbf{g})\varphi(\mathbf{g}') = \int d\mu_C(\bar{\varphi}, \varphi) \, \bar{\varphi}(\mathbf{g})\bar{\varphi}(\mathbf{g}') = 0.$$

In this paper we consider the Fourier transform of the field φ to momentum space, $\varphi_{p_1 p_2 p_3 p_4 p_5}$, defined by

$$\varphi(e^{i\theta_1}, \ldots, e^{i\theta_5}) = \sum_{p_1, \ldots, p_5} \varphi_{p_1 p_2 p_3 p_4 p_5} e^{i \sum_{j=1}^{5} \theta_j p_j}$$

with $\theta_i \in [0, 2\pi)$. We impose for every $a = 1, \ldots, 5$ the condition $|p_a| \leq \Lambda_a$, $\Lambda_a \in \mathbb{N}$, on the field φ, keeping in mind that we will eventually take the limits $\Lambda_a \to \infty$. The truncation of φ shall be here denoted by φ^Λ, and its conjugate field, also compatible with the momentum truncation, by $\bar{\varphi}^\Lambda$. Accordingly, from the full symmetry $\mathrm{U}(\infty)^{\otimes 5}$ remains $\otimes_{i=1}^{5} \mathrm{U}(N_i)$, being $N_a := 2\Lambda_a + 1$. To this symmetry corresponds, as we will see, a set of Ward-Takahashi identities.

For any $a = 1, 2, \cdots, 5$ we let a unitarity $W^{(a)}$ in the factor $\mathrm{U}(N_a)$ of $\otimes_{i=1}^{5} \mathrm{U}(N_i)$ act on the fields as follows:

$$\varphi^\Lambda_{p_1 p_2 \cdots p_5} \to [W^{(a)}\varphi^\Lambda]_{p_1 p_2 \ldots p_5} = \sum_{p'_a = -\Lambda_a}^{\Lambda_a} W^{(a)}_{p_a p'_a} \varphi^\Lambda_{p_1 \ldots p'_a \ldots p_5}, \tag{5}$$

$$\bar{\varphi}^\Lambda_{p_1 p_2 \cdots p_5} \to [\bar{\varphi}^\Lambda W^{(a)\dagger}]_{p_1 p_2 \ldots p_5} = \sum_{p'_a = -\Lambda_a}^{\Lambda_a} \overline{W}^{(a)}_{p_a p'_a} \bar{\varphi}^\Lambda_{p_1 \ldots p'_a \ldots p_5}. \tag{6}$$

There might be some common values of N_a for different indices $a = \{1, \ldots, 5\}$, but the point is to think of each unitary group $\mathrm{U}(N_a)$ as acting exclusively on the the a-th index. Given a $W^{(a)} \in \mathrm{U}(N_a)$, denote by $B^{(a)}$ its (Hermitian) generator, i.e.

$$W^{(a)}_{pp'} = \delta^{(a)}_{pp'} + iB^{(a)}_{pp'} + O(B^2), \quad \bar{W}^{(a)}_{pp'} = \delta^{(a)}_{pp'} - i\bar{B}^{(a)}_{pp'} + O(\bar{B}^2). \tag{7}$$
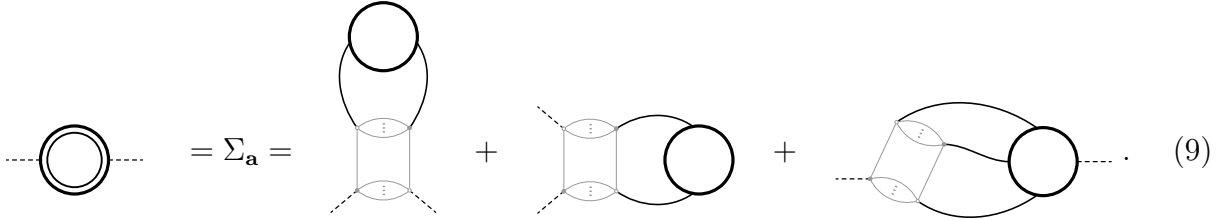
We take the limits $N_a \to \infty$ and consider the variation of the connected partition function with respect to $B^{(a)}$, $\delta(\ln \mathcal{Z})/\delta B^{(a)}_{mn}$. From the invariance of the measure $d\varphi d\bar\varphi$, this variation vanishes. The resulting Ward Takahashi identities were also obtained in detail, for arbitrary-rank TGFT's, in Section 2 of [50]. For $a = 1$ and for the two-point function they read
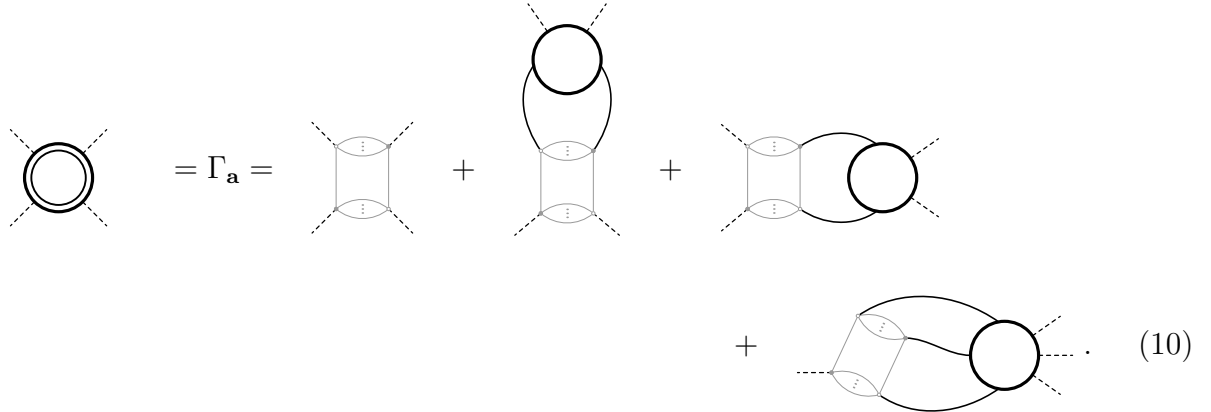
$$\sum_{p_2,p_3,p_4,p_5} \left(C^{-1}_{mp_2p_3p_4p_5} - C^{-1}_{np_2p_3p_4p_5}\right) G^{\text{ins}}_{[mn]p_2p_3p_4p_5} = G_{np_2p_3p_4p_5} - G_{mp_2p_3p_4p_5}, \tag{8}$$

where $C_{p_1\cdots p_5}$ denotes the propagator in the momentum space. Similar identities for arbitray a hold after trivial index reordering. The correlation functions with insertion of strands here are $G^{\text{ins}}_{[mn]p_2p_3p_4p_5} := \langle \varphi_{mqp_2p_3p_4p_5} \bar\varphi_{np_2p_3p_4p_5} \sum_{q_2,q_3,q_4,q_5} \varphi_{nq_2q_3q_4q_5} \bar\varphi_{mq_2q_3q_4q_5} \rangle_c$. The model (1) is (just) renormalizable to all orders of perturbation theory. See Refs [29]-[34] for more detail.

