

# 8D-spectral triple on 4D-Moyal space and the vacuum of noncommutative gauge theory

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## Abstract

Observing that the Hamiltonian of the renormalisable scalar field theory on 4-dimensional Moyal space  $\mathcal{A}$  is the square of a Dirac operator  $\mathcal{D}$  of spectral dimension 8, we complete  $(\mathcal{A}, \mathcal{D})$  to a compact 8-dimensional spectral triple. We add another Connes-Lott copy and compute the spectral action of the corresponding  $U(1)$ -Yang-Mills-Higgs model. We find that in the Higgs potential the square  $\phi^2$  of the Higgs field is shifted to  $\phi \star \phi + \text{const} \cdot X_\mu \star X^\mu$ , where  $X_\mu$  is the covariant coordinate. The classical field equations of our model imply that the vacuum is no longer given by a constant Higgs field, but both the Higgs and gauge fields receive non-constant vacuum expectation values.

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## 0 Preface

In September 2007 we published the first version of this manuscript as preprint arXiv:hep-th/0709.0095v1. We had found dimensional arguments why previous attempts to construct a *four-dimensional* spectral triple for renormalisable scalar field theory on Moyal space with harmonic oscillator potential [1] had to fail. These arguments showed the necessity of a doubling of the dimension, and indeed we were able to identify an 8-dimensional Dirac operator  $\mathcal{D}$  which together with the Moyal algebra gave rise to a reasonable spectral triple. We also computed the decisive part of the resulting spectral action and completed it by gauge invariance.

In preparing a talk for the Oberwolfach meeting on “Noncommutative Geometry” a few days later, one of us (R.W.) realised that the dimensionality of our spectral triple is much more intricate [47]. While  $|\mathcal{D}|^{-8}$  is indeed of Dixmier trace class, localising it with the operator  $L_*(f)$  of left Moyal multiplication by a Schwartz function  $f$  one has  $L_*(f)|\mathcal{D}|^{-4}$  of Dixmier trace class. This dimension drop was already visible in the different parts of the spectral action. We thus concluded that the *metric* dimension of our spectral triple remains  $d = 4$  whereas the *KO-dimension* is  $k = 8$ .

It became apparent that our spectral triple proposed in arXiv:hep-th/0709.0095v1 was the shadow of a very rich mathematical structure which had to be explored. Working out the details, the corrected manuscript deviated more and more into a completely different paper. Additionally, as the computation of the dimension spectrum faced enormous difficulties, the commutative case was treated first in [48]. An important achievement of [48] was to understand that there are in fact two Dirac operators  $\mathcal{D}_1, \mathcal{D}_2$  which both relate to supersymmetric quantum mechanics and which together permit a realisation of the orientability axiom. In arXiv:hep-th/0709.0095v1 we had still pointed out that orientability cannot be recovered.

Eventually, all difficulties with the dimension spectrum in the Moyal case, and several other mathematical issues, have been recently solved in joint work of one of us with V. Gayral [49]. In that paper the factorisation property of the Moyal algebra<sup>1</sup> is heavily used to prove rigorous  $\mathcal{L}^p$ -estimates for all appearing operators. Using these Hölder type estimates a completely different computation of the spectral action is given, which up to typos confirms the result of arXiv:hep-th/0709.0095v1.

In summary, the paper [49] supersedes arXiv:hep-th/0709.0095v1 in all mathematical aspects. But arXiv:hep-th/0709.0095v1 contains the precious heuristic discussion of the dimensionality of the Dirac operator and a useful overview of renormalisable field theories on Moyal space which both are lost in [49]. We therefore think that these parts of arXiv:hep-th/0709.0095v1 and the original technique for computing the spectral action are interesting enough to justify, in spite of four years of delay, a corrected version. Although we know many things better now, we limit ourselves to error corrections. In particular, the historical introduction and the notation is unchanged. We silently correct the typos in the spectral action as identified in [49, footnote 3]. The original section about solutions of the field equation (and comments in the introduction referring to it) is completely removed. As pointed out to us by A. Marcillaud de Goursac, our formula for the Moyal product in radial

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<sup>1</sup>For any  $f \in \mathcal{S}(\mathbb{R}^d)$  there are  $f_1, f_2 \in \mathcal{S}(\mathbb{R}^d)$  with  $f_1 \star f_2 = f$ .

coordinates was wrong and with it our original conclusions. For a discussion of the vacuum configuration of this type of action we refer to [50].

## 1 Introduction

### 1.1 Renormalisable field theories on Moyal space

Renormalisable field theories on Moyal space are by now in mature state. In the first renormalisation proof [1], the matrix base of the Moyal plane was a central philosophy, because we wanted to avoid convergence subtleties with the oscillating integrals in momentum space. We traded the simple matrix product interaction in for a complicated (but manifestly positive) propagator and used exact renormalisation group equations to estimate the ribbon graphs. The technically most challenging part was a brute-force analysis [2] of all possible contractions of ribbon graphs. The scale analysis led to the existence of an additional marginal coupling in the  $\phi^4$ -model, which corresponds to a harmonic oscillator potential for the free field. Later on, we interpreted this term as required by Langmann-Szabo duality [3]. A summary of these ideas can be found in [4].

The renormalisation proof was considerably simplified by switching to multi-scale analysis as the renormalisation scheme. The first version still relied on the matrix base [5]. Once the bounds for the sliced propagator being proven (which is tedious), one obtains in an efficient way the power-counting theorem in terms of the topology of the graph. Subsequently, the renormalisation proof was also achieved by multi-scale analysis in position space (which is equivalent to momentum space by Langmann-Szabo duality) [6], showing the equivalence of various renormalisation schemes. Recently, the position space amplitude of an arbitrary orientable graph was expressed as an integral over Symanzik type hyperbolic polynomials [7]. With all inner integrations carried out, this is the most condensed way of writing Feynman graph amplitudes. See also [8] for the more complicated case of “critical” models.

Additionally, we noticed that the  $\beta$ -function of the renormalisable noncommutative  $\phi^4$ -model tends to zero at large energy scales. This is opposite to the commutative case and supports the hope that a non-perturbative construction of the model is within reach [9, 10]. The one-loop  $\beta$ -function was first computed in [11] (its peculiar feature was noticed in [4]). Roughly speaking, there is a one-loop wavefunction renormalisation in the model (absent in the commutative case), which for large energy scales exactly compensates the renormalisation of the four-point function. Then, in [12] it was shown that at the self-duality point  $\Omega = 1$  (where  $\Omega$  is the frequency of the harmonic oscillator potential in natural units), the  $\beta$ -function vanishes up to three-loop order. Eventually, in [13] the vanishing of the  $\beta$ -function (at  $\Omega = 1$ ) was proven to all orders, which means that the Landau ghost is absent in noncommutative  $\phi^4$ -theory: Wave function renormalisation exactly compensates the renormalisation of the four-point function, so that the flow between the bare and the renormalised coupling is bounded. The main tool in this proof is a clever combination of the Ward identity relative to unitary transformations with the Schwinger-Dyson equations. Strictly speaking, the proof requires  $\Omega = 1$ , but using the bounds established in [5], it is plausible that the renormalisation flow of the coupling is bounded for  $0 < \Omega < 1$ , too.

A good review of these exciting developments is [14]. The relation to previous attempts to renormalise noncommutative field theories is discussed in [15].

The importance of the self-duality case was first noticed in [16, 17] where an exact non-perturbative solution of a complex scalar field theory on Moyal space with critical magnetic background field was constructed. The UV-fixed point of this model is trivial. In [18, 19] a non-trivial exactly solvable (and just renormalisable) field theory was obtained, the noncommutative  $\phi_6^3$ -model at the self-duality point. Here, self-duality relates this model to the Kontsevich-model. For  $\phi_4^3$ , see [20].

There is also considerable progress with other than scalar field models on Moyal space. In [21, 22] renormalisation to all orders of the duality-covariant orientable Gross-Neveu model was shown. To put it into context with the work we present here, it is important to stress that the Dirac operator in [21, 22] is *not* the square root of the harmonic oscillator Hamiltonian appearing in the  $\phi^4$ -model of [1] and following treatments. It is precisely in this paper where we construct such a square root and analyse its properties. The Dirac operator of the Gross-Neveu model is of the type studied (for scalar fields) in [16, 17], just describing the influence of a constant magnetic background field. Its spectrum is very different from the harmonic oscillator (there is e.g. infinite degeneracy). This fact can also be seen from a different structure of the propagator in position space [23], which made the renormalisation of the Gross-Neveu model technically more difficult. In some sense, the magnetic background field is not needed for renormalisation of complex scalar fields, as already argued in [24] (in the massive case a new counterterm is generated, though). See [25] for the one-loop  $\beta$ -function of this model.

The most interesting field theories are Yang-Mills theories, which we also would like to see in renormalisable form on Moyal space. Usual Yang-Mills theory on Moyal space (without modifications of the action by something similar to an oscillator potential) is known to be not renormalisable [26]. Yang-Mills theories in noncommutative geometry [27] are naturally obtained from the spectral action principle [28, 29] relative to an appropriate Dirac operator. In this way, a beautiful reformulation of the standard model of particle physics was obtained, see [30] for its most recent version. Moyal space with undeformed Dirac operator is a (non-compact) spectral triple [31]. The corresponding spectral action was computed in [32], with the result that it is the usual Yang-Mills action on Moyal space (which is not renormalisable). The magnetic background field Dirac operator of the Gross-Neveu model gives the same usual Yang-Mills action, too.

To obtain a gauge theory with sort of oscillator potential via the spectral action principle, we need a Dirac operator with similar spectrum as the square root of the harmonic oscillator. Unfortunately, all attempts to produce such a Dirac operator failed so far, and here we can report progress in this paper. As workaround we translated the physical interpretation of the spectral action (to describe a one-loop effective action of fermions in a classical external gauge field) from fermions to scalar fields. In [33] this method was already worked out for general (isospectral) Rieffel deformations [34]. We finished the computation almost simultaneously in position space [35] and in the matrix base [36]. See also [37, 38]. As a result, there are two additional terms to the Yang-Mills action, namely the integral over  $\tilde{X}_\mu \star \tilde{X}^\mu$  and over its square, where  $\tilde{X}_\mu(x) = (\Theta^{-1})_{\mu\nu} x^\nu + A_\mu(x)$  is a covariant coordinate [39]. The existence

of such a term was conjectured in [40, p. 90].

The problem with the effective action derived in [35, 36] is that, expanding  $\tilde{X}_\mu \star \tilde{X}^\mu$  and its square, there is a linear term in the gauge field  $A_\mu$ . The consequence is that  $A_\mu = 0$  *is not a stable solution of the classical field equation*. Any attempt to solve the classical field equations resulting from [35, 36] failed so far. To circumvent the vacuum problem, in [41] an oscillator potential for the gauge field was achieved solely from a generalised ghost sector, in a BRST-invariant way. Although a one-loop calculation is likely to produce the  $\tilde{X}_\mu \star \tilde{X}^\mu$  terms as in [35, 36], the investigations in [41] demonstrate the enormous freedom of constructing the ghost sector, which in some way will be needed to obtain a manageable gauge field propagator.

## 1.2 Strategy of the paper

Our paper starts from a simple observation, so simple that it is embarrassing not having it earlier exploited. The harmonic oscillator Hamiltonian  $H$  in one-dimensional configuration space, thus two-dimensional phase space, has spectrum  $\omega(n + \frac{1}{2})$  with  $n \in \mathbb{N}$ . Thus,  $H^{-1}$  is a noncommutative infinitesimal [28] of order one—the configuration space dimension. The Hamiltonian  $H$  generalises the Laplacian. The central object in noncommutative geometry is the Dirac operator, which is a (generalised) square root of the Laplacian. Now,  $\mathcal{D} = H^{\frac{1}{2}}$  is a noncommutative infinitesimal of order one over two, two being the phase space dimension. Spectral dimension is defined through the Dirac operator so that *the spectral dimension of the harmonic oscillator is the phase space dimension*.