3 Schwinger-Dyson equation in the melonic approximation

We start by writing the Schwinger-Dyson equations for the one-particle irreducible 2- and 4-point functions of the model (1). We use the following graphical conventions: dashed lines symbolize amputated external legs, a black circle represents a connected function whereas two concentric circles stand for a one-particle irreducible function. Finally, in order to lighten equations, we will use the generic vertex of fig. 1b to mean the sum of the five different coloured interactions. Note that (9) has been derived in [50].

$$(9)$$

$$(10)$$

We now want to restrict our attention to the melonic part of the 2- and 4-point functions, that is, we restrict ourselves to the leading graphs $\mathcal{G}$ characterized by a degree $\omega(\mathcal{G}) = 0$ [13].

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For arbitrary graphs, this number $\omega \geq 0$, also called Gurău's degree, is the analogue of the Euler characteristic in the large-N expansion of matrix models and is defined as follows. In $\mathcal{G}$ sit ribbon graphs J with the same vertices and edges as $\mathcal{G}$, but with face set indexed by a cycle $\tau \in \mathfrak{S}_5$, and defined by

$$F_J = \{f \text{ faces in } \mathcal{G} \,|\, f = (\tau^q(0), \tau^{q+1}(0)), q \in \mathbb{Z}_5\}.$$

To each of these graphs corresponds a compact orientable surface on which the ribbon graph J can be drawn with non-intersecting edges. One then canonically associates to a jacket J a genus g_J — the minimal-genus surface on which one can planarly draw J. Then

$$\omega(\mathcal{G}) := \sum_{J \text{ jacket of } \mathcal{G}} g_J. \tag{11}$$

Let $\mathcal{G}$ be a 2- or 4-point Feynman graph of model (1). We impose $\omega(\mathcal{G}) = 0$. We will prove that not all terms of eqs. (9) and (10) contribute to the melonic functions. A simple way of computing the degree ω of a graph is to count its number F of faces. Indeed, the two are related in the following way (in dimension 5, for a degree 4 interaction)[50]:

$$F = 4V + 4 - 2N - \frac{1}{12}(\widetilde{\omega}(\mathcal{G}) - \omega(\partial \mathcal{G})) - (C_{\partial \mathcal{G}} - 1), \tag{12}$$

where V is the number of vertices of $\mathcal{G}$, N its number of external legs, and $C_{\partial \mathcal{G}}$ is the number of connected components of its boundary graph $\partial \mathcal{G}$ and $\widetilde{\omega}(\mathcal{G}) = \sum_{\widetilde{J} \subset \mathcal{G}} g_{\widetilde{J}}$ with $\widetilde{J}$ the pinched jacket associated with a jacket J of $\mathcal{G}$. Recall that the Feynman graphs here are so-called uncoloured graphs and, as a consequence, a face is a cycle of colours $0i$, $i \in \{1, 2, \ldots, 5\}$ [14]. A detailed analysis of coloured graphs [32, 50] allows to prove that $F(\mathcal{G}) = F_{\max}(\mathcal{G}) = 4V + 4 - 2N$, if and only if $\widetilde{\omega}(\mathcal{G}) = \omega(\partial \mathcal{G}) = C_{\partial \mathcal{G}} - 1 = 0$. Moreover $F \leqslant F_{\max}$. We can thus prove the following

Lemma 1. *The Schwinger-Dyson equations for the* melonic *2- and 4-point functions of model* (1) *are (m stands for melonic):*

$$\tag{13}$$

$$\tag{14}$$

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Proof. The right-hand side of eqs. (9) and (10) involve connected 2-, 4-, and 6-point function insertions and a generic vertex. Let $\mathcal{G}$ be a graph contributing to the left-hand side of (9) or (10) and let F be its number of faces. Let us study a term of the right-hand side of the equation under consideration. The number of faces of a graph contributing to its insertion is written F'. Clearly $F = F' + \delta F$ where $\delta F \geqslant 0$. The additional internal faces are created by closing the external faces of the insertion with the new edges connected to the new vertex. As a consequence, δF is bounded from above by the number of faces of the new vertex which do not contain its external legs. Note also that $F \leqslant F'_{\max} + \delta F$.

Let us now consider eq. (9) and the lying tadpole of its right-hand side (second term). In this case, $\delta F \leqslant 1$. From eq. (12), $F'_{\max} = 4V'$ (V' being the number of vertices of the connected 2-point insertion) and $F \leqslant 4V' + 1 < 4(V' + 1) = F_{\max}$. Thus whatever the insertion, the graph $\mathcal{G}$ cannot be melonic. The same type of argument holds for the other terms but for the sake of clarity, let us repeat it for the last term of eq. (10). Here $\delta F \leqslant 5$ and $F'_{\max} = 4V' - 8$. Their sum never reaches $F_{\max} = 4V - 4 = 4V'$.

The only terms which survive this analysis are the first one of eq. (9), and the first and second ones of eq. (10). Moreover it also proves that for a graph to be melonic, the corresponding insertion needs to be melonic too. Note that a melonic graph necessarily has a melonic boundary [30, 36]. Finally, such arguments also fix the orientation, and the colour, of the boundary graph of the 4-point insertion in the second term on the right-hand side of 14, see fig. 2 for a zoom into this term. $\square$
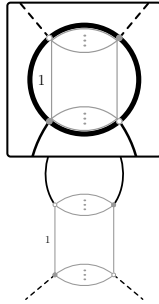

Figure 2: Boundary structure of a melonic 4-point insertion

Note that the Schwinger-Dyson equation (13) and (14) are easy to describe. Taking into account (13) we do not need to write the Ward-Takahashi identities before getting the closed equation of the two-point functions.

4   Two-point correlation functions

We now want to use the melonic approximation to obtain a closed equation for the one-particle irreducible two-point function $\Sigma_{a_1,\ldots,a_5}$. For sake of simplicity write $\mathbf{a} = (a_1, \ldots, a_5) \in \mathbb{Z}^5$. Setting each constant λ_ρ ($\rho = 1, \ldots, 5$) equal to the bare coupling constant, $\lambda_\rho = \lambda$, we can express the one-particle irreducible two-point function $\Sigma_{\mathbf{a}}$ in terms of the renormalized

quantities by using the Taylor expansion

$$\Sigma_{\mathbf{a}} = \Sigma_{\mathbf{0}} + |\mathbf{a}|^2 \frac{\partial \Sigma_{\mathbf{a}}}{\partial |\mathbf{a}|^2}\Big|_{\mathbf{a}=\mathbf{0}} + \Sigma_{\mathbf{a}}^{\mathrm{r}}$$
$$= (Z-1)|\mathbf{a}|^2 + Zm^2 - m_{\mathrm{r}}^2 + \Sigma_{\mathbf{a}}^{\mathrm{r}}, \tag{15}$$

with $|\mathbf{a}|^2 = \sum_{i=1}^{5} a_i^2$ and

$$m^2 = \frac{m_{\mathrm{r}}^2 + \Sigma_{\mathbf{0}}}{Z}, \quad Z = 1 + \frac{\partial \Sigma_{\mathbf{a}}}{\partial |\mathbf{a}|^2}\Big|_{\mathbf{a}=\mathbf{0}}. \tag{16}$$