For field theory we are interested in four-dimensional Moyal configuration space. The isospectral deformation would be a four-dimensional spectral triple [31]. But for renormalisation of the  $\phi_4^4$ -theory we must promote the 4D Laplace operator  $-\Delta$  to the 4D harmonic oscillator Hamiltonian  $H = -\Delta + \Omega^2 \|x\|^2$ . According to the previous discussion, the noncommutative dimension of the 4D harmonic oscillator Hamiltonian is the phase space dimension, which is EIGHT, not four. We thus understand why all attempts to find a 4D Dirac operator for the 4D harmonic oscillator Hamiltonian necessarily failed. On the other hand, it is absolutely trivial to write down an 8D Dirac operator so that its square equals (up to a constant matrix) the 4D harmonic oscillator Hamiltonian. This is what we do in Section 2. Additionally, we show that our 8D-Dirac operator on 4D-Moyal space almost extends to an eight-dimensional spectral triple in the original sense [28]. The orientability axiom is violated. We do not check Poincaré duality.

It is worthwhile to mention that the distinction between configuration space and phase space dimension was crucial for the quantum field theory on projective modules over the noncommutative torus investigated in [42]. There,  $\mathbb{R}^2$  and the 2-dimensional space of holomorphic  $\mathbb{C}^2$ -function were considered as projective modules, i.e. configuration space, over the 4D-noncommutative torus (which extends to a four-dimensional spectral triple). The resulting Hamiltonian was precisely that of the 2D-harmonic oscillator, where the oscillator potential is naturally obtained from the isospectral Dirac operator of the 4D-noncommutative torus. The field theory on 2D-configuration space was shown to be one-loop renormalisable like a 4D-scalar field theory, four being the phase space dimension of the noncommutative

torus. The dimensional relations with Moyal space were discussed to some extent in [42]. It was noticed that the heat kernel traces split into a local integral over field monomials times a *partial trace only* of the propagator (see also [33]). But the true dimensionality of the harmonic oscillator Moyal space was not realised.

Having the 8D-Dirac operator with harmonic oscillator spectrum, we perform the standard procedure [28, 29] of noncommutative geometry to get to the spectral action. To make it a little more interesting, we add in Section 3 another Connes-Lott copy [43] and compute in Section 4 the spectral action for the resulting two  $U(1)$ -Moyal Yang-Mills fields unified with a complex Higgs field to a single noncommutative gauge field. This extends the computation of [35, 36] where the effective scalar field action was (unfortunately) not considered. It turns out that only the inclusion of the Higgs field provides an understanding of the  $\tilde{X}_\mu \star \tilde{X}^\mu$  terms: We find that they appear together with the Higgs field  $\phi$  in a potential of the form  $(\alpha \tilde{X}_\mu \star \tilde{X}^\mu + \beta \phi \star \phi - 1)^2$ , for some positive numbers  $\alpha, \beta$ . Thus, *the origin of the non-trivial gauge field vacuum is nothing but the standard Higgs mechanism*. We experience here a further level of the unification of Higgs and gauge fields through noncommutative geometry: Almost-commutative geometry obtained the potential of the Higgs field as part of the unified Yang-Mills action. Spatial noncommutativity intertwines gauge and Higgs field even further so that the potential combines Higgs and gauge field on an equal footing.

## 2 A spectral triple in dimension 8

The renormalisable real  $\phi^4$ -model on the 4-dimensional Moyal plane is characterised by the appearance of the harmonic oscillator Hamiltonian

$$H_m = -\frac{\partial^2}{\partial x_\mu \partial x^\mu} + \Omega^2 \tilde{x}^\mu \tilde{x}_\mu + m^2 \quad (1)$$

in the action functional [1], where  $\tilde{x}_\mu := 2(\Theta^{-1})_{\mu\nu}x^\nu$ . For simplicity we choose

$$\Theta = \begin{pmatrix} 0 & \theta & 0 & 0 \\ -\theta & 0 & 0 & 0 \\ 0 & 0 & 0 & \theta \\ 0 & 0 & -\theta & 0 \end{pmatrix} =: i\theta\sigma, \quad \theta \in \mathbb{R}, \quad (2)$$

where  $\sigma = \sigma_2 \otimes 1_2$  consists of two copies of the second Pauli matrix. We have  $\Theta^{-1} = \frac{-i}{\theta}\sigma$ . It is then a well-known fact from quantum mechanics that the Hilbert space  $L^2(\mathbb{R}^4)$  has an orthonormal basis  $\{\psi_{\underline{n}}\}_{\underline{n} \in \mathbb{N}^4}$  of eigenfunctions of  $H_m$  with

$$H_m \psi_{\underline{n}} = \frac{4\Omega}{\theta} (|\underline{n}| + 2 + \frac{\theta m^2}{4\Omega}) \psi_{\underline{n}}, \quad |\underline{n}| = n_1 + n_2 + n_3 + n_4 \quad \text{for } \underline{n} = (n_1, n_2, n_3, n_4).$$

The inverse  $H_m^{-1}$  extends to a selfadjoint compact operator on  $L^2(\mathbb{R}^4)$  with eigenvalues

$$\lambda_n(m) = \left( \frac{4\Omega}{\theta} (n + 2 + \frac{\theta m^2}{4\Omega}) \right)^{-1}, \quad n \in \mathbb{N}. \quad (3)$$

The  $n^{\text{th}}$  eigenspace  $E_n$  has dimension  $\dim(E_n) = \binom{n+3}{3}$ , which is the number of possibilities to write  $n$  as a sum of four ordered natural numbers. This means that for  $s > 4$ , the trace

$$\text{Tr}(H_m^{-s}) = \frac{1}{6} \sum_{n=0}^{\infty} (n+3)(n+2)(n+1)(\lambda_n(m))^s \quad (4)$$

exists. The critical value  $s = 4$  characterises  $H^{-4}$  as belonging to the Dixmier trace ideal  $\mathcal{L}^{(1,\infty)}(L^2(\mathbb{R}^4))$  of compact operators [27].

At first sight,  $H^{-4} \in \mathcal{L}^{(1,\infty)}(L^2(\mathbb{R}^4))$  seems to be related to the four dimensional Moyal space under consideration. However, recall that in noncommutative geometry it is the *Dirac operator* which defines the dimension [28]. In a  $d$ -dimensional space we require  $|\mathcal{D}|^{-d} \in \mathcal{L}^{(1,\infty)}(L^2(\mathbb{R}^4))$ . Identifying  $H = |\mathcal{D}|^2$ , we notice the surprising fact that the 4-dimensional Moyal space has actually spectral dimension EIGHT.

In eight dimensions it is very easy to write down an appropriate Dirac operator,

$$\mathcal{D}_8 = i\Gamma^\mu \partial_\mu + \Omega \Gamma^{\mu+4} \tilde{x}_\mu . \quad (5)$$

Here, the  $\Gamma_k \in M_{16}(\mathbb{C})$ ,  $k = 1, \dots, 8$  are the generators of the 8-dimensional real Clifford algebra, satisfying

$$\Gamma_k \Gamma_l + \Gamma_l \Gamma_k = 2\delta_{kl} 1 . \quad (6)$$

We agree that latin indices run from 1 to 8 and greek indices from 1 to 4. Summation over repeated upper and lower indices is self-understood.

Accordingly, we take the Hilbert space  $\mathcal{H}_8 = L^2(\mathbb{R}^4, \mathcal{S})$  of square integrable spinors over FOUR-dimensional euclidean space, where the spinor bundle has typical fibre  $\mathbb{C}^{16}$ . For  $\psi \in \mathcal{H}_8$  we obtain

$$\mathcal{D}_8^2 \psi = ((-\Delta + \Omega^2 \tilde{x}_\mu \tilde{x}^\mu) 1 + \Sigma) \psi , \quad \Sigma := -i\Omega(\Theta^{-1})_{\mu\nu} [\Gamma^\mu, \Gamma^{\nu+4}] , \quad (7)$$

with  $\Delta = \partial_\mu \partial^\mu$ . Assuming a choice of the Clifford algebra where  $\Sigma$  is diagonal, we obtain up to the 16-fold multiplicity of each level and an unimportant shift in the mass exactly the spectrum of the harmonic oscillator Hamiltonian  $H$ . In particular,  $|\mathcal{D}_8|^{-8}$  belongs as required to the Dixmier trace ideal  $\mathcal{L}^{(1,\infty)}(L^2(\mathbb{R}^4, \mathcal{S}))$ .

As algebra  $\mathcal{A}_8$  we take the unitalised Moyal algebra<sup>2</sup>

$$\mathcal{A}_8 = \mathbb{R}_\Theta^4 \oplus \mathbb{C} , \quad (8)$$

where  $\mathbb{R}_\Theta^4$  is a vector space given by the Schwarz class functions on  $\mathbb{R}^4$ , equipped with the Moyal product

$$(f \star g)(x) = \int d^4 y \frac{d^4 k}{(2\pi)^4} f(x + \frac{1}{2}\Theta \cdot k) g(x+y) e^{i\langle k, y \rangle} , \quad f, g \in \mathcal{A}_8 . \quad (9)$$

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<sup>2</sup>This choice of the algebra cannot verify the orientability axiom in any form, because we cannot represent the partition of unity localised at infinity (which belongs to  $\mathcal{A}_8$ ) by derivatives of elements of the algebra (which is not possible with  $\mathcal{A}_8$ ). This can be achieved by an appropriate subalgebra of the multiplier algebra of  $\mathbb{R}_\Theta^4$ , see [31]. But the orientability axiom fails anyway, so it suffices to work with  $\mathcal{A}_8$ .

The Moyal product extends to constant functions using the integral representation of the Dirac distribution.

The algebra  $\mathcal{A}_8$  acts on  $\mathcal{H}_8$  also by componentwise Moyal product,  $\star : \mathcal{A}_8 \times \mathcal{H}_8 \rightarrow \mathcal{H}_8$  (we refer to [31] for the necessary extension of the Moyal product). Clearly, the smooth spinors form a finitely generated projective module over  $\mathcal{A}_8$ .

We compute the commutator of that action with the Dirac operator, taking for smooth spinors the identity  $2x^\mu\psi = x\star\psi + \psi\star x$  into account, as well as the relation  $[x^\nu, f]_\star = i\Theta^{\nu\rho}\partial_\rho f$ :

$$\begin{aligned} & \mathcal{D}_8(f \star \psi) - f \star (\mathcal{D}_8\psi) \\ &= i\Gamma^\mu((\partial_\mu f) \star \psi + f \star \partial_\mu\psi) + \frac{1}{2}\Omega\Gamma^{\mu+4}(\tilde{x}_\mu \star (f \star \psi) + (f \star \psi) \star \tilde{x}_\mu) \\ & \quad - i\Gamma^\mu f \star \partial_\mu\psi - \frac{1}{2}\Omega\Gamma^{\mu+4}(f \star (\tilde{x}_\mu \star \psi) + f \star (\psi \star \tilde{x}_\mu)) \\ &= (i(\Gamma^\mu + \Omega\Gamma^{\mu+4})(\partial_\mu f)) \star \psi . \end{aligned} \tag{10}$$

Thus, just the four-dimensional differential of  $f$  appears, no  $x$ -multiplication! This differential is represented on  $\mathcal{H}_8$  by  $\pi(dx^\mu) = \Gamma^\mu + \Omega\Gamma^{\mu+4}$ , and it is bounded. It commutes with Moyal multiplication from the right, so that the order-one condition is achieved in the usual way. However, the algebra generated by  $[\mathcal{D}_8, \mathcal{A}_8]$  and  $\mathcal{A}_8$  does not contain the chirality matrix  $\Gamma_9$  so that the orientability axiom does not hold. The ingredients of the spectral triple which just rely on the Clifford algebra (dimension table) are automatically satisfied. We do not check Poincaré duality. In conclusion, up to the orientability axiom (and possibly Poincaré duality),  $(\mathcal{A}_8, \mathcal{H}_8, \mathcal{D}_8)$  forms a spectral triple of dimension 8.