Moreover the following renormalization conditions hold, for $\rho = 1, \ldots, 5$:

$$\Sigma_{\mathbf{0}}^{\mathrm{r}} = 0, \quad \frac{\partial \Sigma_{\mathbf{a}}^{\mathrm{r}}}{\partial a_\rho^2}\Big|_{\mathbf{a}=\mathbf{0}} = 0. \tag{17}$$

The propagator C, given explicitly by $C_{\mathbf{p}}^{-1} = Z(|\mathbf{p}|^2 + m^2)$, is related to the dressed propagator $G_{\mathbf{a}}$ by means of the Dyson relation $G_{\mathbf{a}}^{-1} = C_{\mathbf{a}}^{-1} - \Sigma_{\mathbf{a}}$. Then using the Schwinger-Dyson equations for $\Sigma_{\mathbf{a}}$, given in (13), we get

$$\Sigma_{\mathbf{a}} = -Z^2\lambda \sum_{p_1,p_2,p_3,p_4}^{\mathbf{\Lambda}} \left[\frac{1}{C_{a_1p_2p_3p_4}^{-1} - \Sigma_{a_1p_1p_2p_3p_4}} + \frac{1}{C_{p_1a_2p_2p_3p_4}^{-1} - \Sigma_{p_1a_2p_2p_3p_4}} \right.$$
$$+ \frac{1}{C_{p_1p_2a_3p_3p_4}^{-1} - \Sigma_{p_1p_2a_3p_3p_4}} + \frac{1}{C_{p_1p_2p_3a_4p_4}^{-1} - \Sigma_{p_1p_2p_3a_4p_4}}$$
$$\left. + \frac{1}{C_{p_1p_2p_3p_4a_5}^{-1} - \Sigma_{p_1p_2p_3p_4a_5}} \right]. \tag{18}$$

The sums are performed over the integers $p_i \in \mathbb{Z}$ with some cutoff $\mathbf{\Lambda}$. For $\rho = 1, \ldots, 5$, let σ_ρ be the action of $\mathfrak{S}_5$ which permutes the strands with momenta $\mathbf{p}$ as follows:

$$\sigma_1(p_1p_2p_3p_4p_5) = (p_2p_1p_3p_4p_5),$$
$$\sigma_2(p_1p_2p_3p_4p_5) = (p_2p_3p_1p_4p_5),$$
$$\vdots$$
$$\sigma_4(p_1p_2p_3p_4p_5) = (p_2p_3p_4p_5p_1),$$

and σ_5 acts trivially. Notice that the value of the propagator $C_{\mathbf{p}}$ remains invariant under the action of all these σ_ρ, $C_{\sigma_\rho(\mathbf{p})} = C_{\mathbf{p}}$, and since the interaction vertices remain also unaffected, so does $\Sigma_{\mathbf{a}}$. Therefore, by (15), $\Sigma_{\mathbf{a}}^{\mathrm{r}}$ is symmetric under the permutation of its indices. After combining (15) and (18) we can obtain, by using

$$C_{a_1p_1p_2p_3p_4}^{-1} - \Sigma_{a_1p_1p_2p_3p_4} = a_1^2 + \sum_{i=1}^{4} p_i^2 + m_{\mathrm{r}}^2 - \Sigma_{a_1p_1p_2p_3p_4}^{\mathrm{r}}, \tag{19}$$

the expression

$$(Z - 1)|\mathbf{a}|^2 + Zm^2 - m_{\rm r}^2 + \Sigma_{\mathbf{a}}^{\rm r} = -Z^2\lambda \sum_{\rho=1}^{5} \sum_{p_1,p_2,p_3,p_4}^{\Lambda} \frac{1}{(a_\rho^2 + \sum_{i=1}^{4} p_i^2) + m_{\rm r}^2 - \Sigma_{\sigma_\rho(a_\rho p_1 p_2 p_3 p_4)}^{\rm r}}. \tag{20}$$

We now can evaluate at $\mathbf{a} = \mathbf{0}$ to get rid of the term $Zm^2 - m_{\rm r}^2$, which according to this equation is given by

$$Zm^2 - m_{\rm r}^2 = -Z^2\lambda \sum_{p_1,p_2,p_3,p_4}^{\Lambda} \sum_{\rho=1}^{5} \frac{1}{\sum_{i=1}^{4} p_i^2 + m_{\rm r}^2 - \Sigma_{\sigma_\rho(0 p_1 p_2 p_3 p_4)}^{\rm r}}. \tag{21}$$

Replacing the expression (21) in (20), we obtain

$$(Z - 1)|\mathbf{a}|^2 + \Sigma_{\mathbf{a}}^{\rm r} = -Z^2\lambda \sum_{p_1,p_2,p_3,p_4}^{\Lambda} \sum_{\rho=1}^{5} \left[\frac{1}{a_\rho^2 + |\mathbf{p}|^2 + m_{\rm r}^2 - \Sigma_{\sigma_\rho(a_\rho p_1 p_2 p_3 p_4)}^{\rm r}} \right.$$
$$\left. - \frac{1}{|\mathbf{p}|^2 + m_{\rm r}^2 - \Sigma_{\sigma_\rho(0 p_1 p_2 p_3 p_4)}^{\rm r}} \right]. \tag{22}$$

Here we have defined $|\mathbf{p}|^2 := \sum_{i=1}^{4} p_i^2$, with some abuse of notation. The evaluation at $\mathbf{a} = \sigma_\rho(a_\rho 0000)$, namely

$$(Z - 1)a_\rho^2 + \Sigma_{\sigma_\rho(a_\rho 0000)}^{\rm r} = -Z^2\lambda \sum_{\mathbf{p} \in \mathbb{Z}^4}^{\Lambda} \left[\frac{1}{a_\rho^2 + |\mathbf{p}|^2 + m_{\rm r}^2 - \Sigma_{\sigma_\rho(a_\rho p_1 p_2 p_3 p_4)}^{\rm r}} \right. \tag{23}$$
$$\left. - \frac{1}{|\mathbf{p}|^2 + m_{\rm r}^2 - \Sigma_{\sigma_\rho(0 p_1 p_2 p_3 p_4)}^{\rm r}} \right],$$

leads to a splitting of the renormalized one-particle irreducible two-point function as

$$\Sigma_{a_1 a_2 a_3 a_4 a_5}^{\rm r} = \sum_{\rho=1}^{5} \Sigma_{\sigma_\rho(a_\rho 0000)}^{\rm r} \tag{24}$$

as a mere consequence of summing eq. (23) over $\rho = 1, \ldots, 5$ and then comparing the rhs of the resulting equation with that of eq. (22). The wave function renormalization constant Z can be obtained from differentiating (23) with respect to any a_ρ^2 and the subsequent evaluation at $a_\rho = 0$:

$$Z - 1 = Z^2\lambda \left[\sum_{\mathbf{p} \in \mathbb{Z}^4}^{\Lambda} \frac{1}{(|\mathbf{p}|^2 + m_{\rm r}^2 - \Sigma_{0\mathbf{p}}^{\rm r})^2} \right], \quad \Lambda \in \mathbb{Z}^4. \tag{25}$$