### 3 $U(1)$ -Higgs model

In the Connes-Lott spirit [43] we take the tensor product of the 8-dimensional spectral triple  $(\mathcal{A}_8, \mathcal{H}_8, \mathcal{D}_8, \Gamma_9)$  with the finite Higgs spectral triple  $(\mathbb{C} \oplus \mathbb{C}, \mathbb{C}^2, M\sigma_1)$ . The Dirac operator  $\mathcal{D} = \mathcal{D}_8 \otimes 1 + \Gamma_9 \otimes M\sigma_1$  of the product triple becomes

$$\mathcal{D} = \begin{pmatrix} \mathcal{D}_8 & M\Gamma_9 \\ M\Gamma_9 & \mathcal{D}_8 \end{pmatrix} . \tag{11}$$

In this representation, the algebra is  $\mathcal{A}_8 \oplus \mathcal{A}_8 \ni (f, g)$ , which acts on  $\mathcal{H} = \mathcal{H}_8 \oplus \mathcal{H}_8$  by diagonal Moyal multiplication. The commutator of  $\mathcal{D}$  with  $(f, g)$  is

$$[\mathcal{D}, (f, g)] = \begin{pmatrix} i(\Gamma^\mu + \Omega\Gamma^{\mu+4})L_\star(\partial_\mu f) & M\Gamma_9 L_\star(g - f) \\ M\Gamma_9 L_\star(f - g) & i(\Gamma^\mu + \Omega\Gamma^{\mu+4})L_\star(\partial_\mu g) \end{pmatrix} , \tag{12}$$

where  $L_\star(f)\psi = f \star \psi$  is left Moyal multiplication. This shows that selfadjoint fluctuated Dirac operators  $\mathcal{D}_A = \mathcal{D} + \sum_i a_i [\mathcal{D}, b_i]$  are of the form

$$\mathcal{D}_A = \begin{pmatrix} \mathcal{D}_8 + (\Gamma^\mu + \Omega\Gamma^{\mu+4})L_\star(A_\mu) & \Gamma_9 L_\star(\phi) \\ \Gamma_9 L_\star(\bar{\phi}) & \mathcal{D}_8 + (\Gamma^\mu + \Omega\Gamma^{\mu+4})L_\star(B_\mu) \end{pmatrix} , \tag{13}$$



for real fields  $A_\mu, B_\mu \in \mathcal{A}_8$  and a complex field  $\phi \in \mathcal{A}_8$ . The square of  $\mathcal{D}_A$  is

$$\mathcal{D}_A^2 = \begin{pmatrix} (H_0^2 + L_\star(\phi \star \bar{\phi}))1 + \Sigma + F_A & i(\Gamma^\mu + \Omega\Gamma^{\mu+4})\Gamma_9 L_\star(D_\mu\phi) \\ i(\Gamma^\mu + \Omega\Gamma^{\mu+4})\Gamma_9 L_\star(\overline{D_\mu\phi}) & (H_0^2 + L_\star(\bar{\phi} \star \phi))1 + \Sigma + F_B \end{pmatrix}, \quad (14)$$

where

$$D_\mu\phi := \partial_\mu\phi - iA \star \phi + i\phi \star B, \quad (15)$$

$$\begin{aligned} F_A &:= \{\mathcal{D}_8, (\Gamma^\mu + \Omega\Gamma^{\mu+4})L_\star(A_\mu)\} + (\Gamma^\mu + \Omega\Gamma^{\mu+4})(\Gamma^\nu + \Omega\Gamma^{\nu+4})L_\star(A_\mu \star A_\nu) \\ &= \{L_\star(A^\mu), i\partial_\mu + \Omega^2 M_\bullet(\tilde{x}_\mu)\} + (1 + \Omega^2)L_\star(A_\mu \star A^\mu) \\ &\quad + i\left(\frac{1}{4}[\Gamma^\mu, \Gamma^\nu] + \frac{1}{4}\Omega^2[\Gamma^{\mu+4}, \Gamma^{\nu+4}] + \Omega\Gamma^\mu\Gamma^{\nu+4}\right)L_\star(F_{\mu\nu}^A), \end{aligned} \quad (16)$$

and similarly for  $F_B$ . In this expression,  $F_{\mu\nu}^A = \partial_\mu A_\nu - \partial_\nu A_\mu - i(A_\mu \star A_\nu - A_\nu \star A_\mu)$  is the field strength and  $(M_\bullet(\tilde{x}_\mu)\psi)(x) = \tilde{x}_\mu\psi(x)$  is ordinary local multiplication.

## 4 The spectral action

### 4.1 General remarks

According to the spectral action principle [28, 29], the bosonic action depends only on the spectrum of the Dirac operator. Thus, by functional calculus, the most general form of the bosonic action is

$$S(\mathcal{D}_A) = \text{Tr}(\chi(\mathcal{D}_A^2)), \quad (17)$$

for some function  $\chi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  for which the Hilbert space trace exists. By Laplace transformation one has

$$S(\mathcal{D}_A) = \int_0^\infty dt \text{Tr}(e^{-t\mathcal{D}_A^2})\hat{\chi}(t), \quad (18)$$

where  $\hat{\chi}$  is the (inverse) Laplace transform of  $\chi$ , i.e.  $\chi(s) = \int_0^\infty dt e^{-st}\hat{\chi}(t)$ . Assuming the heat kernel has an asymptotic expansion

$$e^{-t\mathcal{D}_A^2} = \sum_{z=-\delta}^\infty a_z(\mathcal{D}_A^2)t^z, \quad \delta \in \mathbb{N}, \quad (19)$$

we obtain

$$S(\mathcal{D}_A) = \sum_{z=-\delta}^\infty \text{Tr}(a_z(\mathcal{D}_A^2)) \int_0^\infty dt t^z \hat{\chi}(t) =: \sum_{z=-\delta}^\infty \chi_z \text{Tr}(a_z(\mathcal{D}_A^2)). \quad (20)$$

For compact manifolds, the most singular order  $\delta$  is half of the dimension according to Weyl's theorem. To compute the  $\chi_z$  we have to distinguish the cases  $z \in \mathbb{N}$  and  $z \notin \mathbb{N}$ . First,

$$\int_0^\infty ds s^{-z-1}\chi(s) = \int_0^\infty ds \int_0^\infty dt e^{-st}s^{-z-1}\hat{\chi}(t) = \Gamma(-z) \int_0^\infty dt t^z \hat{\chi}(t), \quad (21)$$

which yields the coefficients  $\chi_z$  unless  $z \in \mathbb{N}$ . For  $z = k \in \mathbb{N}$  we have instead

$$\begin{aligned} \int_0^\infty dt t^k \hat{\chi}(t) &= \lim_{s \rightarrow 0} \int_0^\infty dt e^{-st} t^k \hat{\chi}(t) = \lim_{s \rightarrow 0} (-1)^k \frac{\partial^k}{\partial s^k} \int_0^\infty dt e^{-st} \hat{\chi}(t) \\ &= \lim_{s \rightarrow 0} (-1)^k \frac{\partial^k \chi}{\partial s^k}(s) = (-1)^k \chi^{(k)}(0). \end{aligned} \quad (22)$$

In summary,

$$\chi_z = \frac{1}{\Gamma(-z)} \int_0^\infty ds s^{-z-1} \chi(s) \quad \text{for } z \notin \mathbb{N}, \quad (23a)$$

$$\chi_k = (-1)^k \chi^{(k)}(0) \quad \text{for } k \in \mathbb{N}. \quad (23b)$$

In a position space basis, the Hilbert space trace is given by

$$\text{Tr}(e^{-t\mathcal{D}_A^2}) = \int_{\mathbb{R}^4} dx \text{tr}((e^{-t\mathcal{D}_A^2})(x, x)), \quad (24)$$

where  $\text{tr}$  denotes the matrix trace (including the Clifford algebra) and  $(e^{-t\mathcal{D}_A^2})(x, y)$  is the heat kernel. To obtain the heat kernel coefficients  $a_z(\mathcal{D}_A^2)$ , we write

$$\mathcal{D}_{A=0}^2 := H_0, \quad \mathcal{D}_A^2 := H_0 - V, \quad (25)$$

and consider the Duhamel expansion (see [44] for more information)

$$\begin{aligned} e^{-t_0(H_0-V)} &= e^{-t_0 H_0} - \int_0^{t_0} dt_1 \frac{d}{dt_1} (e^{-(t_0-t_1)(H_0-V)} e^{-t_1 H_0}) \\ &= e^{-t_0 H_0} + \int_0^{t_0} dt_1 (e^{-(t_0-t_1)(H_0-V)} V e^{-t_1 H_0}) \\ &= e^{-t_0 H_0} + \int_0^{t_0} dt_1 (e^{-(t_0-t_1)H_0} V e^{-t_1 H_0}) \\ &+ \int_0^{t_0} dt_1 \int_0^{t_0-t_1} dt_2 (e^{-(t_0-t_1-t_2)H_0} V e^{-t_2 H_0} V e^{-t_1 H_0}) + \dots \\ &+ \int_0^{t_0} dt_1 \dots \int_0^{t_0-t_1-\dots-t_{n-1}} dt_n (e^{-(t_0-t_1-\dots-t_n)H_0} (V e^{-t_n H_0}) \dots (V e^{-t_1 H_0})) + \dots \\ &= e^{-t_0 H_0} + \sum_{n=1}^{\infty} t_0^n \int_{\Delta^n} d^n \alpha \left( e^{-t_0(1-|\alpha|)H_0} \prod_{j=1}^n (V e^{-t_0 \alpha_j H_0}) \right), \end{aligned} \quad (26)$$

where the integration is performed over the standard  $n$ -simplex  $\Delta^n := \{\alpha := (\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n, \alpha_i \geq 0, |\alpha| := \alpha_1 + \dots + \alpha_n \leq 1\}$ .

## 4.2 Position space kernels

According to (14) we have  $H_0 = H_0 1_{32} + \Sigma 1_2$ . Its position space kernel is

$$\begin{aligned} (e^{-tH_0})(x, y) &= \int d^4 z (e^{-tH_0 1_{32}})(x, z) (e^{-t\Sigma 1_2})(z, y) = e^{-t\Sigma 1_2} (e^{-tH_0})(x, y) \\ &= \left( \frac{\tilde{\Omega}}{2\pi \sinh(2\tilde{\Omega}t)} \right)^2 e^{-t\Sigma 1_2 - \frac{\tilde{\Omega}}{4} (\coth(\tilde{\Omega}t)|x-y|^2 + \tanh(\tilde{\Omega}t)|x+y|^2)} , \end{aligned} \quad (27)$$

where the main part is given by the four-dimensional Mehler kernel (see e.g. [45]), with  $\tilde{\Omega} := \frac{2\Omega}{\theta}$  and  $|x|^2 := x_\mu x^\mu$ . It will be convenient to distinguish the following vertices in (14):

$$V_\phi = -L_\star(\phi \star \phi) 1_{16} , \quad (28a)$$

$$V_{D\phi} = -iL_\star(D_\mu \phi)(\Gamma^\mu + \Omega \Gamma^{\mu+4}) \Gamma_9 , \quad (28b)$$

$$V_A = -(1 + \Omega^2) L_\star(A_\mu \star A^\mu) , \quad (28c)$$

$$V_{DA} = -\{L_\star(A^\mu), i\partial_\mu + \Omega^2 M_\bullet(\tilde{x}_\mu)\} , \quad (28d)$$

$$V_{FA} = -iL_\star(F_{\mu\nu}^A) \left( \frac{1}{4} [\Gamma^\mu, \Gamma^\nu] + \frac{\Omega^2}{4} [\Gamma^{\mu+4}, \Gamma^{\nu+4}] + \frac{\Omega}{2} \Gamma^\mu \Gamma^{\nu+4} - \frac{\Omega}{2} \Gamma^\nu \Gamma^{\mu+4} \right) , \quad (28e)$$

and similarly for  $V_B$ ,  $V_{DB}$  and  $V_{FB}$ .

We compute the necessary position space kernels:

$$\begin{aligned} (L_\star(f)g)(x) &= \int d^4 y \left( \int \frac{d^4 k}{(2\pi)^4} f(x + \frac{1}{2}\Theta \cdot k) e^{i\langle k, y-x \rangle} \right) g(y) \\ &= \int d^4 y \left( \frac{1}{\pi^4 \theta^4} \int d^4 z f(z) e^{2i(\langle x, \Theta^{-1}y \rangle + \langle y, \Theta^{-1}z \rangle + \langle z, \Theta^{-1}x \rangle)} \right) g(y) , \end{aligned} \quad (29)$$

from which we get

$$(L_\star(f))(x, y) = \frac{1}{\pi^4 \theta^4} \int d^4 z f(z) e^{i\langle x-y, \Theta^{-1}(x+y) \rangle + 2i\langle z, \Theta^{-1}(x-y) \rangle} . \quad (30)$$

Next, we compute

$$\begin{aligned} &(\{L_\star(A^\mu), i\partial_\mu + \Omega^2 M_\bullet(\tilde{x}_\mu)\}g)(x) \\ &= \int d^4 y \left( \frac{1}{\pi^4 \theta^4} \int d^4 z A^\mu(z) e^{2i(\langle x, \Theta^{-1}y \rangle + \langle y, \Theta^{-1}z \rangle + \langle z, \Theta^{-1}x \rangle)} \right) \left( i \frac{\partial g}{\partial y^\mu}(y) + \Omega^2 \tilde{y}_\mu g(y) \right) \\ &+ \left( i \frac{\partial}{\partial x^\mu} + \Omega^2 \tilde{x}_\mu \right) \left( \int d^4 y \left( \frac{1}{\pi^4 \theta^4} \int d^4 z A^\mu(z) e^{2i(\langle x, \Theta^{-1}y \rangle + \langle y, \Theta^{-1}z \rangle + \langle z, \Theta^{-1}x \rangle)} \right) g(y) \right) \\ &= \int d^4 y \left( \frac{1}{\pi^4 \theta^4} \int d^4 z (2\tilde{z}^\mu - (1-\Omega^2)(\tilde{x}^\mu + \tilde{y}^\mu)) A_\mu(z) e^{2i(\langle x, \Theta^{-1}y \rangle + \langle y, \Theta^{-1}z \rangle + \langle z, \Theta^{-1}x \rangle)} \right) g(y) . \end{aligned} \quad (31)$$

Therefore, the position space kernel of a Moyal-derivative vertex is

$$\begin{aligned} &\{L_\star(A^\mu), i\partial_\mu + \Omega^2 M_\bullet(\tilde{x}_\mu)\}(x, y) \\ &= \frac{1}{\pi^4 \theta^4} \int d^4 z (2\tilde{z}^\mu - (1-\Omega^2)(\tilde{x}^\mu + \tilde{y}^\mu)) A_\mu(z) e^{2i(\langle x, \Theta^{-1}y \rangle + \langle y, \Theta^{-1}z \rangle + \langle z, \Theta^{-1}x \rangle)} . \end{aligned} \quad (32)$$

### 4.3 Computation of the traces

The first term in the expansion (26), which corresponds to vacuum graphs, has the heat kernel expansion

$$\begin{aligned}\mathrm{Tr}(e^{-tH_0}) &= \mathrm{tr} \int d^4x (e^{-tH_0})(x, x) = \left( \frac{\tilde{\Omega}}{2\pi \sinh(2\tilde{\Omega}t)} \right)^2 2 \mathrm{tr} \int d^4x e^{-t\Sigma - \tilde{\Omega} \tanh(\tilde{\Omega}t)|x|^2} \\ &= \frac{1}{8 \sinh^4(\tilde{\Omega}t)} \mathrm{tr}(e^{-t\Sigma}) .\end{aligned}\quad (33)$$

We need the traces of the lowest powers of  $\Sigma$ :

$$\mathrm{tr}(\Sigma^0) = 16 , \quad \mathrm{tr}(\Sigma^2) = 16 \cdot \frac{16\Omega^2}{\theta^2} , \quad \mathrm{tr}(\Sigma^4) = 16 \cdot \frac{640\Omega^4}{\theta^4} .\quad (34)$$

All odd powers of  $\Sigma$  are traceless. Therefore,

$$\mathrm{Tr}(e^{-tH_0}) = \frac{\theta^4}{8\Omega^4 t^4} + \frac{2\theta^2}{3\Omega^2 t^2} + \frac{52}{45} + \mathcal{O}(t^2) .\quad (35)$$

This reconfirms that the noncommutative space under consideration is of dimension 8.

In the appendix we compute the first and second order  $x$ - $y$  integrals

$$\begin{aligned}&\int d^4x d^4y (e^{-tH_0})(y, x) V(x, y) , \\ &\int d^4x_1 d^4y_1 d^4x_2 d^4y_2 (e^{-(t-t_2)H_0})(y_2, x_1) V(x_1, y_1) (e^{-t_2H_0})(y_1, x_2) V'(x_2, y_2) ,\end{aligned}\quad (36)$$

where  $V, V'$  stand for combinations of the Moyal and Moyal-derivative vertices. In second order, we also perform a Taylor expansion about coinciding external positions. It is remarkable that only terms of order  $t^{-1}$  and regular terms in  $t$  appear, just as in 4D-Yang-Mills theory. Only the vacuum graphs behave like a 8D-model, for proper graphs only partial 4D-traces appear.

In the following, we only consider the trace of the 16-dimensional upper left corner containing the  $A$ -field and the structure  $\phi \star \bar{\phi}$ . At the very end we add the lower right corner where  $A$  is replaced by  $B$  and  $\phi \leftrightarrow \bar{\phi}$ .

With one  $V_A$  or  $V_{DA}$  vertex we see from (49) and (50) that the leading divergence after  $t_1$ -integration is  $\sim t^{-1}$ . Therefore, the  $\Sigma$  matrix gives no contribution up to order  $t^0$ , so that the leading terms are

$$\begin{aligned}S_{(A+DA)}(t) &:= \mathrm{Tr} \left( \int_0^t dt_1 (e^{-(t_0-t_1)(H_0)} (V_A + V_{DA}) e^{-t_1 H_0}) \right) \\ &= \frac{1}{\pi^2 (1 + \Omega^2)^2} \int d^4z \left\{ - \frac{4\Omega^2}{(1 + \Omega^2)} t^{-1} \tilde{z}^\mu A_\mu(z) + \frac{4\Omega^4}{(1 + \Omega^2)^2} \tilde{z}^\mu A_\mu(z) \tilde{z}^\nu \tilde{z}_\nu \right. \\ &\quad \left. - (1 + \Omega^2) t^{-1} (A_\mu \star A^\mu)(z) + \Omega^2 (A_\mu \star A^\mu)(z) \tilde{z}^\nu \tilde{z}_\nu \right\} + \mathcal{O}(t) .\end{aligned}\quad (37)$$

A single  $V_{FA}$ -vertex gets a non-vanishing trace of order  $t^0$  together with one  $\Sigma$ -matrix, but the resulting integral  $\int dz F_{\mu\nu}(z)$  vanishes. With two  $V_{FA}$ -vertices and the trace

$$\begin{aligned} & \text{tr} \left( i \left( \frac{1}{4} [\Gamma^\mu, \Gamma^\nu] + \frac{1}{4} \Omega^2 [\Gamma^{\mu+4}, \Gamma^{\nu+4}] + \frac{1}{2} \Omega \Gamma^\mu \Gamma^{\nu+4} - \frac{1}{2} \Omega \Gamma^\nu \Gamma^{\mu+4} \right) \right. \\ & \quad \times \left. i \left( \frac{1}{4} [\Gamma^\rho, \Gamma^\sigma] + \frac{1}{4} \Omega^2 [\Gamma^{\rho+4}, \Gamma^{\sigma+4}] + \frac{1}{2} \Omega \Gamma^\rho \Gamma^{\sigma+4} - \frac{1}{2} \Omega \Gamma^\sigma \Gamma^{\rho+4} \right) \right) \\ & = 4(1 + \Omega^2)^2 (\delta_{\mu\rho} \delta_{\nu\sigma} - \delta_{\mu\sigma} \delta_{\nu\rho}) \end{aligned} \quad (38)$$

we find with (61)

$$\begin{aligned} S_{(FA)^2}(t) & := \text{Tr} \left( \int_0^t dt_1 \int_0^{t-t_1} dt_2 e^{-(t_0-t_1-t_2)H_0} V_{FA} e^{-t_2 H_0} V_{FA} e^{-t_1 H_0} \right) \\ & = \frac{1}{4\pi^2} t^0 \int d^4 z F_{\mu\nu}^A(z) F_A^{\mu\nu}(z) + \mathcal{O}(t) . \end{aligned} \quad (39)$$

For two  $V_{DA}$ -vertices we obtain from (65) after some integrations by parts

$$\begin{aligned} S_{(DA)^2}(t) & := \text{Tr} \left( \int_0^t dt_1 \int_0^{t-t_1} dt_2 e^{-(t_0-t_1-t_2)(H_0)} V_{DA} e^{-t_2 H_0} V_{DA} e^{-t_1 H_0} \right) \\ & = 16 \int_0^t dt_1 \int_0^{t-t_1} dt_2 \frac{1}{(4\pi t)^2 (1 + \Omega^2)^4} \int d^4 z \\ & \quad \times \left( \frac{2(1 - \Omega^2)^2 (1 + \Omega^2)}{t} A_\mu(z) A^\mu(z) - 2\Omega^2 (1 - \Omega^2)^2 A_\mu(z) A^\mu(z) |\tilde{z}|^2 \right. \\ & \quad \left. + A^\mu(z) (\partial^\nu \partial_\nu A_\mu)(z) \left( 2(1 - \Omega^2)^4 \frac{t_2(t - t_2)}{t^2} + 2\Omega^2 (1 - \Omega^2)^2 \right) \right. \\ & \quad \left. + 16\Omega^4 \tilde{z}^\mu A_\mu(z) \tilde{z}^\nu A_\nu(z) + (1 - \Omega^2)^4 \frac{t^2 - 4t_2 t + 4t_2^2}{t^2} (\partial_\nu A_\mu)(z) (\partial^\mu A^\nu)(z) \right) \\ & = \frac{1}{\pi^2 (1 + \Omega^2)^2} \int d^4 z \left( \frac{(1 - \Omega^2)^2}{1 + \Omega^2} t^{-1} A_\mu \star A^\mu \right. \\ & \quad \left. - \frac{(1 - \Omega^2)^4}{6(1 + \Omega^2)^2} ((\partial^\nu A^\mu) \star (\partial_\nu A_\mu) - (\partial^\nu A^\mu) \star (\partial_\mu A_\nu)) \right. \\ & \quad \left. - \frac{\Omega^2 (1 - \Omega^2)^2}{(1 + \Omega^2)^2} A_\mu \star A^\mu |\tilde{z}|^2 + \frac{8\Omega^4}{(1 + \Omega^2)^2} \left( (\tilde{z} \cdot A) \star (\tilde{z} \cdot A) \right) (z) \right) . \end{aligned} \quad (40)$$

We have used

$$A^\mu(z) \tilde{z}_\nu \tilde{z}^\nu = A^\mu(z) \star (\tilde{z}_\nu \tilde{z}^\nu) + i(\partial_\nu A^\mu)(z) \tilde{z}^\nu + (\partial_\nu \partial^\nu A^\mu)(z) \quad (41)$$

as well as  $\int d^4 z A_\mu(z) (\partial_\nu A^\mu)(z) \tilde{z}^\nu = 0$ .