Here (17) has been used. Insertion of this value for $(Z - 1)$ into eq. (23) yields, setting $\tilde{\lambda} = Z^2\lambda$ and using (24) again,

$$\Sigma_{a\mathbf{0}}^{\rm r} = -\tilde{\lambda} \sum_{\mathbf{p} \in \mathbb{Z}^4}^{\Lambda} \left[\frac{1}{a^2 + |\mathbf{p}|^2 + m_{\rm r}^2 - \Sigma_{a\mathbf{0}}^{\rm r} - \Sigma_{0\mathbf{p}}^{\rm r}} + \frac{a^2}{(|\mathbf{p}|^2 + m_{\rm r}^2 - \Sigma_{0\mathbf{p}}^{\rm r})^2} - \frac{1}{|\mathbf{p}|^2 + m_{\rm r}^2 - \Sigma_{0\mathbf{p}}^{\rm r}} \right]. \tag{26}$$

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The above equation could lead to a divergence in the limit where $\mathbf{\Lambda} \to \infty$. To deal with this we introduce later, in sec. 6, a regularization method. We now pass to a continuum limit in which the discrete momenta $a \in \mathbb{Z}$, $\mathbf{p} \in \mathbb{Z}^4$ become continuous. We do this here in a formal manner. A rigorous treatment should first view the regularized dual of $\mathrm{U}(1)^5$ as a toroidal lattice $(\mathbb{Z}/2\Lambda\mathbb{Z})^5$, then take an appropriate scaling limit to the 5-torus $[-\Lambda, \Lambda]^5$ with periodic boundary conditions, and finally $\Lambda \to \infty$. These steps should give for (26):

$$\Sigma^{\mathrm{r}}_{a\mathbf{0}} = -\tilde{\lambda} \int_{\mathbb{R}^4} d\mathbf{p} \left[\frac{a^2}{(|\mathbf{p}|^2 + m_{\mathrm{r}}^2 - \Sigma^{\mathrm{r}}_{0\mathbf{p}})^2} + \frac{1}{a^2 + |\mathbf{p}|^2 + m_{\mathrm{r}}^2 - \Sigma^{\mathrm{r}}_{a\mathbf{0}} - \Sigma^{\mathrm{r}}_{0\mathbf{p}}} - \frac{1}{|\mathbf{p}|^2 + m_{\mathrm{r}}^2 - \Sigma^{\mathrm{r}}_{0\mathbf{p}}} \right] \tag{27}$$

with $d\mathbf{p} = dp_1 dp_2 dp_3 dp_4$. Because of (24), i.e. $\Sigma^{\mathrm{r}}_{0\mathbf{p}} = \sum_{i=1}^4 \Sigma^{\mathrm{r}}_{p_i \mathbf{0}}$, (27) is a closed equation for the function $\Sigma^{\mathrm{r}}_{a\mathbf{0}}$. Using Taylor's formula we can equivalently write this equation as

$$\Sigma^{\mathrm{r}}_{a\mathbf{0}} = -\tilde{\lambda} \int_0^{a^2} dt\, (a^2 - t) \int_{\mathbb{R}^4} d\mathbf{p}\, \frac{d^2}{dt^2} \left(\frac{1}{m_{\mathrm{r}}^2 + t - \Sigma^{\mathrm{r}}_{\sqrt{t}\mathbf{0}} + \sum_{i=1}^4 (p_i^2 - \Sigma^{\mathrm{r}}_{p_i \mathbf{0}})} \right). \tag{28}$$

The equation (28) is the analogue of the fixed point equation [47, eq. (4.48)] for the boundary 2-point function G_{a0} of the quartic matrix model: In both situations the decisive function satisfies a non-linear integral equation for which we can at best expect an approximative numerical solution. Finding a suitable method, implementing it in a computer program and and running the computation needs time. We intend to report results in a future publication. At the moment we have to limit ourselves to a perturbative investigation of this equation, see sec. 6.

5 Closed equation of the one-particle irreducible four-point functions

In this section we prove that the coupling constant $\tilde{\lambda}$ is finite in the UV regime. It will be convenient to briefly discuss first the index structure of the four-point function. Γ^4 has 10 indices: Each external coloured line of $\varphi_{\mathbf{a}}$ and $\varphi_{\mathbf{b}}$ should be paired with one of the complex conjugate fields $\bar{\varphi}_{\mathbf{c}}$ and $\bar{\varphi}_{\mathbf{d}}$ in the vertex $\varphi_{\mathbf{a}}\bar{\varphi}_{\mathbf{c}}\varphi_{\mathbf{b}}\bar{\varphi}_{\mathbf{d}}$. That is to say that $\mathbf{c}$ and $\mathbf{d}$ are expressed[1] in terms of $(\mathbf{a}, \mathbf{b})$. For instance, for the vertex of colour 1, represented in fig. (1a), $\mathbf{c} = (a_5 a_4 a_3 a_2 b_1)$, and $\mathbf{d} = (b_5 b_4 b_3 b_2 a_1)$. The external lines for that vertex look as follows:

---

[1]More precisely, $\mathbf{c} = (\pi_1 \circ \varrho)(\mathbf{a}, \mathbf{b})$ and $\mathbf{d} = (\pi_2 \circ \varrho)(\mathbf{a}, \mathbf{b})$ where, $(\mathbf{a}, \mathbf{b}) \in \mathbb{Z}^{10}$, π_1 and π_2 are the projections in the first or second factor of $\mathbb{Z}^5 \oplus \mathbb{Z}^5$, and ϱ is a permutation in $\mathfrak{S}_{10}$ that allows colour conservation.
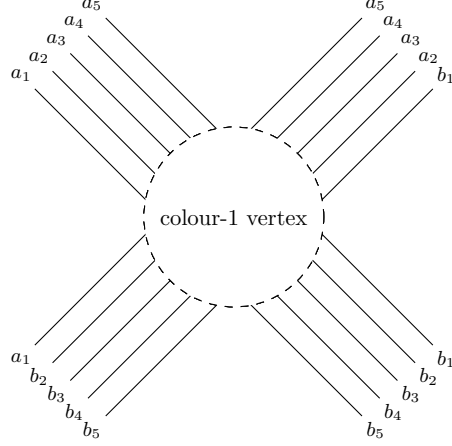
We now excise the vertex in the rhs of the melonic approximation of the Schwinger-Dyson equation for the four-point function and write its value, $-Z^2\lambda$, instead. The first graph in the rhs of eq. (14) is precisely the vertex. In the second graph, after removing the vertex, a jump in the colour 1 occurs; this can be understood as an insertion, whose value we give now. The removal of the colour-1 vertex in that graph leaves the following graph, where the upper dotted lines have indices $\mathbf{a} = (a_1 a_2 a_3 a_4 a_5)$ and $\mathbf{c} = (a_5 a_4 a_3 a_2 b_1)$.