The  $A$ -linear and  $A$ -bilinear part of the spectral action are given by the sum  $S_{(A+DA)} + S_{(FA)^2} + S_{(DA)^2}$ . As the spectral action is manifestly gauge invariant, we simply complete the  $A$ -trilinear and  $A$ -quadrilinear terms in a gauge-invariant way. Introducing covariant coordinates

$$\tilde{X}_A^\mu(z) := \frac{\tilde{z}^\mu}{2} + A^\mu(z) , \quad (42)$$

with  $\tilde{X}_0^\mu(z) = \frac{\tilde{z}^\mu}{2}$ , we obtain the pure  $A$ -part of the spectral action to

$$\begin{aligned}
S_A(t) &= \frac{1}{\pi^2(1+\Omega^2)^2} \int d^4z \left\{ -\frac{4\Omega^2}{1+\Omega^2} t^{-1} (\tilde{X}_A^\mu \star \tilde{X}_{A\mu} - \tilde{X}_0^\mu \star \tilde{X}_{0\mu}) \right. \\
&\quad + \frac{t^0}{2} \left( \frac{4\Omega^2}{1+\Omega^2} \right)^2 (\tilde{X}_A^\mu \star \tilde{X}_{A\mu} \star \tilde{X}_A^\nu \star \tilde{X}_{A\nu} - \tilde{X}_0^\mu \star \tilde{X}_{0\mu} \star \tilde{X}_0^\nu \star \tilde{X}_{0\nu}) \\
&\quad \left. + \left( \frac{(1+\Omega^2)^2}{4} - \frac{(1-\Omega^2)^2}{12(1+\Omega^2)^2} \right) t^0 F_{\mu\nu}^A \star F_A^{\mu\nu} \right\} (z) + \mathcal{O}(t). \tag{43}
\end{aligned}$$

The scalar field potential becomes

$$\begin{aligned}
S_{(\phi+\phi^2)}(t) &= \text{Tr} \left( \int_0^t dt_1 (e^{-(t_0-t_1)H_0} V_\phi e^{-t_1 H_0}) \right) \\
&\quad + \text{Tr} \left( \int_0^t dt_1 \int_0^{t-t_1} dt_2 e^{-(t_0-t_1-t_2)H_0} V_\phi e^{-t_2 H_0} V_\phi e^{-t_1 H_0} \right) \\
&= \frac{1}{\pi^2(1+\Omega^2)^2} \int d^4z \left( -t^{-1} \phi \star \bar{\phi} + \frac{\Omega^2 |\tilde{z}|^2}{1+\Omega^2} \phi \star \bar{\phi} + \frac{1}{2} \phi \star \bar{\phi} \star \phi \star \bar{\phi} \right) (z). \tag{44}
\end{aligned}$$

The usual kinetic term of the scalar field comes from two  $D\phi$ -vertices:

$$\begin{aligned}
S_{(D\phi)^2}(t) &= \text{Tr} \left( \int_0^t dt_1 \int_0^{t-t_1} dt_2 e^{-(t_0-t_1-t_2)H_0} V_{D\phi} e^{-t_2 H_0} V_{\overline{D\phi}} e^{-t_1 H_0} \right) \\
&= \frac{1}{2\pi^2(1+\Omega^2)} t^0 \int d^4z (D_\mu \phi \star \overline{D_\mu \phi}) (z) + \mathcal{O}(t). \tag{45}
\end{aligned}$$

It remains the combination of  $V_\phi$  with  $V_A$  and  $V_{DA}$ , namely  $V_\phi V_A$ ,  $V_A V_\phi$  as well as  $V_{DA} V_\phi$ ,  $V_\phi V_{DA}$  and  $V_\phi V_{DA} V_{DA}$ ,  $V_{DA} V_\phi V_{DA}$ ,  $V_{DA} V_{DA} V_\phi$ . At first order in  $A$  we get from (67)

$$\begin{aligned}
S_{(DA\phi+\phi DA)}(t) &= \text{Tr} \left( \int_0^t dt_1 \int_0^{t-t_1} dt_2 (e^{-(t_0-t_1-t_2)H_0} (V_\phi e^{-t_2 H_0} V_A e^{-t_1 H_0} + V_A e^{-t_2 H_0} V_\phi e^{-t_1 H_0})) \right) \\
&= \frac{4\Omega^2}{\pi^2(1+\Omega^2)^3} \int d^4z (\phi \star \bar{\phi} \star (\tilde{z}^\mu A_\mu)) (z) + \mathcal{O}(t). \tag{46}
\end{aligned}$$

Completing the the  $AA\phi\bar{\phi}$ -term by gauge invariance, the scalar field part of the spectral action becomes

$$\begin{aligned}
S_\phi(t) &= \frac{1}{\pi^2(1+\Omega^2)^2} \int d^4z \left( -t^{-1} \phi \star \bar{\phi} + \frac{1}{2} D_\mu \phi \star \overline{D_\mu \phi} \right. \\
&\quad \left. + \frac{1}{2} \phi \star \bar{\phi} \star \left( \phi \star \bar{\phi} + 2 \frac{4\Omega^2}{1+\Omega^2} \tilde{X}_A^\mu \star \tilde{X}_{A\mu} \right) \right) (z). \tag{47}
\end{aligned}$$

To obtain the spectral action, we convert the Laplace-transform variable  $t^n$  into  $\chi_n$  and add the lower  $B$ -corner. The result (including the vacuum contribution is

$$S = \frac{\theta^4 \chi_{-4}}{8\Omega^4} + \frac{2\theta^2 \chi_{-2}}{3\Omega^2} + \frac{52\chi_0}{45}$$

$$\begin{aligned}
& + \frac{\chi_0}{2\pi^2(1+\Omega^2)^2} \int d^4 z \left\{ \left( \frac{(1+\Omega^2)^2}{2} - \frac{(1-\Omega^2)^4}{6(1+\Omega^2)^2} \right) (F_{\mu\nu}^A \star F_A^{\mu\nu} + F_{\mu\nu}^B \star F_B^{\mu\nu}) \right. \\
& + \left( \phi \star \bar{\phi} + \frac{4\Omega^2}{1+\Omega^2} \tilde{X}_A^\mu \star \tilde{X}_{A\mu} - \frac{\chi_{-1}}{\chi_0} \right)^2 + \left( \bar{\phi} \star \phi + \frac{4\Omega^2}{1+\Omega^2} \tilde{X}_B^\mu \star \tilde{X}_{B\mu} - \frac{\chi_{-1}}{\chi_0} \right)^2 \\
& \left. - 2 \left( \frac{4\Omega^2}{1+\Omega^2} \tilde{X}_0^\mu \star \tilde{X}_{0\mu} - \frac{\chi_{-1}}{\chi_0} \right)^2 + 2(1+\Omega^2) D_\mu \phi \star \overline{D_\mu \phi} \right\} (z) + \mathcal{O}(\chi_1). \tag{48}
\end{aligned}$$

The most important conclusion is that the squared covariant derivatives combine with the Higgs field to a non-trivial potential. This was not noticed in [35, 36].

## A Appendix: Moyal integrals

### A.1 One Moyal vertex

We compute a generic trace term with a change of variables  $u = x - y$ ,  $v = x + y$  with Jacobian  $\frac{1}{16}$  (see [6]):

$$\begin{aligned}
V_1(f) & := \int d^4 x d^4 y (e^{-tH_0})(y, x) (L_\star(f))(x, y) \\
& = \frac{\tilde{\Omega}^2}{4\pi^2 \sinh^2(2\tilde{\Omega}t)} \frac{1}{(2\pi\theta)^4} \int d^4 z f(z) \int d^4 u d^4 v e^{-\frac{\tilde{\Omega}}{4} \left( \frac{|u|^2}{\tanh(\tilde{\Omega}t)} + \frac{|v|^2}{\coth(\tilde{\Omega}t)} \right) + i\langle u, \Theta^{-1}(v-2z) \rangle} \\
& = \frac{1}{\cosh^4(\tilde{\Omega}t)} \frac{1}{(2\pi\theta)^4} \int d^4 z f(z) \int d^4 v e^{-\frac{\tilde{\Omega} \tanh(\tilde{\Omega}t)}{4} (|v|^2 + \frac{4}{\tilde{\Omega}^2 \theta^2} |v-2z|^2)} \\
& = \frac{1}{\cosh^4(\tilde{\Omega}t)} \frac{1}{(2\pi\theta)^4} \int d^4 z f(z) \int d^4 v e^{-\frac{\tanh(\tilde{\Omega}t)}{2\theta\tilde{\Omega}} ((1+\Omega^2)|v - \frac{2}{1+\Omega^2}z|^2 + \frac{4\Omega^2}{1+\Omega^2}|z|^2)} \\
& = \frac{\tilde{\Omega}^2}{4\pi^2(1+\Omega^2)^2 \sinh^2(2\tilde{\Omega}t)} \int d^4 z f(z) e^{-\frac{\tilde{\Omega} \tanh(\tilde{\Omega}t)}{1+\Omega^2} |z|^2}. \tag{49}
\end{aligned}$$

### A.2 One Moyal+derivative vertex

After a change of variables  $u = x - y$ ,  $v = x + y$  with Jacobian  $\frac{1}{16}$ , we have

$$\begin{aligned}
V_1(A) & := \int d^4 x d^4 y (e^{-tH_0})(y, x) \{ L_\star(A^\mu), i\partial_\mu + \Omega^2 M_\bullet(\tilde{x}_\mu) \} (x, y) \\
& = \frac{\tilde{\Omega}^2}{4\pi^2 \sinh^2(2\tilde{\Omega}t)} \frac{1}{(2\pi\theta)^4} \int d^4 u d^4 v d^4 z A_\mu(z) (2\tilde{z}^\mu - (1-\Omega^2)\tilde{v}^\mu) \\
& \times e^{-\frac{\tilde{\Omega}}{4} \left( \frac{|u|^2}{\tanh(\tilde{\Omega}t)} + \frac{|v|^2}{\coth(\tilde{\Omega}t)} \right) + i\langle u, \Theta^{-1}(v-2z) \rangle} \\
& = \frac{\tilde{\Omega}^2}{4\pi^2 \sinh^2(2\tilde{\Omega}t)} \frac{1}{(2\pi\theta)^4} \int d^4 u \left( \int d^4 v d^4 z A_\mu(z) \left( 2\tilde{z}^\mu + 2i(1-\Omega^2) \frac{\partial}{\partial w_\mu} \right) \right. \\
& \left. \times e^{-\frac{\tilde{\Omega}}{4} \left( \frac{|u|^2}{\tanh(\tilde{\Omega}t)} + \frac{|v|^2}{\coth(\tilde{\Omega}t)} \right) + i\langle w, \Theta^{-1}v \rangle - 2i\langle u, \Theta^{-1}z \rangle} \Big|_{w=u} \right)
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{\sinh^4(\tilde{\Omega}t)} \frac{1}{(2\pi\theta)^4} \int d^4u \left( \int d^4z A_\mu(z) \left( 2\tilde{z}^\mu - \frac{2i(1-\Omega^2)}{\Omega\theta \tanh(\tilde{\Omega}t)} w^\mu \right) \right. \\
&\quad \left. \times e^{-\frac{\Omega|u|^2}{2\theta \tanh(\tilde{\Omega}t)} - \frac{|w|^2}{2\Omega\theta \tanh(\tilde{\Omega}t)} - 2i\langle u, \Theta^{-1}z \rangle} \Big|_{w=u} \right) \\
&= \frac{1}{(2\pi\theta)^4} \int d^4u d^4z A_\mu(z) \left( 2\tilde{z}^\mu - \frac{(1-\Omega^2)}{\Omega\theta \tanh(\tilde{\Omega}t)} \Theta^{\mu\nu} \frac{\partial}{\partial z^\nu} \right) \frac{e^{-\frac{(1+\Omega^2)|u|^2}{2\Omega\theta \tanh(\tilde{\Omega}t)} - 2i\langle u, \Theta^{-1}z \rangle}}{\sinh^4(\tilde{\Omega}t)} \\
&= \frac{\Omega^2}{(\pi\theta)^2(1+\Omega^2)^2} \int d^4z A_\mu(z) \left( 2\tilde{z}^\mu - \frac{(1-\Omega^2)}{\Omega\theta \tanh(\tilde{\Omega}t)} \Theta^{\mu\nu} \frac{\partial}{\partial z^\nu} \right) \frac{e^{-\frac{\tilde{\Omega} \tanh(\tilde{\Omega}t)}{(1+\Omega^2)}|z|^2}}{\sinh^2(2\tilde{\Omega}t)} \\
&= \frac{4\Omega^4}{(\pi\theta)^2(1+\Omega^2)^3 \sinh^2(2\tilde{\Omega}t)} \int d^4z \tilde{z}^\mu A_\mu(z) e^{-\frac{\tilde{\Omega} \tanh(\tilde{\Omega}t)}{1+\Omega^2}|z|^2}. \tag{50}
\end{aligned}$$

This term gives the complete  $A$ -linear part. It vanishes for  $\Omega = 0$ , as expected.

### A.3 Two Moyal vertices

To simplify the notations in this case we let  $\tau_1 := \tanh(\tilde{\Omega}(t-t_2))$  and  $\tau_2 := \tanh(\tilde{\Omega}t_2)$ . The change of variables  $u_i = x_i - y_i$  and  $v_i = x_i + y_i$  for  $i = 1, 2$  leads to

$$\begin{aligned}
V_2(f, g) &:= \int d^4x_1 d^4y_1 d^4x_2 d^4y_2 (e^{-(t-t_2)H_0})(y_2, x_1)(L_\star(f))(x_1, y_1) \\
&\quad \times (e^{-t_2H_0})(y_1, x_2)(L_\star(g))(x_2, y_2) \\
&= \left( \frac{\tilde{\Omega}^2(1-\tau_1^2)(1-\tau_2^2)}{16\pi^2\tau_1\tau_2} \right)^2 \frac{1}{(2\pi\theta)^8} \int d^4u_1 d^4v_1 d^4u_2 d^4v_2 d^4z_1 d^4z_2 f(z_1)g(z_2) \\
&\quad \times e^{-\frac{\tilde{\Omega}}{16\tau_1}|u_1+v_1+u_2-v_2|^2 - \frac{\tilde{\Omega}\tau_1}{16}|u_1+v_1-u_2+v_2|^2 - \frac{\tilde{\Omega}}{16\tau_2}|u_1-v_1+u_2+v_2|^2 - \frac{\tilde{\Omega}\tau_2}{16}|-u_1+v_1+u_2+v_2|^2} \\
&\quad \times e^{i\langle u_1, \Theta^{-1}v_1 \rangle - 2i\langle u_1, \Theta^{-1}z_1 \rangle + i\langle u_2, \Theta^{-1}v_2 \rangle - 2i\langle u_2, \Theta^{-1}z_2 \rangle}. \tag{51}
\end{aligned}$$

Defining

$$\begin{aligned}
C &:= \begin{pmatrix} 1+\tau_1\tau_2 & 1-\tau_1\tau_2 & -\frac{\tau_1-\tau_2}{\tau_1+\tau_2}(1-\tau_1\tau_2) & \frac{\tau_1-\tau_2}{\tau_1+\tau_2}(1+\tau_1\tau_2) \\ 1-\tau_1\tau_2 & 1+\tau_1\tau_2 & -\frac{\tau_1-\tau_2}{\tau_1+\tau_2}(1+\tau_1\tau_2) & \frac{\tau_1-\tau_2}{\tau_1+\tau_2}(1-\tau_1\tau_2) \\ -\frac{\tau_1-\tau_2}{\tau_1+\tau_2}(1-\tau_1\tau_2) & -\frac{\tau_1-\tau_2}{\tau_1+\tau_2}(1+\tau_1\tau_2) & 1+\tau_1\tau_2 & -(1-\tau_1\tau_2) \\ \frac{\tau_1-\tau_2}{\tau_1+\tau_2}(1+\tau_1\tau_2) & \frac{\tau_1-\tau_2}{\tau_1+\tau_2}(1-\tau_1\tau_2) & -(1-\tau_1\tau_2) & 1+\tau_1\tau_2 \end{pmatrix} \\
G &:= \begin{pmatrix} 0 & 0 & -\frac{4\tau_1\tau_2}{\Omega(\tau_1+\tau_2)} & 0 \\ 0 & 0 & 0 & -\frac{4\tau_1\tau_2}{\Omega(\tau_1+\tau_2)} \\ \frac{4\tau_1\tau_2}{\Omega(\tau_1+\tau_2)} & 0 & 0 & 0 \\ 0 & \frac{4\tau_1\tau_2}{\Omega(\tau_1+\tau_2)} & 0 & 0 \end{pmatrix}, \quad X := \begin{pmatrix} u_1 \\ u_2 \\ v_1 \\ v_2 \end{pmatrix}, \quad Z := \begin{pmatrix} z_1 \\ z_2 \\ 0 \\ 0 \end{pmatrix}, \tag{52}
\end{aligned}$$

and  $Q := C \otimes 1_4 + G \otimes \sigma$ , we obtain

$$\begin{aligned}
V_2(f, g) &= \left( \frac{\tilde{\Omega}^2(1-\tau_1^2)(1-\tau_2^2)}{16\pi^2\tau_1\tau_2} \right)^2 \frac{1}{(2\pi\theta)^8} \int d^{16}X d^8Z f(z_1)g(z_2) \\
&\quad \times e^{-\frac{\Omega(\tau_1+\tau_2)}{8\theta\tau_1\tau_2} X^t Q X - \frac{2}{\theta} X^t \sigma Z}
\end{aligned}$$



$$\begin{aligned}
&= \left( \frac{\tilde{\Omega}^2(1-\tau_1^2)(1-\tau_2^2)}{16\pi^2\tau_1\tau_2} \right)^2 \left( \frac{4\tau_1\tau_2}{\Omega(\tau_1+\tau_2)} \right)^8 \det(C \otimes 1_4 + G \otimes \sigma)^{-\frac{1}{2}} \\
&\times \int d^8 Z f(z_1)g(z_2) e^{-\frac{8\tau_1\tau_2}{\Omega\theta(\tau_1+\tau_2)} Z^t \sigma (C \otimes 1_4 + G \otimes \sigma)^{-1} \sigma Z}.
\end{aligned} \tag{53}$$

In [7] it was proven that

$$\det(Q) = \det(G + C)^4, \tag{54}$$

$$Q^{-1} = \frac{1}{2}((G + C)^{-1} + ((G + C)^{-1})^t) \otimes 1_4 + \frac{1}{2}((G + C)^{-1} - ((G + C)^{-1})^t) \otimes \sigma. \tag{55}$$

We find

$$\det(G + C) = \left( \frac{16(1 + \Omega^2)\tau_1^2\tau_2^2}{\Omega^2(\tau_1 + \tau_2)^2} \right)^2 \left( 1 + \frac{\Omega^2(\tau_1 - \tau_2)^2}{(1 + \Omega^2)^2\tau_1\tau_2} \right), \tag{56}$$

which suggests to introduce

$$T := 1 + \frac{\Omega^2(\tau_1 - \tau_2)^2}{(1 + \Omega^2)^2\tau_1\tau_2}, \tag{57}$$

and further

$$\begin{aligned}
&\frac{1}{2}((G + C)^{-1} + ((G + C)^{-1})^t) = \frac{\Omega^2(\tau_1 + \tau_2)^2}{16\tau_1^2\tau_2^2(1 + \Omega^2)T} \\
&\times \begin{pmatrix} 1 + \tau_1\tau_2 & -(1 - \tau_1\tau_2) & \frac{1 - \Omega^2}{1 + \Omega^2} \frac{\tau_1 - \tau_2}{\tau_1 + \tau_2} (1 - \tau_1\tau_2) & \frac{1 - \Omega^2}{1 + \Omega^2} \frac{\tau_1 - \tau_2}{\tau_1 + \tau_2} (1 + \tau_1\tau_2) \\ -(1 - \tau_1\tau_2) & 1 + \tau_1\tau_2 & -\frac{1 - \Omega^2}{1 + \Omega^2} \frac{\tau_1 - \tau_2}{\tau_1 + \tau_2} (1 + \tau_1\tau_2) & -\frac{1 - \Omega^2}{1 + \Omega^2} \frac{\tau_1 - \tau_2}{\tau_1 + \tau_2} (1 - \tau_1\tau_2) \\ \frac{1 - \Omega^2}{1 + \Omega^2} \frac{\tau_1 - \tau_2}{\tau_1 + \tau_2} (1 - \tau_1\tau_2) & -\frac{1 - \Omega^2}{1 + \Omega^2} \frac{\tau_1 - \tau_2}{\tau_1 + \tau_2} (1 + \tau_1\tau_2) & 1 + \tau_1\tau_2 & 1 - \tau_1\tau_2 \\ \frac{1 - \Omega^2}{1 + \Omega^2} \frac{\tau_1 - \tau_2}{\tau_1 + \tau_2} (1 + \tau_1\tau_2) & -\frac{1 - \Omega^2}{1 + \Omega^2} \frac{\tau_1 - \tau_2}{\tau_1 + \tau_2} (1 - \tau_1\tau_2) & 1 - \tau_1\tau_2 & 1 + \tau_1\tau_2 \end{pmatrix}
\end{aligned} \tag{58}$$

as well as

$$\begin{aligned}
&\frac{1}{2}((G + C)^{-1} - ((G + C)^{-1})^t) = \frac{\Omega^2(\tau_1 + \tau_2)^2}{8\tau_1^2\tau_2^2(1 + \Omega^2)^2T} \\
&\times \begin{pmatrix} 0 & \Omega(\tau_1 - \tau_2) & \frac{2(1 + \Omega^2)\tau_1\tau_2 + \Omega^2(\tau_1 - \tau_2)^2}{\Omega(\tau_1 + \tau_2)} & 0 \\ -\Omega(\tau_1 - \tau_2) & 0 & 0 & \frac{2(1 + \Omega^2)\tau_1\tau_2 + \Omega^2(\tau_1 - \tau_2)^2}{\Omega(\tau_1 + \tau_2)} \\ -\frac{2(1 + \Omega^2)\tau_1\tau_2 + \Omega^2(\tau_1 - \tau_2)^2}{\Omega(\tau_1 + \tau_2)} & 0 & 0 & \Omega(\tau_1 - \tau_2) \\ 0 & -\frac{2(1 + \Omega^2)\tau_1\tau_2 + \Omega^2(\tau_1 - \tau_2)^2}{\Omega(\tau_1 + \tau_2)} & -\Omega(\tau_1 - \tau_2) & 0 \end{pmatrix}.
\end{aligned} \tag{59}$$

We thus conclude

$$\begin{aligned}
V_2(f, g) &= \left( \frac{\tilde{\Omega}^2(1 - \tau_1^2)(1 - \tau_2^2)}{16(1 + \Omega^2)^2\pi^2 T \tau_1\tau_2} \right)^2 \int d^4 z_1 d^4 z_2 f(z_1)g(z_2) \\
&\times e^{-\frac{\Omega(\tau_1 + \tau_2)}{2\theta\tau_1\tau_2(1 + \Omega^2)T} (|z_1 - z_2|^2 + \tau_1\tau_2|z_1 + z_2|^2) - \frac{2\Omega^2(\tau_1^2 - \tau_2^2)}{\theta\tau_1\tau_2(1 + \Omega^2)^2 T} z_1 \sigma z_2}.
\end{aligned} \tag{60}$$

For  $\tau_1, \tau_2 \rightarrow 0$  the integrand is regular unless  $z_1 = z_2$ . To capture the singularity at  $z_1 = z_2$ , we expand  $g(z_2) = g(z_1) + (z_2 - z_1) \int d\xi (\partial_\mu g)(z_1 + \xi(z_2 - z_1))$  and consider the leading term  $g(z_1)$ . After a shift  $z_2 \mapsto z_2 + z_1$  we have

$$V_2(f, g)^0 = \left( \frac{\tilde{\Omega}(1 - \tau_1^2)(1 - \tau_2^2)}{4\pi(1 + \Omega^2)(\tau_1 + \tau_2)(1 + \tau_1\tau_2)} \right)^2 \int d^4 z_1 f(z_1) g(z_1) e^{-\frac{\tilde{\Omega}(\tau_1 + \tau_2)}{(1 + \tau_1\tau_2)(1 + \Omega^2)} |z_1|^2}. \quad (61)$$

It can be shown that  $(z_2 - z_1) \int d\xi (\partial_\mu g)(z_1 + \xi(z_2 - z_1))$  is subleading.