$$
G^{-1}_{a_1 a_2 a_3 a_4 a_5} G^{-1}_{a_5 a_4 a_3 a_2 b_1} G^{\text{ins}}_{[a_1 b_1] a_2 a_3 a_4 a_5} \quad = \quad
$$

$$(29)$$

According to (8), the value of that insertion is

$$
G^{\text{ins}}_{[a_1 b_1] a_2 a_3 a_4 a_5} = \frac{1}{Z(a_1^2 - b_1^2)} \left(G_{a_1 a_2 a_3 a_4 a_5} - G_{a_5 a_4 a_3 a_2 b_1} \right).
$$

In general any of the vertex in this model has a privileged colour i (i.e. the colour i is with the neighbour vertically and the remaining colours are connected with the other neighbouring field, sidewards). The excised graph for the 'colour i'-vertex has then the following value:

$$
G^{-1}_{a_1 a_2 a_3 a_4 a_5} G^{-1}_{a_5 \ldots b_1 \hat{a}_i \ldots a_1} G^{\text{ins}}_{[a_i b_i] a_1 \ldots \hat{a}_i \ldots a_5} = \frac{1}{Z(a_i^2 - b_i^2)} \left[\frac{1}{G_{a_5 \ldots b_i \hat{a}_i \ldots a_2 a_1}} - \frac{1}{G_{a_1 a_2 a_3 a_4 a_5}} \right],
$$

where $\hat{a}_i$ means omission of $\hat{a}_i$ (and this index is substituted by b_i) and, accordingly, $\mathbf{c} = (a_5 \ldots b_i \hat{a}_i \ldots a_2 a_1)$. Then the full equation for $\Gamma^{4,\text{ren}}_{a_1 a_2 a_3 a_4 a_5 b_1 b_2 b_3 b_4 b_5}$ is given by the sum over

these two kinds of graphs over all the vertices of the model, to wit

$$\Gamma_{\mathbf{a},\mathbf{b}}^{4,\text{ren}} = \sum_{i=1}^{5} -Z^2\lambda(1 + G_{a_1a_2a_3a_4a_5}^{-1} G_{a_5\ldots b_1\hat{a}_i\ldots a_1}^{-1} G_{[a_ib_i]a_1\ldots\hat{a}_i\ldots a_5}^{\text{ins}})$$

$$= -Z^2\lambda\left(5 + \frac{1}{Z(a_1^2 - b_1^2)}\left[\frac{1}{G_{a_5a_4a_3a_2b_1}} - \frac{1}{G_{a_1a_2a_3a_4a_5}}\right]\right.$$

$$+ \frac{1}{Z(a_2^2 - b_2^2)}\left[\frac{1}{G_{a_5a_4a_3b_2a_1}} - \frac{1}{G_{a_1a_2a_3a_4a_5}}\right] + \frac{1}{Z(a_3^2 - b_3^2)}\left[\frac{1}{G_{a_5a_4b_3a_2a_1}} - \frac{1}{G_{a_1a_2a_3a_4a_5}}\right]$$

$$\left. + \frac{1}{Z(a_4^2 - b_4^2)}\left[\frac{1}{G_{a_5b_4a_3a_2a_1}} - \frac{1}{G_{a_1a_2a_3a_4a_5}}\right] + \frac{1}{Z(a_5^2 - b_5^2)}\left[\frac{1}{G_{b_5a_4a_3a_2a_1}} - \frac{1}{G_{a_1a_2a_3a_4a_5}}\right]\right).$$

By inserting the value for $G_{\mathbf{q}}$ given by (19), and by imposing the renormalization conditions, taking the limit $\mathbf{a}, \mathbf{b} \to 0$ one readily obtains

$$\Gamma_0^{4,\text{ren}} = -5\widetilde{\lambda}\left(1 + \frac{1}{Z}\right). \tag{30}$$

By imposing the cutoff Λ, we can show (perturbatively) that the wave function renormalization (for a similar computation cf. Lemma 5 in [32]) takes the form

$$Z = 1 + x\lambda\log(\Lambda) + \mathcal{O}(\lambda^2), \quad x \in \mathbb{R}. \tag{31}$$

Then one has

$$-\lambda_r = \Gamma_0^{4,\text{ren}} \to -5\widetilde{\lambda}. \tag{32}$$

6 Solution of the integral equation

The integral equation (28) is a non-linear integro-partial differential equation. We therefore opt for a numerical approach. We introduce the following dimensionless variables:

$$\alpha \equiv \frac{a}{m_r}, \qquad \tau \equiv \frac{t}{m_r^2}, \qquad \boldsymbol{\rho} \equiv \frac{\mathbf{p}}{m_r}, \qquad \text{and} \qquad \gamma \equiv 1 + \tau + \sum_{i=1}^{4}\rho_i^2,$$

and, accordingly, we rescale the two-point function $\sigma(\alpha) \equiv \Sigma_{a0000}^r/m_r^2$. Equation (28) can be thus reworded:

$$\sigma(\alpha) = -\widetilde{\lambda}\int d\boldsymbol{\rho}\int_0^{\alpha^2} d\tau(\alpha^2 - \tau)\frac{\partial^2}{\partial\tau^2}\left\{\frac{1}{1 + \tau + |\boldsymbol{\rho}|^2 - \sigma(\sqrt{\tau},\boldsymbol{\rho})}\right\}. \tag{33}$$

Expanding the solution in $\widetilde{\lambda}$, $\sigma(\alpha) = \sum_{n=0}^{\infty}\sigma_n(\alpha)\widetilde{\lambda}^n$, it readily follows $\sigma_0(\alpha) = 0$. To proceed with the computation of the non-trivial orders, we invert the power series (in $\widetilde{\lambda}$) appearing in the denominator (33) after factoring out γ, namely $(1 - \sigma(\sqrt{\tau},\boldsymbol{\rho})/\gamma)$. First, we treat this series as a formal power series, then we care about convergence. The idea is that in order to compute σ_{n+1}, for which we need σ_i, $i \leq n$, we approximate the latter functions by near-to-'principal diagonal' Padé approximants, i.e. by quotients of polynomials of almost equal

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degree; this approximation is valid in a certain domain and would lead to the convergence of the series there. Shortly, a second advantage of the Padé approximants will be evident.

We use the following result for the power of a series (cf. sec. 3.5 in [49]): For any $r \in \mathbb{C}$, the r-th power of a formal power series $1 + g_1 t^1 + \frac{1}{2!} g_2 t^2 + \ldots$ can be expanded as follows:

$$\left(1 + \sum_{n \geq 1} g_n \frac{t^n}{n!} \right)^r = 1 + \sum_{n \geq 1} \left(\mathbb{P}_n^{(r)} \frac{t^n}{n!} \right), \tag{34}$$

where the $\mathbb{P}_n^{(r)}$, the so-called *potential polynomials*, are given in terms of the Bell polynomials $\mathbb{B}_{p,q}$:

$$\mathbb{P}_n^{(r)} = \sum_{1 \leq k \leq n} (r)_k \mathbb{B}_{n,k}(g_1, \ldots, g_{n-k+1})$$

$$= \sum_{1 \leq k \leq n} (-1)^k k! \mathbb{B}_{n,k}(g_1, \ldots, g_{n-k+1}).$$

In our case, the Pochammer symbol appearing there, $(r)_k$, becomes $(-1)_k = (-1)^k k!$ As for the Bell polynomials, they are defined by

$$\mathbb{B}_{n,k}(x_1, \ldots, x_{n-k+1}) = \sum_{c_j} \frac{n!}{c_1! c_2! \cdots c_{n-k+1}!} \left(\frac{x_1}{1!} \right)^{c_1} \left(\frac{x_2}{2!} \right)^{c_2} \cdots \left(\frac{x_{n-k+1}}{n-k+1!} \right)^{c_{n-k+1}}.$$

The sum here runs over all the non-negative integers c_l such that the conditions

$$\sum_{i=1}^{n-k+1} c_i = k \qquad \text{and} \qquad \sum_{q=1}^{n-k+1} q c_q = n \tag{35}$$

are fulfilled. It will be useful to rescale the k-th variable x_k in the Bell polynomials by $x'_k = x_k \cdot w \cdot k!$, for a number $w \neq 0$, to obtain a simpler expression in the lhs:

$$\mathbb{B}_{n,k}(x'_1, \ldots, x'_{n-k+1}) = \mathbb{B}_{n,k}(x_1 \cdot w \cdot 1!, x_2 \cdot w \cdot 2!, \ldots, x_{n-k+1} \cdot w \cdot (n-k+1)!)$$

$$= w^k \sum_{c_j} \frac{n!}{c_1! c_2! \cdots c_{n-k+1}!} x_1^{c_1} \cdots (x_{n-k+1})^{c_{n-k+1}}. \tag{36}$$

Remark. After taking the reciprocal of the power series, the convergence of each coefficient of $\tilde{\lambda}^n$, $\sigma_n(\alpha)$, is not guaranteed. We denote by $\tilde{\sigma}_n(\alpha)$ those probably divergent coefficients, which need to be renormalized. Thus, taking the $(n+1)$-order in $\tilde{\lambda}$ of $\tilde{\sigma}(\alpha)$, $\tilde{\sigma}_{n+1}(\alpha)$, boils down to integrate

$$\tilde{\sigma}_{n+1}(\alpha) = - \int_{\mathbb{R}^4} d\boldsymbol{\rho} \int_0^{\alpha^2} d\tau (\alpha^2 - \tau) \frac{\partial^2}{\partial \tau^2}$$

$$\left[\frac{1}{n! \gamma} \sum_{1 \leq k \leq n} (-1)^k k! \mathbb{B}_{n,k} \left(-1! \frac{\sigma_1(\zeta)}{\gamma}, -2! \frac{\tilde{\sigma}_2(\zeta)}{\gamma}, \ldots, -(n-k+1)! \frac{\tilde{\sigma}_{n-k+1}(\zeta)}{\gamma} \right) \right]$$

$$= - \int_{\mathbb{R}^4} d\boldsymbol{\rho} \int_0^{\alpha^2} d\tau (\alpha^2 - \tau) \frac{\partial^2}{\partial \tau^2} \left[\sum_{1 \leq k \leq n} \frac{k!}{\gamma^{k+1}} \sum_{\mathbf{c}(k,n)} \prod_{j=1}^{n-k+1} \left(\frac{\tilde{\sigma}_j(\zeta)^{c_j}}{c_j!} \right) \right]. \tag{37}$$

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Here $\zeta = (\sqrt{\tau}, \boldsymbol{\rho})$ and we have made use of (36) with the nowhere-vanishing $w = -\gamma^{-1}$. To obtain expressions for higher-order solutions we use the explicit form of the Bell polynomials

$$\mathbb{B}_{1,1}(x_1) = x_1, \quad \mathbb{B}_{2,1}(x_1, x_2) = x_2, \quad \mathbb{B}_{3,1}(x_1, x_2, x_3) = x_3, \qquad \mathbb{B}_{4,1}(x_1, x_2, x_3, x_4) = x_4,$$
$$\mathbb{B}_{2,2}(x_1, x_2) = x_1^2, \quad \mathbb{B}_{3,2}(x_1, x_2, x_3) = 3x_1 x_2, \quad \mathbb{B}_{4,2}(x_1, x_2, x_3, x_4) = 4x_1 x_3 + 3x_2^2,$$
$$\mathbb{B}_{3,3}(x_1, x_2, x_3) = x_1^3, \qquad \mathbb{B}_{4,3}(x_1, x_2, x_3, x_4) = 6x_1^2 x_2,$$
$$\mathbb{B}_{4,4}(x_1, x_2, x_3, x_4) = x_1^4.$$

The first order in perturbation theory can be given exactly —and without using the Padé approximants, nor regularization — and is given by

$$\sigma_1(\alpha) = -2\text{Vol}(\mathbb{S}^3) \int_0^{\alpha^2} d\tau (\alpha^2 - \tau) \int_0^\infty d\rho \frac{\rho^2}{(1 + \tau + \rho^3)^2}$$
$$= -2(2\pi^2) \int_0^{\alpha^2} d\tau \frac{\alpha^2 - \tau}{4(1 + \tau)} = -\pi^2[(\alpha^2 + 1)\log(\alpha^2 + 1) - \alpha^2].$$