#### A.4 Two Moyal-derivative vertices

To complete the  $A$ -bilinear part, we also need the contribution with two vertices of Moyal+derivative type. We use as far as possible the same notation as in the previous calculation. Defining the auxiliary vector  $W = (0, 0, w_3, w_4)^t$ , this gives

$$\begin{aligned} V_2(A, A) &:= \int d^4 x_1 d^4 y_1 d^4 x_2 d^4 y_2 (e^{-(t-t_2)H_0})(y_2, x_1) \{L_\star(A^\mu), i\partial_\mu + \Omega^2 M_\bullet(\tilde{x}_\mu)\}(x_1, y_1) \\ &\times (e^{-t_2 H_0})(y_1, x_2) \{L_\star(A^\mu), i\partial_\mu + \Omega^2 M_\bullet(\tilde{x}_\mu)\}(x_2, y_2) \\ &= \left( \frac{\tilde{\Omega}^2(1 - \tau_1^2)(1 - \tau_2^2)}{16\pi^2\tau_1\tau_2} \right)^2 \frac{1}{(2\pi\theta)^8} \int d^{16} X d^8 Z A_\mu(z_1) A_\nu(z_2) \\ &\times \left( 2\tilde{z}_1^\mu - (1 - \Omega^2)(\Theta^{-1})^{\mu\rho} \frac{\partial}{\partial w_3^\rho} \right) \left( 2\tilde{z}_2^\nu - (1 - \Omega^2)(\Theta^{-1})^{\nu\sigma} \frac{\partial}{\partial w_4^\sigma} \right) \\ &\times e^{-\frac{\Omega(\tau_1 + \tau_2)}{8\theta\tau_1\tau_2} X^t Q X - \frac{2}{\theta} X^t \sigma Z + 2X^t W} \Big|_{W=0} \\ &= \left( \frac{\tilde{\Omega}^2(1 - \tau_1^2)(1 - \tau_2^2)}{16\pi^2\tau_1\tau_2} \right)^2 \left( \frac{4\tau_1\tau_2}{\Omega(\tau_1 + \tau_2)} \right)^8 (\det Q)^{-1/2} \int d^8 Z A_\mu(z_1) A_\nu(z_2) \\ &\times \left( \frac{8\theta\tau_1\tau_2}{\Omega(\tau_1 + \tau_2)} (1 - \Omega^2)^2 (\Theta^{-1})^{\mu\rho} (\Theta^{-1})^{\nu\sigma} ((Q^{-1})_{\sigma\rho}^{43} + (Q^{-1})_{\rho\sigma}^{34}) \right. \\ &+ \left( 2\tilde{z}_1^\mu + \frac{8i\theta\tau_1\tau_2(1 - \Omega^2)}{\Omega(\tau_1 + \tau_2)} (\Theta^{-1})^{\mu\rho} (Q^{-1}\tilde{Z})_\rho^3 \right) \\ &\quad \times \left( 2\tilde{z}_2^\nu + \frac{8i\theta\tau_1\tau_2(1 - \Omega^2)}{\Omega(\tau_1 + \tau_2)} (\Theta^{-1})^{\nu\sigma} (Q^{-1}\tilde{Z})_\sigma^4 \right) \\ &\times e^{-\frac{8\tau_1\tau_2}{\Omega\theta(\tau_1 + \tau_2)} (Z(Q^{-1})Z + i\theta^2 W Q^{-1}\tilde{Z} - \theta^2 W Q^{-1}W)} \Big|_{W=0} \\ &= \left( \frac{\tilde{\Omega}^2(1 - \tau_1^2)(1 - \tau_2^2)}{16(1 + \Omega^2)^2 \pi^2 T \tau_1\tau_2} \right)^2 \int d^4 z_1 d^4 z_2 A_\mu(z_1) A_\nu(z_2) \\ &\times \left( \frac{\Omega(1 - \Omega^2)^2 (\tau_1 + \tau_2)(1 - \tau_1\tau_2)}{\theta(1 + \Omega^2) T \tau_1\tau_2} \delta^{\mu\nu} \right. \\ &+ \left( -\frac{i\tilde{\Omega}(\tau_1 - \tau_2)}{2\tau_1\tau_2 T} \frac{(1 - \Omega^2)^2}{(1 + \Omega^2)^2} ((z_1^\mu - z_2^\mu) - \tau_1\tau_2(z_1^\mu + z_2^\mu)) + \frac{\Omega^2(\tau_1 + \tau_2)^2}{(1 + \Omega^2)\tau_1\tau_2 T} \tilde{z}_1^\mu \right) \\ &\times \left( -\frac{i\tilde{\Omega}(\tau_1 - \tau_2)}{2\tau_1\tau_2 T} \frac{(1 - \Omega^2)^2}{(1 + \Omega^2)^2} ((z_1^\nu - z_2^\nu) + \tau_1\tau_2(z_1^\nu + z_2^\nu)) + \frac{\Omega^2(\tau_1 + \tau_2)^2}{(1 + \Omega^2)\tau_1\tau_2 T} \tilde{z}_2^\nu \right) \end{aligned}$$

$$\times e^{-\frac{\Omega(\tau_1+\tau_2)}{2\theta\tau_1\tau_2(1+\Omega^2)T}(|z_1-z_2|^2+\tau_1\tau_2|z_1+z_2|^2)-\frac{2\Omega^2(\tau_1^2-\tau_2^2)}{\theta\tau_1\tau_2(1+\Omega^2)^2T}z_1\sigma z_2}. \quad (62)$$

We write  $(z_1 - z_2) \pm \tau_1 \tau_2 (z_1 + z_2)$  as derivative of the exponential, plus appropriate corrections, and integrate by parts:

$$\begin{aligned} V_2(A, A) &= \left( \frac{\tilde{\Omega}^2(1-\tau_1^2)(1-\tau_2^2)}{16(1+\Omega^2)^3\pi^2T\tau_1\tau_2} \right)^2 \int d^4z_1 d^4z_2 A_\mu(z_1)A_\nu(z_2) \\ &\times \left( \frac{2\tilde{\Omega}(1-\Omega^2)^2(1+\Omega^2)(1-\tau_1\tau_2)}{(\tau_1+\tau_2)} \delta_{\mu\nu} - 2i\Omega^2(1-\Omega^2)^2 \frac{(\tau_1^2-\tau_2^2)}{\tau_1\tau_2T} (\Theta^{-1})^{\mu\nu} \right. \\ &+ \left. \left( -i(1-\Omega^2)^2 \frac{\tau_1-\tau_2}{\tau_1+\tau_2} \frac{\partial}{\partial z_{2\mu}} + 4\Omega^2 \tilde{z}_1^\mu \right) \left( i(1-\Omega^2)^2 \frac{\tau_1-\tau_2}{\tau_1+\tau_2} \frac{\partial}{\partial z_{1\nu}} + 4\Omega^2 \tilde{z}_2^\nu \right) \right) \\ &\times e^{-\frac{\Omega(\tau_1+\tau_2)}{2\theta\tau_1\tau_2(1+\Omega^2)T}(|z_1-z_2|^2+\tau_1\tau_2|z_1+z_2|^2)-\frac{2\Omega^2(\tau_1^2-\tau_2^2)}{\theta\tau_1\tau_2(1+\Omega^2)^2T}z_1\sigma z_2} \\ &= \left( \frac{\tilde{\Omega}^2(1-\tau_1^2)(1-\tau_2^2)}{16(1+\Omega^2)^3\pi^2T\tau_1\tau_2} \right)^2 \int d^4z_1 d^4z_2 \\ &\times \left( \frac{2\tilde{\Omega}(1-\Omega^2)^2(1+\Omega^2)(1-\tau_1\tau_2)}{\tau_1+\tau_2} A_\mu(z_1)A^\mu(z_2) + 16\Omega^4 \tilde{z}_1^\mu A_\mu(z_1) \tilde{z}_2^\nu A_\nu(z_2) \right. \\ &+ (1-\Omega^2)^4 \frac{(\tau_1-\tau_2)^2}{(\tau_1+\tau_2)^2} (\partial_\nu A_\mu)(z_1) (\partial_\mu A_\nu)(z_2) \\ &+ 4i\Omega^2(1-\Omega^2)^2 \frac{\tau_1-\tau_2}{\tau_1+\tau_2} (A^\mu(z_1) \tilde{z}_2^\nu (\partial_\mu A_\nu)(z_2) - \tilde{z}_1^\mu (\partial_\nu A_\mu)(z_1) A^\nu(z_2)) \\ &- 2i\Omega^2(1-\Omega^2)^2 \frac{\tau_1-\tau_2}{\tau_1+\tau_2} \left( 4 + \frac{(\tau_1+\tau_2)^2}{\tau_1\tau_2T} \right) (\Theta^{-1})^{\mu\nu} A_\mu(z_1)A_\nu(z_2) \left. \right) \\ &\times e^{-\frac{\tilde{\Omega}(\tau_1+\tau_2)}{4\tau_1\tau_2(1+\Omega^2)T}(|z_1-z_2|^2+\tau_1\tau_2|z_1+z_2|^2)-\frac{2i\Omega^2(\tau_1^2-\tau_2^2)}{\tau_1\tau_2(1+\Omega^2)^2T}\langle z_1, \Theta^{-1}z_2 \rangle}. \quad (63) \end{aligned}$$