With (37) in our hands, other low-order terms can be obtained:

$$\widetilde{\sigma}_0(\alpha) = \sigma_0(\alpha) = 0$$
$$\widetilde{\sigma}_1(\alpha) = \sigma_1(\alpha) = -\pi^2[(\alpha^2 + 1)\log(\alpha^2 + 1) - \alpha^2]$$
$$\widetilde{\sigma}_2(\alpha) = -\int_{\mathbb{R}_\Lambda^4} d\boldsymbol{\rho} \int_0^{\alpha^2} d\tau (\alpha^2 - \tau) \frac{\partial^2}{\partial \tau^2} \left(\frac{1}{\gamma^2} \widetilde{\sigma}_1(\zeta) \right)$$
$$\widetilde{\sigma}_3(\alpha) = -\int_{\mathbb{R}_\Lambda^4} d\boldsymbol{\rho} \int_0^{\alpha^2} d\tau (\alpha^2 - \tau) \frac{\partial^2}{\partial \tau^2} \left\{ \frac{1}{\gamma^3} \left(\widetilde{\sigma}_1^2(\zeta) + \gamma \widetilde{\sigma}_2(\zeta) \right) \right\}$$
$$\widetilde{\sigma}_4(\alpha) = -\int_{\mathbb{R}_\Lambda^4} d\boldsymbol{\rho} \int_0^{\alpha^2} d\tau (\alpha^2 - \tau) \frac{\partial^2}{\partial \tau^2} \left\{ \frac{1}{\gamma^4} \left(\widetilde{\sigma}_1^3(\zeta) + 2\gamma \widetilde{\sigma}_1(\zeta) \widetilde{\sigma}_2(\zeta) + \gamma^2 \widetilde{\sigma}_3(\zeta) \right) \right\}$$
$$\widetilde{\sigma}_5(\alpha) = -\int_{\mathbb{R}_\Lambda^4} d\boldsymbol{\rho} \int_0^{\alpha^2} d\tau (\alpha^2 - \tau) \frac{\partial^2}{\partial \tau^2} \left\{ \frac{1}{\gamma^5} \left(\widetilde{\sigma}_1^4(\zeta) + 3\gamma \widetilde{\sigma}_1(\zeta)^2 \widetilde{\sigma}_2(\zeta) \right. \right.$$
$$\left. \left. + 2\gamma^2 (\widetilde{\sigma}_1(\zeta) \widetilde{\sigma}_3(\zeta) + \widetilde{\sigma}_2^2(\zeta)) + \gamma^3 \widetilde{\sigma}_4(\zeta) \right) \right\}.$$

In all these expressions $\widetilde{\sigma}_i(\zeta) = \sum_{j=1}^4 \widetilde{\sigma}_i(p_j) + \widetilde{\sigma}_i(\sqrt{\tau})$, with $\zeta_0 = \sqrt{\tau}, \zeta_1 = \rho_1, \ldots, \zeta_4 = \rho_4$. Notice that the non-linearity is evident from the third order on.

To shed some light on the procedure to extract the divergence occurring in the integral (37), we consider the second order and then extend the method to higher orders. The most dangerous term in

$$\widetilde{\sigma}_2(\alpha) \;=\; -\int_{\mathbb{R}_\Lambda^4} d\boldsymbol{\rho} \int_0^{\alpha^2} d\tau (\alpha^2 - \tau) \left[\frac{6\widetilde{\sigma}_1(\zeta)}{\gamma^4} - \frac{4\widetilde{\sigma}_1'(\sqrt{\tau})}{\gamma^3} + \frac{\widetilde{\sigma}_1''(\sqrt{\tau})}{\gamma^2} \right] \qquad (38)$$

is the last summand. We write the Taylor expansion of $\gamma^{-2} \widetilde{\sigma}_1''(\sqrt{\tau})$ at first order and get the renormalized expression $\sigma_2(\alpha)$ as

$$\sigma_2(\alpha) \;=\; -\int_{\mathbb{R}_\Lambda^4} d\boldsymbol{\rho} \int_0^{\alpha^2} d\tau (\alpha^2 - \tau) \left[\frac{6\sigma_1(\zeta)}{\gamma^4} - \frac{4\sigma_1'(\sqrt{\tau})}{\gamma^3} + \frac{\sigma_1''(\sqrt{\tau})}{\gamma^2} - \frac{\sigma_1''(\sqrt{\tau})}{(1 + |\boldsymbol{\rho}|^2)^2} \right]$$

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$$= -\int_{\mathbb{R}^4_\Lambda} d\boldsymbol{\rho} \int_0^{\alpha^2} d\tau (\alpha^2 - \tau)\Big[\frac{6\sigma_1(\zeta)}{\gamma^4} - \frac{4\sigma_1'(\sqrt{\tau})}{\gamma^3}\Big]$$

$$+\pi^2 \int_0^{\alpha^2} d\tau (\alpha^2 - \tau)\sigma_1''(\sqrt{\tau}) \log(1+\tau)$$

$$= -\int_{\mathbb{R}^4_\Lambda} d\boldsymbol{\rho} \int_0^{\alpha^2} d\tau (\alpha^2 - \tau)\Big[\frac{6\sigma_1(\zeta)}{\gamma^4} - \frac{4\sigma_1'(\sqrt{\tau})}{\gamma^3}\Big]$$

$$+\pi^4 \Big\{ (1+\alpha^2) \log(1+\alpha^2) - \alpha^2 - \frac{1}{2}(1+\alpha^2)\big[\log(1+\alpha^2)\big]^2 \Big\}.$$

The above integral is convergent and therefore $\sigma_2(\alpha)$ is well defined in the limit where $\Lambda \to \infty$. Consider now

$$\widetilde{\sigma}_{n+1}(\alpha) \; = \; -\int_{\mathbb{R}^4} d\boldsymbol{\rho} \int_0^{\alpha^2} d\tau (\alpha^2 - \tau)\frac{\partial^2}{\partial \tau^2} \left[\sum_{1\le k\le n} \frac{k!}{\gamma^{k+1}} \sum_{\mathbf{c}(k,n)} \prod_{j=1}^{n-k+1} \left(\frac{\widetilde{\sigma}_j(\zeta)^{c_j}}{c_j!}\right) \right]. \quad (39)$$

The integral leads to the logarithmically divergence which could be removed. We get

$$\sigma_{n+1}(\alpha) \; = \; -\int_{\mathbb{R}^4} d\boldsymbol{\rho} \int_0^{\alpha^2} d\tau (\alpha^2 - \tau)\Big\{ \frac{\partial^2}{\partial \tau^2} \Big[\sum_{1\le k\le n} \frac{k!}{\gamma^{k+1}} \sum_{\mathbf{c}(k,n)} \prod_{j=1}^{n-k+1} \Big(\frac{\sigma_j(\zeta)^{c_j}}{c_j!}\Big)\Big]$$

$$-\frac{\sigma_n''(\sqrt{\tau})}{(1+|\boldsymbol{\rho}|^2)^2} \Big\}. \quad (40)$$

The above integral is convergent in the limit where $\Lambda \to \infty$ using (almost) equal degree Padé approximation. The solution of the integral equation, for small values of the coupling constant, is given in fig. 3, fig. 4 and fig. 5 . Those plots show $\sigma(\alpha)$, computed to second order in $\widetilde{\lambda}$. We have used $Mathematica^{TM}$ to obtain the Padé approximants and to plot the solution. Their advantage over partial Taylor sums to approximate the σ_i's becomes now clear— those had been otherwise divergent and the only term we introduced in order to control the divergence would not have been enough.