Again, the integrand is regular for  $z_1 \neq z_2$ , so that we expand

$$\begin{aligned} A_\nu(z_2) &= A_\nu(z_1) + (z_2^\rho - z_1^\rho)(\partial_\rho A_\nu)(z_1) + \frac{1}{2}(z_2^\rho - z_1^\rho)(z_2^\sigma - z_1^\sigma)(\partial_\rho \partial_\sigma A_\nu)(z_1) \\ &+ \frac{1}{2}(z_2^\rho - z_1^\rho)(z_2^\sigma - z_1^\sigma)(z_2^\kappa - z_1^\kappa) \int_0^1 d\xi (1-\xi)^2 (\partial_\rho \partial_\sigma \partial_\kappa A_\nu)(z_1 + \xi(z_2 - z_1)), \quad (64) \end{aligned}$$

and similarly for  $(\partial_\mu A_\nu)(z_2)$ . In leading  $t$ -order, we must expand  $A_\mu(z_1)A^\mu(z_2)$  up to second order (due to the appearance of  $(\tau_1 + \tau_2)^{-1}$ ) and all other terms only up to zeroth order. These leading terms become after a shift  $z_2 \mapsto z_2 + z_1$

$$\begin{aligned} V_2(A, A)^0 &= \left( \frac{\tilde{\Omega}^2(1-\tau_1^2)(1-\tau_2^2)}{16(1+\Omega^2)^3\pi^2T\tau_1\tau_2} \right)^2 \int d^4z_1 d^4z_2 \\ &\times \left( \frac{2\tilde{\Omega}(1-\Omega^2)^2(1+\Omega^2)(1-\tau_1\tau_2)}{\tau_1+\tau_2} \left( A_\mu(z_1)A^\mu(z_1) + A_\mu(z_1)(\partial_\rho A^\mu)(z_1) \frac{\partial}{\partial w_\rho} \right. \right. \\ &\left. \left. + \frac{1}{2} A_\mu(z_1)(\partial_\rho \partial_\sigma A^\mu)(z_1) \frac{\partial^2}{\partial w_\rho \partial w_\sigma} \right) \right) \end{aligned}$$

$$\begin{aligned}
& + 16\Omega^4 \tilde{z}_1^\mu A_\mu(z_1) \tilde{z}_1^\nu A_\nu(z_1) + (1 - \Omega^2)^4 \frac{(\tau_1 - \tau_2)^2}{(\tau_1 + \tau_2)^2} (\partial_\nu A_\mu)(z_1) (\partial_\mu A_\nu)(z_1) \\
& + 2(\Theta^{-1})^{\nu\rho} \left( 4i\Omega^2(1 - \Omega^2)^2 \frac{\tau_1 - \tau_2}{\tau_1 + \tau_2} A^\mu(z_1) (\partial_\mu A_\nu)(z_1) + 16\Omega^4 \tilde{z}_1^\mu A_\mu(z_1) A^\nu(z_1) \right) \frac{\partial}{\partial w^\rho} \\
& \times e^{-\frac{\tilde{\Omega}(\tau_1 + \tau_2)}{4\tau_1\tau_2(1 + \Omega^2)T} (1 + \tau_1\tau_2)|z_2|^2 + 4\tau_1\tau_2 \langle z_2, z_1 \rangle + 4\tau_1\tau_2 |z_1|^2 + \frac{2i\Omega^2(\tau_1^2 - \tau_2^2)}{\tau_1\tau_2(1 + \Omega^2)^2 T} \langle z_2, \Theta^{-1}z_1 \rangle + \langle w, z_2 \rangle} \Big|_{w=0} \\
& = \left( \frac{\tilde{\Omega}(1 - \tau_1^2)(1 - \tau_2^2)}{4\pi(1 + \Omega^2)^2(\tau_1 + \tau_2)(1 + \tau_1\tau_2)} \right)^2 \int d^4 z_1 e^{-\frac{\tilde{\Omega}(\tau_1 + \tau_2)}{(1 + \Omega^2)(1 + \tau_1\tau_2)} |z_1|^2} \\
& \times \left( \frac{2\tilde{\Omega}(1 - \Omega^2)^2(1 + \Omega^2)(1 - \tau_1\tau_2)}{\tau_1 + \tau_2} \left( A_\mu(z_1) A^\mu(z_1) \right. \right. \\
& \quad + A^\mu(z_1) (\partial_\nu A_\mu)(z_1) \left( -\frac{2\tau_1\tau_2}{1 + \tau_1\tau_2} z_1^\nu + \frac{i\theta\Omega(\tau_1 - \tau_2)}{(1 + \Omega^2)(1 + \tau_1\tau_2)} \tilde{z}_1^\nu \right) \\
& \quad + \frac{A_\mu(z_1) (\partial_\rho \partial_\sigma A^\mu)(z_1)}{2(1 + \tau_1\tau_2)^2} \left( 2\tau_1\tau_2 z_1^\rho - \frac{i\theta\Omega(\tau_1 - \tau_2)}{(1 + \Omega^2)} \tilde{z}_1^\rho \right) \left( 2\tau_1\tau_2 z_1^\sigma - \frac{i\theta\Omega(\tau_1 - \tau_2)}{(1 + \Omega^2)} \tilde{z}_1^\sigma \right) \\
& \quad \left. \left. + A^\mu(z_1) (\partial^\nu \partial_\nu A_\mu)(z_1) \frac{\tau_1\tau_2(1 + \Omega^2)T}{\tilde{\Omega}(\tau_1 + \tau_2)(1 + \tau_1\tau_2)} \right) \right) \\
& + 16\Omega^4 \tilde{z}_1^\mu A_\mu(z_1) \tilde{z}_1^\nu A_\nu(z_1) + (1 - \Omega^2)^4 \frac{(\tau_1 - \tau_2)^2}{(\tau_1 + \tau_2)^2} (\partial_\nu A_\mu)(z_1) (\partial_\mu A_\nu)(z_1) \\
& + \left( 4i\Omega^2(1 - \Omega^2)^2 \frac{\tau_1 - \tau_2}{\tau_1 + \tau_2} A^\mu(z_1) (\partial_\mu A_\nu)(z_1) + 16\Omega^4 \tilde{z}_1^\mu A_\mu(z_1) A_\nu(z_1) \right) \\
& \quad \times \left( -\frac{2\tau_1\tau_2}{1 + \tau_1\tau_2} \tilde{z}_1^\nu - \frac{2i\tilde{\Omega}(\tau_1 - \tau_2)}{(1 + \Omega^2)(1 + \tau_1\tau_2)} z_1^\nu \right). \tag{65}
\end{aligned}$$

### A.5 Moyal vertex plus Moyal-derivative vertex

This combination is (among others) necessary for a new type of coupling between scalar field and gauge field. We use as far as possible the same notation as in the previous calculation. Defining the auxiliary vector  $W = (0, 0, w_3, w_4)^t$ , we have

$$\begin{aligned}
V_2(A, f) & = \int d^4 x_1 d^4 y_1 d^4 x_2 d^4 y_2 (e^{-(t-t_2)H_0})(y_2, x_1) (L_\star(f))(x_1, y_1) \\
& \quad \times (e^{-t_2 H_0})(y_1, x_2) \{ L_\star(A^\mu), i\partial_\mu + \Omega^2 M_\bullet(\tilde{x}_\mu) \}(x_1, y_1) \\
& = \left( \frac{\tilde{\Omega}^2(1 - \tau_1^2)(1 - \tau_2^2)}{16\pi^2\tau_1\tau_2} \right)^2 \frac{1}{(2\pi\theta)^8} \int d^{16} X d^8 Z f(z_1) A_\mu(z_2) \\
& \quad \times \left( 2\tilde{z}_2^\mu - (1 - \Omega^2)(\Theta^{-1})^{\mu\rho} \frac{\partial}{\partial w_4^\rho} \right) e^{-\frac{\Omega(\tau_1 + \tau_2)}{8\theta\tau_1\tau_2} X^t Q X - \frac{2}{\theta} X^t \sigma Z + 2X^t W} \Big|_{W=0} \\
& = \left( \frac{\tilde{\Omega}^2(1 - \tau_1^2)(1 - \tau_2^2)}{16\pi^2\tau_1\tau_2} \right)^2 \left( \frac{4\tau_1\tau_2}{\Omega(\tau_1 + \tau_2)} \right)^8 (\det Q)^{-1/2} \int d^8 Z f(z_1) A_\mu(z_2) \\
& \quad \times \left( 2\tilde{z}_2^\mu + \frac{8i\theta\tau_1\tau_2(1 - \Omega^2)}{\Omega(\tau_1 + \tau_2)} (\Theta^{-1})^{\mu\rho} (Q^{-1} \tilde{Z})_\rho^4 \right) e^{-\frac{8\tau_1\tau_2}{\Omega\theta(\tau_1 + \tau_2)} Z(Q^{-1})Z}
\end{aligned}$$

$$\begin{aligned}
&= \left( \frac{\tilde{\Omega}^2(1-\tau_1^2)(1-\tau_2^2)}{16(1+\Omega^2)^2\pi^2 T\tau_1\tau_2} \right)^2 \int d^4 z_1 d^4 z_2 f(z_1) A_\mu(z_2) \\
&\times \left( -\frac{i\tilde{\Omega}(\tau_1-\tau_2)(1-\Omega^2)^2}{2\tau_1\tau_2 T(1+\Omega^2)^2} ((z_1^\mu - z_2^\mu) + \tau_1\tau_2(z_1^\mu + z_2^\mu)) + \frac{\Omega^2(\tau_1+\tau_2)^2}{(1+\Omega^2)\tau_1\tau_2 T} \tilde{z}_2^\mu \right) \\
&\times e^{-\frac{\Omega(\tau_1+\tau_2)}{2\theta\tau_1\tau_2(1+\Omega^2)T}(|z_1-z_2|^2+\tau_1\tau_2|z_1+z_2|^2) - \frac{2\Omega^2(\tau_1^2-\tau_2^2)}{\theta\tau_1\tau_2(1+\Omega^2)^2 T} z_1\sigma z_2} \\
&= \left( \frac{\tilde{\Omega}^2(1-\tau_1^2)(1-\tau_2^2)}{16(1+\Omega^2)^2\pi^2 T\tau_1\tau_2} \right)^2 \int d^4 z_1 d^4 z_2 \\
&\times \left( -i\frac{(1-\Omega^2)^2}{1+\Omega^2} \frac{\tau_1-\tau_2}{\tau_1+\tau_2} (\partial^\mu f)(z_1) A_\mu(z_2) + \frac{4\Omega^2}{1+\Omega^2} f(z_1) \tilde{z}_2^\mu A_\mu(z_2) \right) \\
&\times e^{-\frac{\Omega(\tau_1+\tau_2)}{2\theta\tau_1\tau_2(1+\Omega^2)T}(|z_1-z_2|^2+\tau_1\tau_2|z_1+z_2|^2) - \frac{2\Omega^2(\tau_1^2-\tau_2^2)}{\theta\tau_1\tau_2(1+\Omega^2)^2 T} z_1\sigma z_2} . \tag{66}
\end{aligned}$$

As before, for  $\tau_1, \tau_2 \rightarrow 0$  the integrand is regular unless  $z_1 = z_2$ , so that we expand  $A_\mu(z_2) = A_\mu(z_1) + (z_2^\nu - z_1^\nu) \int d\xi (\partial_\nu A_\mu)(z_1 + \xi(z_2 - z_1))$  and consider the leading term  $A_\mu(z_1)$ . After a shift  $z_2 \mapsto z_2 + z_1$  we have, neglecting the subleading summand  $\tilde{z}_2^\mu$ ,

$$\begin{aligned}
V_2(A, f)^0 &= \left( \frac{\tilde{\Omega}(1-\tau_1^2)(1-\tau_2^2)}{4\pi(1+\Omega^2)(\tau_1+\tau_2)(1+\tau_1\tau_2)} \right)^2 \int d^4 z_1 e^{-\frac{\tilde{\Omega}(\tau_1+\tau_2)}{(1+\Omega^2)(1+\tau_1\tau_2)}|z_1|^2} \\
&\times \left( i\frac{(1-\Omega^2)^2}{1+\Omega^2} \frac{\tau_1-\tau_2}{\tau_1+\tau_2} f(z_1) (\partial^\mu A_\mu)(z_1) + \frac{4\Omega^2}{1+\Omega^2} f(z_1) \tilde{z}_1^\mu A_\mu(z_1) \right) . \tag{67}
\end{aligned}$$

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