7   Conclusion

In this paper we have considered the just renormalizable φ_5^4 tensorial group field theory with the propagator of the form $1/\mathbf{p}^2$. We have introduced the melonic approximation of the Schwinger-Dyson equation of the two and four-point functions. This is made possible, by suppressing the non-melonic graphs, to obtain a closed equation for the two-point functions. This equation is solved perturbatively. It would be interesting to apply the melonic approximation to other tensor models supporting a large-N expansion, e.g. to multi-orientable tensor models [18].

For future investigation remains the numerical study of the four-point function we treated in section 5. We also plan to address the criticality of the model. Concretely, at certain value of the coupling constant, namely about $\lambda \approx -2.125 \times 10^{-2}$, the behaviour of the two-point function noticeably bifurcates. Thus, some criticality is promissory in fig. 4. To claim this
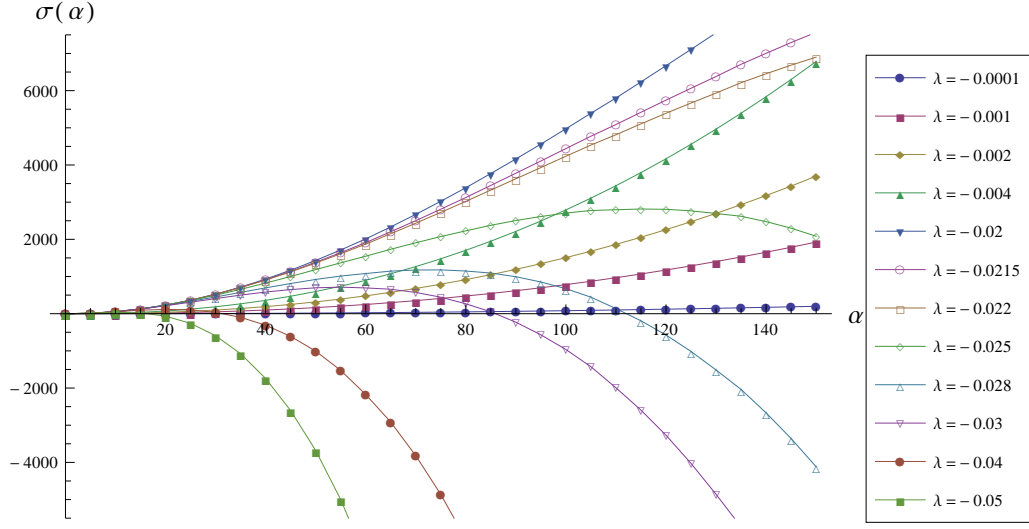
Figure 3: Plot of $\sigma(\alpha)$ with different negative values of λ. The curves are interpolations of discrete data obtained for the two-point function of the φ_5^4-model (with m_r set to 1) to second order in $\widetilde{\lambda}$.
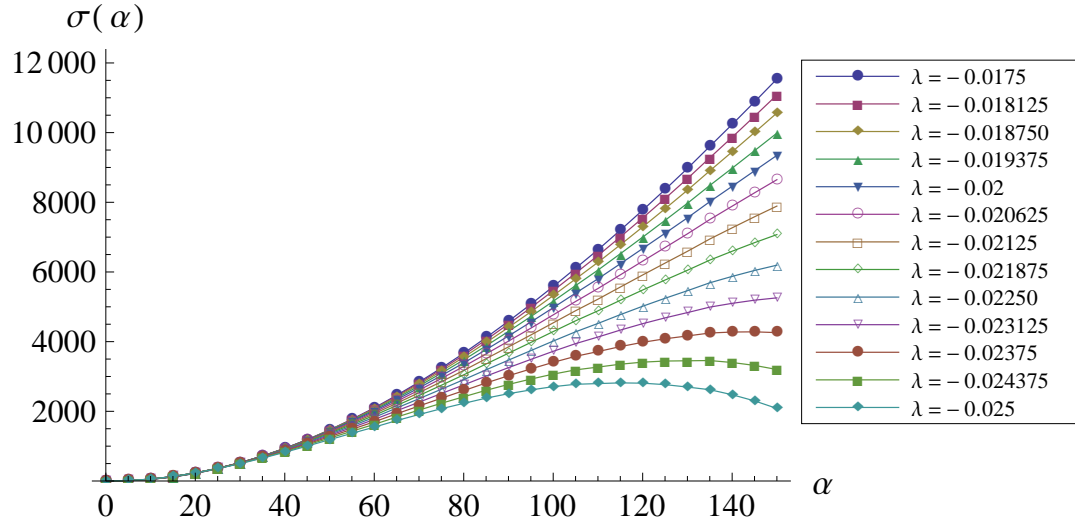

Figure 4: This is a zoom to the region where criticality might take place. It shows how the behaviour of the two-point function bifurcates from a certain value for the coupling constant about $\lambda \approx -0.002125$.
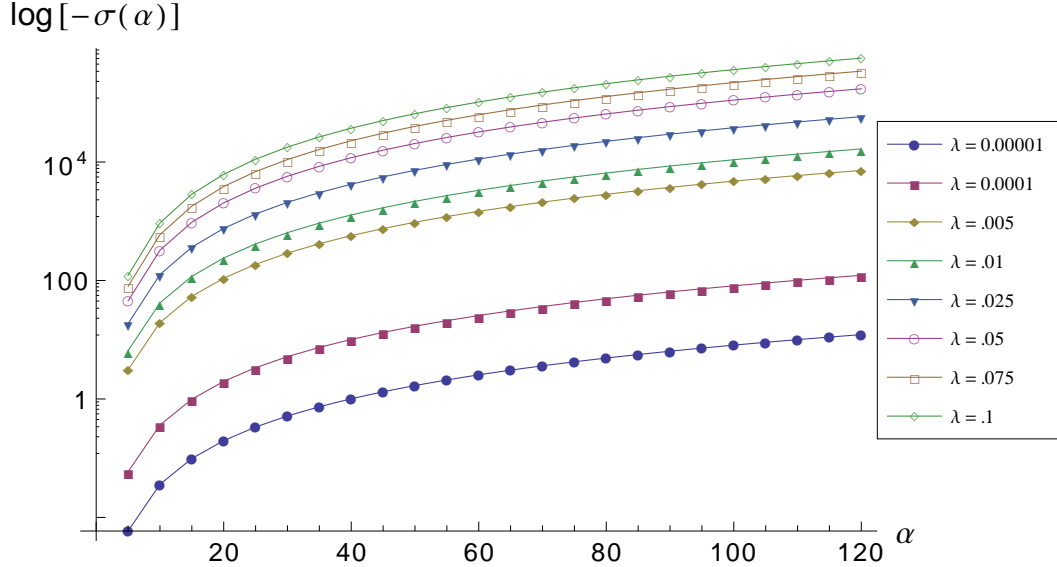
Figure 5: Plot of $\log[-\sigma(\alpha)]$ with different positive values of λ. Just as in the previous plot, we interpolated a discrete graph.

we need a new, more detailed study, though; for instance, by solving for higher values of α. The phase transitions and the critical behaviour of the model could physically relevant, and in particular, interesting for applications in cosmology.

Acknowledgements

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